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The Numerical Range of $C_\psi^\star C_\varphi$ and $C_\varphi C_\psi^\star$

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Abstract: In this paper we investigate the numerical range of $C^\star_{b\varphi^m}C_{a\varphi^n}$ and $C_{a\varphi^n}C^\star_{b\varphi^m}$ on the Hardy space where φ is an inner function fixing the origin and a and b are points in the open unit disc. In the case when $|a| = |b| = 1$ we characterize the numerical range of these operators by constructing lacunary polynomials of unit norm whose image under the quadratic form incrementally foliate the numerical range. In the case when a and b are small we show numerical range of both operators is equal to the numerical range of the operator restricted to a 3-dimensional subspace.

Keywords: Composition operator, Numerical range, Inner function, Weighted shift

MSC: 47B33, 47A12, 15A60

1 Introduction

The Hardy space, $H^2(\mathbb{D})$, consists of all holomorphic functions f on the open unit disc $\mathbb{D}$ whose radial means are uniformly bounded in $L^2(\mathbb{T})$. The Hardy space is the domain for the class of composition operators C_φ defined by

$$C_\varphi(f) = f \circ \varphi,$$

where φ is a holomorphic self-map of the unit disc called the *inducing map* of the composition operator. Littlewood's Subordination Principle [8] asserts that composition operators are continuous on the Hardy space and that their norms satisfy the following inequality:

$$\sup_{\|f\|=1} \langle C_\varphi f, C_\varphi f\rangle = \|C_\varphi\|^2 \le \frac{1+|\varphi(0)|}{1-|\varphi(0)|}. \tag{1}$$

In particular, if $\varphi(0) = 0$, then $\|C_\varphi\| = 1$. For more information on composition operators, please see Cowen and MacClure [5], Martinez and Rosenthal [10], and Shapiro [12].

The *numerical range* of an operator T on a Hilbert space H is

$$W(T) = \{\langle Tx, x\rangle_H : \|x\|_H = 1\}, \tag{2}$$

and the *numerical radius* of T is $\omega(T) = \sup\{|\langle Tx, x\rangle_H| : \|x\|_H = 1\}$. The numerical range of composition operators was first studied by Matache in [9] and by Bourdon and Shapiro in [1, 2].

In what follows, we will investigate the numerical range of the product of a composition operator with the adjoint of a composition operator for a special class of inducing maps. Our problem seeks to characterize all possible pairings

$$\langle C_\varphi f, C_\psi f\rangle = \langle C_\psi^\star C_\varphi f, f\rangle \quad \text{and} \quad \langle C_\psi^\star f, C_\varphi^\star f\rangle = \langle C_\varphi C_\psi^\star f, f\rangle$$

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in terms of the inducing maps φ and ψ, subject to the constraint $\sum_{n=0}^{\infty} |\hat{f}(n)|^2 = \|f\|^2 = 1$.

2 Background

2.1 The numerical range

Some properties of the numerical range of a bounded linear operator T on a Hilbert space H that we will use in this paper are listed below.

1. $W(xT + yI) = xW(T) + y$ for any complex numbers x and y.
2. Suppose that $T = \oplus T_n$ on $H = \oplus H_n$. Then $W(T) = Co\left[\cup W(H_n)\right]$.
3. If $A = (a_{ij})$ and $B = (b_{ij})$ are n by n matrices with nonnegative terms and $a_{ij} \le b_{ij}$ then $\omega(A) \le \omega(B)$. If, in addition, $\frac{B+B^*}{2}$ is irreducible and $A \ne B$, then $\omega(A) < \omega(B)$.

This collection of results is a minimal set of tools that we will employ in our analysis. For a more comprehensive set of results and proofs of the aforementioned properties, we refer the reader to Gustafson and Rao, [6].

2.2 Sums of shifts

Let S_n be the backward shift on $\mathbb{C}^n$, that is $S_n(e_m) = e_{m-1}$, with matrix representation

$$S_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

It is well known that the numerical range of S_n is the closed disc centered at the origin with radius $\cos(\pi/(n+1))$ and the numerical range of the shift S on ℓ^2 is the open unit disc [7]. More generally, the numerical range of a weighted shift is a disc centered at the origin; thus, the numerical radius of a weighted shift determines the interior of the numerical range. Stout's foundational work in [15] characterizes the numerical radius of a finite weighted shift as the smallest root of a polynomial given in Theorem 2.

In order to set the stage for our main result we will first consider the operator $T = \oplus_{n=2}^{\infty} x^n S_n$ acting on $\ell^2 = \oplus_n \mathbb{C}^n$ for $0 < x \le 1$. By property (2), the numerical range of T is the convex hull of the union of $W(x^n S_n)$. The numerical range of $x^n S_n$ is a closed disc centered at the origin of radius $x^n \cos(\pi/(n+1))$. Thus $W(T)$ is a closed disc centered at the origin with radius $\sup_{n\in\mathbb{N}}(x^n \cos(\pi/(n+1)))$.

Theorem 1. *Suppose $T = \oplus_{n=2}^{\infty}(x^n S_n)$ and $\ell^2 = \oplus_{n=2}^{\infty}\mathbb{C}^n$. If*

$$\cos(\pi/n)/\cos(\pi/(n+1)) \le x \le \cos(\pi/(n+1))/\cos(\pi/(n+2))$$

then $\omega(T) = \omega(T_n) = x^n \cos(\pi/(n+1))$.

Proof. Fix $n_0 \in \mathbb{N}$. Let

$$\cos(\pi/n_0)/\cos(\pi/(n_0+1)) \le x \le \cos(\pi/(n_0+1))/\cos(\pi/(n_0+2)). \tag{3}$$

We will show $x^n \cos(\pi/(n+1)) \le x^{n_0} \cos(\pi/(n_0+1))$ for all $n \ne n_0$. The second inequality in (3) is equivalent to

$$x^{n_0+1} \cos(\pi/(n_0+2)) \le x^{n_0} \cos(\pi/(n_0+1)). \tag{4}$$

Now, if $x \le \cos(\pi/(n_0+1))/\cos(\pi/(n_0+2))$ then $x \le \cos(\pi/(n_0+2))/\cos(\pi/(n_0+3))$ because $\cos(\pi/n)/\cos(\pi/(n+1))$ is an increasing sequence. Rewriting $x \le \cos(\pi/(n_0+2))/\cos(\pi/(n_0+3))$ as

$$x^{n_0+2}\cos(\pi/(n_0+3)) \le x^{n_0+1}\cos(\pi/(n_0+2)), \tag{5}$$

we can conclude by inequalities (4) and (5), that

$$x^{n_0+2}\cos(\pi/(n_0+3)) \le x^{n_0}\cos(\pi/(n_0+1)). \tag{6}$$

Proceeding in this way we see

$$x^{n}\cos(\pi/(n+1)) \le x^{n_0}\cos(\pi/(n_0+1)) \quad \text{for all} \quad n \ge n_0.$$

The first inequality in (3) is equivalent to

$$x^{n_0-1}\cos(\pi/n_0) \le x^{n_0}\cos(\pi/(n_0+1)).$$

A similar argument to the one above shows

$$x^{n}\cos(\pi/(n+1)) \le x^{n_0}\cos(\pi/(n_0+1)) \quad \text{for all} \quad n \le n_0.$$

We then may conclude that

$$\omega(T) = \sup_{n\in\mathbb{N}}(x^n\cos(\pi/(n+1))) = x^{n_0}\cos(\pi/(n_0+1)) = \omega(x^{n_0}S_{n_0}).$$

□

2.3 The operators $C^{\star}_{\psi}C_{\varphi}$ and $C_{\varphi}C^{\star}_{\psi}$

In this section we introduce the operators $C^{\star}_{\varphi^m}C_{\varphi^n}$ and $C_{\varphi^n}C^{\star}_{\varphi^m}$ where φ is an inner function fixing the origin.

In the case when $\varphi = e_m$, where $e_m(z) = z^m$, the adjoint $C^{\star}_{e_m}$ has the following defining property:

$$C^{\star}_{e_m}(e_n) = \begin{cases} e_k & \text{if } n = mk \text{ for some integer } k \\ 0 & \text{otherwise.} \end{cases}$$

From here, it is immediate that

$$C_{e_\ell}C^{\star}_{e_m}(e_n) = \begin{cases} e_{\ell k} & \text{if } n = mk \text{ for some integer } k \\ 0 & \text{otherwise.} \end{cases}$$

In [11], Nordgren showed that a composition operator induced by an inner function fixing the origin is an isometry; that is, $C^{\star}_{\varphi}C_{\varphi} = I$. Using this fact and the observation that $C_{\varphi^n} = C_{\varphi}C_{e_n}$ we see

$$C^{\star}_{\varphi^m}C_{\varphi^n} = C^{\star}_{e_m}C_{e_n}. \tag{7}$$

Now suppose that $f = \sum_{k=0}^{\infty} a_k e_k$ is a function in $H^2(\mathbb{D})$. We have

$$C^{\star}_{e_m}C_{e_n}f = \sum_{k=0}^{\infty} a_{m_0k}e_{n_0k},$$

where $m_0 = m/GCD(m,n)$ and $n_0 = n/GCD(m,n)$.

We observe that if m divides n, then $C^{\star}_{\varphi^m}C_{\varphi^n} = C_{e_{n/m}}$, but that $C_{\varphi^n}C^{\star}_{\varphi^m}$ is a composition operator if and only if $\varphi^m(z) = e^{i\xi}z$ for some $\xi \in \mathbb{R}$. With this in mind, we recall the following theorem by Matache [9] using Stout's characterization of the numerical radius in [15].

Theorem 2. *The numerical range of $C_{\alpha e_n}$ for $n \in \mathbb{N}$ and $\alpha \in \mathbb{D}$ is the convex hull of the point $\{1\}$ and the closed disc with radius $1/\sqrt{t}$, where t is the smallest positive root of the entire function*

$$F(z) = \sum S_l(\dots, a_{-1}^2, a_0^2, a_1^2, \dots)(-1/4)^l z^l,$$

where $S_l(\dots, a_{-1}^2, a_0^2, a_1^2, \dots)$ are the circularly symmetric functions corresponding to the weight sequence $(a_n)_{n\in\mathbb{Z}}$, $a_n = |\alpha|^{k^n}$ if $n \geq 0$ and $a_n = 0$ if $n < 0$.

In what circumstances may we invoke this result, i.e., when is either $C_\varphi^\star C_\psi$ or $C_\varphi C_\psi^\star$ equal to $C_{\alpha e_n}$? As appears to be typical, the order $C_\varphi C_\psi^\star$ is more tractable. We will require the following result from [3]:

Theorem 3. *Suppose φ and ψ are holomorphic self-maps of $\mathbb{D}$ such that φ is non-constant and $C_\varphi = C_\psi T$ for some $T \in B(H^2(\mathbb{D}))$. Then there is a holomorphic self-map γ of $\mathbb{D}$ such that $T = C_\gamma$. Consequently, $\varphi = \gamma \circ \psi$.*

We recall that the adjoint of a composition operator maps a $H^2(\mathbb{D})$ reproducing kernel to a reproducing kernel,

$$C_\varphi^\star K_p = K_{\varphi(p)},$$

where $K_p(z) = 1/(1 - \overline{p}z)$ is the $H^2(\mathbb{D})$ reproducing kernel.

Theorem 4. *Let φ, ψ and θ be holomorphic self-maps of $\mathbb{D}$ and suppose $C_\varphi C_\psi^\star = C_\theta$. Then*

1. *if φ and θ are both nonconstant then $\psi(z) = e^{i\xi}z$ for some $\xi \in \mathbb{R}$;*
2. *if φ is constant, then θ is identically zero.*

Proof. 1. For any nonconstant inducing map the associated composition operator is injective on $H^2(\mathbb{D})$. If θ and φ are nonconstant, then C_θ and consequently $C_\psi^\star$ are injective because $C_\varphi C_\psi^\star = C_\theta$. Thus C_ψ is surjective which implies C_ψ is invertible. It follows from the invertibility of C_ψ that ψ is a conformal automorphism of the unit disc [10] that is there exist $\xi \in \mathbb{R}$ and $\beta \in \mathbb{D}$ with

$$\psi(z) = e^{i\xi}\frac{z-\beta}{1-\overline{\beta}z}.$$

By Theorem 3, $C_\psi^\star = C_\gamma$ for some holomorphic self-map γ of $\mathbb{D}$. Using Cowen's adjoint formula [4],

$$C_\psi^\star = M_{\frac{1}{e^{-i\xi}\overline{\beta}z+1}} C_{\frac{e^{-i\xi}z+\beta}{e^{-i\xi}\overline{\beta}z+1}} M_{1-\overline{\beta}z}^\star.$$

Applying this formula to $f(z) = 1$,

$$1 = C_\gamma(f(z)) = C_\psi^\star(f(z)) = \frac{(e^{-i\xi} - \beta e^{-i\xi})z}{(e^{-i\xi}z + \beta)(e^{-i\xi}\overline{\beta}z + 1)} + \frac{-\beta}{-\beta - e^{-i\xi}z},$$

from which one can deduce that $\beta = 0$. Therefore, $\psi(z) = e^{i\xi}z$.

2. Let $\varphi(z) = w$ for some $w \in \mathbb{D}$ and for all $z \in \mathbb{D}$. Set $K_p(z) = 1/(1 - \overline{p}z)$ for $p \in \mathbb{D}$ fixed and all $z \in \mathbb{D}$. We have that

$$\frac{1}{1-\overline{p}\theta(z)} = C_\theta K_p = C_\varphi C_\psi^\star K_p = C_\varphi K_{\psi(p)} = \frac{1}{1-\overline{\psi(p)}\varphi(z)} = \frac{1}{1-\overline{\psi(p)}w}.$$

We obtain $\theta(z) = \frac{\overline{\psi(p)}}{\overline{p}}w$ for nonzero p. Since the right-hand side is independent of z, we must have $\theta(z) = 0$ for all $z \in \mathbb{D}$. □

For the sake of completeness, we address the case in which θ is constant and φ is nonconstant in the statement of Theorem 4. Injectivity of C_φ forces the range of $C_\psi^\star$ to be one-dimensional, which in turn implies that C_ψ must have one-dimensional range, and so ψ must be constant.

Corollary 5. *Let φ and ψ be holomorphic self-maps of $\mathbb{D}$. Then $C_\varphi C_\psi^\star = C_{\alpha e_n}$ for $n \in \mathbb{N}$ and $\alpha \in \mathbb{D}\backslash\{0\}$ if and only if $\psi(z) = e^{i\xi}z$ for some $\xi \in \mathbb{R}$ and $\varphi(z) = e^{i\xi}\alpha z^n$. Consequently, the numerical range of $C_\varphi C_\psi^\star$ is a closed polygonal region, whose vertices form a subset of $\{\alpha^n : n \geq 0\}$. If $\alpha \in \mathbb{R}$ then $W(C_\psi^\star C_\varphi)$ or $W(C_\varphi C_\psi^\star)$ is $(0, 1]$ for $\alpha > 0$ and $[a, 1]$ for $\alpha \leq 0$.*

Proof. If $\psi(z) = e^{i\xi}z$ for some $\xi \in \mathbb{R}$ and $\varphi(z) = e^{i\xi}\alpha z^n$,

$$C_\varphi C_\psi^\star = C_{e^{i\xi}\alpha e_n} C_{e^{i\xi}z}^\star = C_{e^{i\xi}\alpha e_n} C_{e^{-i\xi}z} = C_{\alpha e_n}.$$

Now suppose that $C_\varphi C_\psi^\star = C_{\alpha e_n}$ for $n \in \mathbb{N}$ and $\alpha \in \mathbb{D}\backslash\{0\}$. Assuming φ to be nonconstant, the reverse direction follows by applying (1) in Theorem 4, and the concluding statement by applying Theorem 2. Assuming $\varphi(z)$ is identically equal to a constant $w \in \mathbb{D}$, (2) in Theorem 4 yields a contradiction since $\alpha \neq 0$. □

We can currently say very little in the case where $C_\varphi^\star C_\psi = C_\theta$. If θ is inner and $\theta(0) = 0$, then the main result of section 4 in [3] shows that φ must be inner with $\varphi(0) = 0$ and $\psi = \theta \circ \varphi$.

3 The Numerical Range of $C_{\varphi^n} C_{\varphi^m}^\star$ and $C_{\varphi^m}^\star C_{\varphi^n}$ with φ an inner function fixing the origin

We start by computing the point spectrum of the two products $C_{\varphi^n} C_{\varphi^m}^\star$ and $C_{\varphi^m}^\star C_{\varphi^n}$. Equation (7) shows us that if φ is inner with $\varphi(0) = 0$, $C_{\varphi^m}^\star C_{\varphi^n} = C_{e_m}^\star C_{e_n}$. Furthermore, if $k = GCD(m, n)$ and $n_0 = n/k$, $m_0 = m/k$ as in Section 2,

$$C_{e_m}^\star C_{e_n} = C_{e_{m_0}}^\star C_{e_k}^\star C_{e_k} C_{e_{n_0}} = C_{e_{m_0}}^\star C_{e_{n_0}}. \tag{8}$$

We retain the notation for the proof of Theorem 6.

Although no similar reductions exist for $C_{\varphi^n} C_{\varphi^m}^\star$, we may decompose $H^2(\mathbb{D})$ in such a manner that the questions we are interested in reduce to questions about $C_{e_n} C_{e_m^\star}$.

Theorem 6. *Suppose that φ is an inner function that fixes the origin. Then*

1. *The point spectrum of $C_{\varphi^m}^\star C_{\varphi^n}$ is*
   - $\{1\}$ *if n is a multiple of m;*
   - $\mathbb{D} \cup \{1\}$ *if n is not a multiple of m but m is a multiple of n;*
   - $\{0, 1\}$ *if neither n is a multiple of m nor m a multiple of n;*
2. *The point spectrum of $C_{\varphi^n} C_{\varphi^m}^\star$ is $\{1\}$ if $\varphi(z) = e^{i\xi}z$ for some $\xi \in \mathbb{R}$ and $m = 1$; otherwise, the point spectrum is $\{0, 1\}$.*

Proof. Clearly 1 is in the point spectrum of any composition operator with corresponding eigenfunction 1, and the calculation $C_{\varphi^m}^\star 1 = C_{\varphi^m}^\star K_0 = K_{\varphi^m(0)} = K_0 = 1$ shows 1 is in the point spectrum of $C_{\varphi^m}^\star$. Thus 1 is in the point spectrum of both $C_{\varphi^n} C_{\varphi^m}^\star$ and $C_{\varphi^m}^\star C_{\varphi^n}$.

1. For the operator $C_{\varphi^m}^\star C_{\varphi^n}$, we know that $C_{\varphi^m}^\star C_{\varphi^n} = C_{e_{m_0}}^\star C_{e_{n_0}}$ from Equation (8). If m_0 is not equal to 1 (corresponding to the case where n is not a multiple of m), the function z is in the null space of $C_{e_{m_0}}^\star C_{e_{n_0}}$, as e_{n_0} is in the null space of $C_{e_{m_0}}^\star$, so 0 is in the point spectrum. If $m_0 = 1$ (corresponding to the case where n is a multiple of m), then $C_{e_{m_0}}^\star C_{e_{n_0}} = C_{e_{n_0}}$ is an isometry, and so 0 is not in the point spectrum of $C_{e_{n_0}}$.

Let $0 \neq f = \sum_{k=0}^\infty a_k e_k \in H^2(\mathbb{D})$ and suppose $C_{e_{m_0}}^\star C_{e_{n_0}} f = \lambda f$ for some $\lambda \neq 0, 1$. Then

$$\lambda \sum_{k=0}^\infty a_k e_k = \lambda f = C_{e_{m_0}}^\star C_{e_{n_0}} f = \sum_{k=0}^\infty a_{m_0 k} e_{n_0 k}.$$

We see that $a_k = 0$ if k is not a multiple of n_0. In light of this fact, we recalculate:

$$\lambda \sum_{k=0}^\infty a_{n_0 k} e_{n_0 k} = \lambda f = C_{e_{m_0}}^\star C_{e_{n_0}} f = \sum_{k=0}^\infty a_{m_0 n_0 k} e_{n_0^2 k},$$

and so $a_k = 0$ if k is not a multiple of n_0^2. Continuing in this fashion, if $n_0 \neq 1$, we must have that f is constant, and hence $\lambda = 1$, a contradiction. If $n_0 = 1$, then $C_{e_{m_0}}^\star C_{e_{n_0}} = C_{e_{m_0}}^\star$ whose spectrum is $\{1\}$ if $m_0 = 1$. If $n_0 = 1$ and $m_0 \neq 1$, then we reduce to finding the point spectrum of $C_{e_{m_0}}^\star$. Take $\lambda \in \mathbb{D}$, and $l \in \mathbb{N}$, $l > 1$, with $GCD(l, m_0) = 1$. The function

$$f_\lambda(z) = \sum_{k=0}^\infty \lambda^k z^{l m_0^k},$$

is an eigenvector for $C^\star_{e_{m_0}}$ corresponding to the eigenvalue λ. By considering similar sums, we can see that if $|\lambda| = 1$ and $\lambda \neq 1$, then λ is not in the point spectrum of $C^\star_{e_{m_0}}$, and so we obtain that the $\mathbb{D} \cup \{1\}$ is the point spectrum of $C^\star_{e_{m_0}}$

2. For the operator $C_{\varphi^n} C^\star_{\varphi^m}$, if $\varphi(z) \neq e^{i\xi} z$ for some $\xi \in \mathbb{R}$, then $C^\star_\varphi$ (and hence $C^\star_{\varphi^m}$ for $m > 1$) is not injective, so 0 is in the point spectrum of $C_{\varphi^n} C^\star_{\varphi^m}$. If $\varphi(z) = e^{i\xi} z$ for some $\xi \in \mathbb{R}$ and $m > 1$, $C^\star_{\varphi^m} z = 0$ and so 0 is in the point spectrum of $C_{\varphi^n} C^\star_{\varphi^m}$. If $\varphi(z) = e^{i\xi} z$ for some $\xi \in \mathbb{R}$ and $m = 1$, $C_{\varphi^n} C^\star_\varphi$ is injective for all $n \in \mathbb{N}$, and hence 0 is not in the point spectrum.

Finally, suppose $\lambda \in \mathbb{C} \setminus \{1\}$ and there exists a non-zero function $f \in H^2(\mathbb{D})$ such that

$$C_{\varphi^n} C^\star_{\varphi^m} f = \lambda f.$$

Clearly f must be in the range of C_{φ^n}, which in turn is a subspace of

$$C_\varphi H^2(\mathbb{D}) = \{f \, : \, f = \sum_{k=0}^{\infty} a_k \varphi^k\}.$$

Thus we will assume that $f = \sum_{k=0}^{\infty} a_k \varphi^k$. Then

$$C_{\varphi^n} C^\star_{\varphi^m} f = \sum_{k=0}^{\infty} a_{mk} \varphi^{nk}$$

Now, equating coefficients for λf and $C_{\varphi^n} C^\star_{\varphi^m} f$ and using the same argument as for $C^\star_{e_m} C_{e_n}$ we see that they all are zero save for a_0.

□

In order to capture the numerical range of $C_{\varphi^n} C^\star_{\varphi^m}$ and $C^\star_{\varphi^m} C_{\varphi^n}$, we wish to reduce to $\varphi(z) = z$; this reduction will be useful in Section 4. The following lemma accomplishes this goal.

Lemma 7. *If φ is an inner function that fixes the origin, then $W(C^\star_{\varphi^m} C_{\varphi^n}) = W(C^\star_{e_m} C_{e_n})$. If $m = n = 1$ and $\varphi(z) \neq e^{i\xi} z$ for some $\xi \in \mathbb{R}$, $W(C_{e_n} C^\star_{e_m}) \subset W(C_{\varphi^n} C^\star_{\varphi^m})$; otherwise $W(C_{\varphi^n} C^\star_{\varphi^m}) = W(C_{e_n} C^\star_{e_m})$.*

Proof. The first claim is immediate by Equation (7). For the second, take $f = \sum_{k=0}^{\infty} a_k e_k \in H^2(\mathbb{D})$ of norm 1. Then

$$\begin{aligned} \langle C_{e_n} C^\star_{e_m} f, f \rangle &= \langle C_{e_n} C^\star_{e_m} C^\star_\varphi C_\varphi f, C^\star_\varphi C_\varphi f \rangle \\ &= \langle C_\varphi C_{e_n} C^\star_{e_m} C^\star_\varphi C_\varphi f, C_\varphi f \rangle \\ &= \langle C_{\varphi^n} C^\star_{\varphi^m} f \circ \varphi, f \circ \varphi \rangle. \end{aligned}$$

Since C_φ is isometric from $H^2(\mathbb{D})$ to $C_\varphi H^2(\mathbb{D})$, $\|f \circ \varphi\|_2 = 1$, and so

$$W(C_{e_n} C^\star_{e_m}) \subseteq W(C_{\varphi^n} C^\star_{\varphi^m}).$$

For the reverse inclusion, the previous calculation shows that the numerical range of the restriction of $C_{\varphi^n} C^\star_{\varphi^m}$ to $C_\varphi(H^2(\mathbb{D}))$ is equal to the numerical range of $C_{e_n} C^\star_{e_m}$. However, $C_\varphi(H^2(\mathbb{D}))^\perp$ is contained in the nullspace of $C_{\varphi^n} C^\star_{\varphi^m}$, and so the only potential contribution to the numerical range from $C_\varphi(H^2(\mathbb{D}))^\perp$ is 0. By Theorem 6, if $m \neq 1$, then 0 is the point spectrum of $C_{e_n} C^\star_{e_m}$, which is contained in $W(C_{e_n} C^\star_{e_m})$. If $m = 1$ but $n \neq 1$, then $\langle C_{e_n} z, z \rangle = 0$, and so 0 is in the numerical range of C_{e_n}. Using the third listed property of the numerical range from Section 2.1, $W(C_{\varphi^n} C^\star_{\varphi^m}) \subseteq W(C_{e_n} C^\star_{e_m})$ and so we obtain equality of the numerical ranges if $m \neq 1$ or $n \neq 1$. If $m = n = 1$ and $\varphi(z) = e^{i\xi} z$ for some $\xi \in \mathbb{R}$, then $C_\varphi C^\star_\varphi = C_{e_1} C^\star_{e_1} = I$ and the numerical ranges are equal.

Now suppose $m = n = 1$ and $\varphi(z) \neq e^{i\xi} z$ for some $\xi \in \mathbb{R}$. Then C_φ is not unitary, so $0 \in W(C_\varphi C^\star_\varphi)$, but $C_{e_1} C^\star_{e_1} = I$ and $W(I) = \{1\}$. □

Theorem 8. *Let φ be an inner function that fixes the origin. Then*

1. *if $m \neq 1 \neq n$, $W(C_{\varphi^n} C^\star_{\varphi^m}) = W(C^\star_{\varphi^m} C_{\varphi^n}) = \mathbb{D} \cup \{1\}$;*

2. *if* $n = m = 1$ *and* $\varphi(z) = e^{i\xi}z$ *for some* $\xi \in \mathbb{R}$, *then*

$$W(C_\varphi C_\varphi^\star) = W(C_\varphi^\star C_\varphi) = \{1\};$$

3. *if* $n = m = 1$ *and* $\varphi(z) \neq e^{i\xi}z$ *for some* $\xi \in \mathbb{R}$, *then*

$$\{1\} = W(C_\varphi^\star C_\varphi) \subset W(C_\varphi C_\varphi^\star) = [0, 1].$$

Proof. 1. By the previous lemma, we reduce to $\varphi(z) = z$. Since 1 is in the point spectrum of both $C_{e_n}C_{e_m}^\star$ and $C_{e_m}^\star C_{e_n}$, with corresponding eigenfunction 1, we can conclude that 1 is in the numerical range of both operators.

The family of polynomials

$$p_N(z) = \frac{1}{\sqrt{N+1}} \sum_{k=0}^{N} e^{-ik\theta} z^{m^{N-k}n^k}, \quad N \in \mathbb{N} \text{ and } \theta \in \mathbb{R}, \tag{9}$$

all have norm one, so $\langle C_{e_m}^\star C_{e_n} p_N, p_N\rangle \in W(C_{e_m}^\star C_{e_n})$. A calculation shows

$$\langle C_{e_m}^\star C_{e_n} p_N, p_N\rangle = \langle p_N(e_n), p_N(e_m)\rangle = \frac{N}{N+1} e^{i\theta}, \quad \theta \in \mathbb{R}.$$

It follows that $\left(1 - \frac{1}{N+1}\right)\overline{\mathbb{D}}$ is contained in $W(C_{e_m}^\star C_{e_n})$ for every $N > 0$, from which we see $\mathbb{D} \subseteq W(C_{e_m}^\star C_{e_n})$. A similar argument shows that $\mathbb{D} \subseteq W(C_{e_n}C_{e_m}^\star)$.

Lastly, we will show that if $|\lambda| = 1$ and $\lambda \neq 1$ then λ is not in $W(C_{e_n}C_{e_m}^\star)$ or $W(C_{e_m}^\star C_{e_n})$. Suppose that $\lambda \in W(C_{e_m}^\star C_{e_n})$; that is, there exists a function f with norm one such that $\langle C_{e_m}^\star C_{e_n} f, f\rangle = \lambda$. By the Cauchy-Schwartz Inequality we know that

$$1 = |\langle C_{e_m}^\star C_{e_n} f, f\rangle| \leq \|C_{e_m}^\star\|\|C_{e_n}\| = 1.$$

Equality in the Cauchy-Schwartz inequality implies λ is in the point spectrum with corresponding eigenfunction f. This contradicts the fact that λ is not in the point spectrum of $C_{e_m}^\star C_{e_n}$. A similar argument holds for the operator $C_{e_n}C_{e_m}^\star$.

2. If $n = m = 1$ and $\varphi(z) = e^{i\xi}z$ for some $\xi \in \mathbb{R}$, then C_φ is unitary, so $W(C_\varphi C_\varphi^\star) = W(C_\varphi^\star C_\varphi) = W(I) = \{1\}$.

3. If $n = m = 1$ and $\varphi(z) \neq e^{i\xi}z$ for some $\xi \in \mathbb{R}$, then C_φ is a non-unitary isometry, so $0 < C_\varphi C_\varphi^\star < I$ is an orthogonal projection with numerical range equal to $[0, 1]$.

□

4 The Numerical Range of $C_{a\varphi^n}C_{b\varphi^m}^\star$ and $C_{b\varphi^m}^\star C_{a\varphi^n}$ with φ an inner function fixing the origin and $a, b \in \mathbb{D}$

In this section, φ will be an inner function that fixes the origin and $a, b \in \mathbb{D}$. The principal result of this section is to characterize the numerical range of $C_{a\varphi^n}C_{b\varphi^m}^\star$ and $C_{b\varphi^m}^\star C_{a\varphi^n}$ in terms of a and b in the case when the modulus of $a\bar{b}$ is sufficiently small (more precisely, when it is slightly away from 1). First, we will make a reduction to the case when $\varphi(z) = z$. Indeed, since $C_{a\varphi^n} = C_\varphi C_{ae_n}$ we see that $C_{b\varphi^m}^\star C_{a\varphi^n} = C_{be_m}^\star C_{ae_n}$. When the order of the operators is reversed we no longer have equality of the operators $C_{a\varphi^n}C_{b\varphi^m}^\star$ and $C_{ae_n}C_{be_m}^\star$, but Lemma 7 can be easily modified to show that $W(C_{a\varphi^n}C_{b\varphi^m}^\star) = W(C_{ae_n}C_{be_m}^\star)$. So, we will perform all subsequent computations for the operators in the case when $\varphi(z) = z$.

Assume that neither n nor m are one, otherwise we can reduce the problem to the case of a single composition operator and appeal to work of Matache, [9]. Applying the notation from Section 3, we write $m = km_0$ and $n = kn_0$ where $GCD(m, n) = k$ is the greatest common divisor of m and n and $GCD(m_0, n_0) = 1$. Then,

$$C_{be_m}^\star C_{ae_n} = C_{be_{m_0}}^\star C_{e_k}^\star C_{e_k} C_{ae_{n_0}} = C_{be_{m_0}}^\star C_{ae_{n_0}}.$$

When the adjoint comes first, we have

$$C_{ae_n} C^{\star}_{be_m} = C_{e_k} C_{ae_{n_0}} C^{\star}_{be_{m_0}} C^{\star}_{e_k}.$$

From this factorization, we see that for any $f \in H^2(\mathbb{D})$,

$$\langle C_{ae_n} C^{\star}_{be_m} f, f\rangle = \langle C_{ae_{n_0}} C^{\star}_{be_{m_0}} C^{\star}_{e_k} f, C^{\star}_{e_k} f\rangle.$$

This implies that we only need to consider functions f in $C_{e_k}(zH^2(\mathbb{D}))$ when studying the numerical range of this operator. If $k \neq 1$, our subsequent analysis for the operator $C_{ae_n} C^{\star}_{be_m}$ can be modified by applying C_{e_k} to our subspace bases, so we will therefore assume $GCD(m, n) = 1$ for a more elegant presentation.

We will now decompose $H^2(\mathbb{D})$ into finite dimensional subspaces on which the operators $C^{\star}_{be_m} C_{ae_n}$ and $C_{ae_n} C^{\star}_{be_m}$ are forward weighted shifts. Since 1 is in the point spectrum of either of these operators and their adjoints, $\mathbb{C}$ is a reducing subspace for $C^{\star}_{be_m} C_{ae_n}$ and $C_{ae_n} C^{\star}_{be_m}$. This allows us to focus on $zH^2(\mathbb{D})$. Let $H_{N,q}$ be the subspace of $zH^2(\mathbb{D})$ with orthonormal basis functions $\{e_{m^N q}, e_{m^{N-1} nq}, \ldots, e_{mn^{N-1} q}, e_{n^N q}\}$, where $q \geq 1$ is not divisible by m or n, that is,

$$H_{N,q} = \operatorname{span}\{e_{m^N q}, e_{m^{N-1} nq}, \ldots, e_{mn^{N-1} q}, e_{n^N q}\}.$$

Then

$$zH^2(\mathbb{D}) = \left(\bigoplus_{q} \bigoplus_{N=1}^{\infty} H_{N,q}\right) \bigoplus M,$$

where M is the orthogonal complement of $\bigoplus_q \bigoplus_{N=1}^{\infty} H_{N,q}$ in $zH^2(\mathbb{D})$.

Each monomial e_k is in exactly one of the subspaces $H_{N,q}$ or M. Indeed, if k has either m or n as a divisor, then $e_k \in H_{N,q}$ for some N and q. If neither m nor n are divisors of k, then $e_k \in M$ since m and n are relatively prime. Now if $e_k \in M$ then k is not divisible by m, so M is clearly a subset of the null space of $C_{ae_n} C^{\star}_{be_m}$. Similarly, we can see that M is a subset of the null space of $C^{\star}_{be_m} C_{ae_n}$ since n and m are relatively prime. M will therefore contribute nothing more than zero to the numerical range of these operators. This allows us to focus on the numerical range of these operators restricted to the spaces $H_{N,q}$.

Let $T_{N,q}$ be the restriction of $C^{\star}_{be_m} C_{ae_n}$ to $H_{N,q}$ and let $S_{N,q}$ be the restriction of $C_{ae_n} C^{\star}_{be_m}$ to $H_{N,q}$ for $N, q \geq 1$. Throughout the rest of the paper we will also let $T_{N,q}$ and $S_{N,q}$ be the matrix representation of the restricted operators with respect to the orthonormal basis $\{e_{m^N q}, e_{m^{N-1} nq}, \ldots, e_{mn^{N-1} q}, e_{n^N q}\}$.

Lemma 9. *For all $q \geq 1$, the operator $C_{ae_n} C^{\star}_{be_m}$ restricted to $H_{N,q}$ is a weighted forward shift with matrix representation*

$$S_{N,q} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ v^{m^{N-1} q} & 0 & \cdots & 0 & 0 \\ 0 & v^{m^{N-2} nq} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & v^{n^{N-1} q} & 0 \end{pmatrix}. \tag{10}$$

where $v = a\bar{b}$. The operator $C^{\star}_{be_m} C_{ae_n}$ restricted to $H_{N,q}$ is also a weighted forward shift with matrix representation

$$T_{N,q} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \delta^{m^{N-1} q} & 0 & \cdots & 0 & 0 \\ 0 & \delta^{m^{N-2} nq} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \delta^{n^{N-1} q} & 0 \end{pmatrix}. \tag{11}$$

where $\delta = a^m \overline{b}^n$.

Proof. The matrix representations (10) and (11) immediately follow from the pair of calculations

$$\begin{aligned} C_{ae_n}C^\star_{be_m}e_{m^{N-k}n^kq} &= a^{m^{N-k-1}n^kq}(\overline{b})^{m^{N-k-1}n^kq}e_{m^{N-k-1}n^{k+1}q} \\ &= (a\bar{b})^{m^{N-k-1}n^kq}e_{m^{N-k-1}n^{k+1}q} \\ &= \nu^{m^{N-k-1}n^kq}e_{m^{N-k-1}n^{k+1}q} \end{aligned}$$

and

$$\begin{aligned} C^\star_{be_m}C_{ae_n}e_{m^{N-k}n^kq} &= a^{m^{N-k}n^kq}(\overline{b})^{m^{N-k-1}n^{k+1}q}e_{m^{N-k-1}n^{k+1}q} \\ &= (a^m(\overline{b})^n)^{m^{N-k-1}n^kq}e_{m^{N-k-1}n^{k+1}q} \\ &= \delta^{m^{N-k-1}n^kq}e_{m^{N-k-1}n^{k+1}q}. \end{aligned}$$

Each of these calculations is valid for all $k = 0, 1, \ldots, N-1$ and all $0 < |\delta|, |\nu| < 1$, which gives the desired result.

□

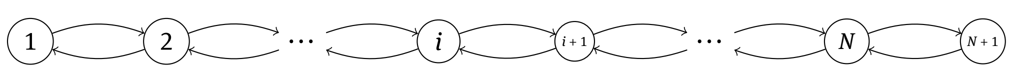
We now see by lemma 9 that $T_{N,q}$ and $S_{N,q}$ are all weighted forward shifts on the appropriate subspace of $zH^2(\mathbb{D})$. Since any weighted shift is unitarily similar to a weighted shift with positive weights, we may assume that both δ and ν are real and positive in the remainder of the analysis. We will now show that for $q > 1$, the numerical ranges of the operators $T_{N,q}$ and $S_{N,q}$ are strictly contained in the numerical ranges of $T_{N,1}$ and $S_{N,1}$ respectively. This allows us to focus our attention on the numerical range of $T_{N,1}$ or $S_{N,1}$. To this end, define $T_N = T_{N,1}$ and $S_N = S_{N,1}$.

Lemma 10. *Let $q > 1$. The numerical radius of $T_{N,q}$ is strictly less then the numerical radius of T_N, that is, $w(T_{N,q}) < w(T_N)$ for all $0 < \delta < 1$. Similarly, the numerical radius of $S_{N,q}$ is strictly less then the numerical radius of S_N for all $0 < \nu < 1$*

Proof. Symmetrize the operator T_N,

$$T_N + T_N^\star = \begin{pmatrix} 0 & \delta^{m^{N-1}} & 0 & \cdots & 0 & 0 \\ \delta^{m^{N-1}} & 0 & \delta^{m^{N-2}n} & 0 & \cdots & 0 \\ 0 & \delta^{m^{N-2}n} & 0 & \cdots & 0 & \vdots \\ 0 & 0 & \delta^{m^{N-3}n^2} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \delta^{n^{N-1}} \\ 0 & 0 & \cdots & 0 & \delta^{n^{N-1}} & 0 \end{pmatrix}.$$

The associated graph $\mathcal{G}$ has an edge from vertex i to $i+1$ and from vertex $i+1$ to i for all $1 \le i \le N$. Visually,



Therefore, $\mathcal{G}$ is strongly connected and so $\frac{T_N+T_N^\star}{2}$ is irreducible. The lemma follows immediately from (3) in Section 2.1 with $A = T_{N,q}$ and $B = T_N$. Replacing δ with ν gives the result for S_N and $S_{N,q}$. □

In the next theorem we prove that the numerical range of the operators $C^\star_{be_m}C_{ae_n}$ and $C_{ae_n}C^\star_{be_m}$ can be exactly characterized when δ or ν are sufficiently small. A calculation shows that

$$\begin{aligned} \omega(T_1) &= \frac{\delta}{2}, \quad \omega(T_2) = \frac{\sqrt{\delta^{2m} + \delta^{2n}}}{2} \\ \omega(S_1) &= \frac{\nu}{2}, \quad \omega(S_2) = \frac{\sqrt{\nu^{2m} + \nu^{2n}}}{2}. \end{aligned}$$

Let τ_1 be the unique positive root of the equation $x^{2m-2} + x^{2n-2} = 1$ on the interval $[0, 1]$. We have the following characterization,

Theorem 11. *If $\delta < \tau_1$, then*

$$W(C^{\star}_{be_m} C_{ae_n}) = Co\left(\frac{\delta}{2}\overline{\mathbb{D}} \cup \{1\}\right).$$

Similarly, if $\nu < \tau_1$, then

$$W(C_{ae_n} C^{\star}_{be_m}) = Co\left(\frac{\nu}{2}\overline{\mathbb{D}} \cup \{1\}\right).$$

Proof. We will prove this theorem for the operator $C^{\star}_{be_m} C_{ae_n}$. The proof when the adjoint comes first is similar. Let $f \in H_N$ be a function of unit norm. By Lemma 10 it suffices to show that $|\langle T_N f, f\rangle| < \delta/2$. Let $N > 2$, then we see from (11) that

$$|\langle T_N f, f\rangle| = \left|\sum_{k=0}^{N-1} a_k \overline{a_{k+1}} \delta^{m^{N-k-1} n^k}\right| \le \frac{1}{2}\sum_{k=0}^{N-1}(|a_k|^2 + |a_{k+1}|^2)\delta^{m^{N-k-1} n^k}, \tag{12}$$

where $f = \sum_{k=0}^{N} a_k e_{m^{N-k} n^k}$. Grouping the common terms in the sum on the right-hand side of Equation (12), we see that it is less than or equal to

$$\frac{\delta}{2}\left(|a_0|^2 + |a_N|^2 + \sum_{k=1}^{N-1} |a_k|^2 (\delta^{m^{N-k} n^{k-1} - 1} + \delta^{m^{N-k-1} n^k - 1})\right).$$

Now, it is easy to see that $\delta^{m^{N-k} n^{k-1} - 1} + \delta^{m^{N-k-1} n^k - 1} < \delta^{2m-2} + \delta^{2n-2}$ for all $1 \le k \le N-1$ provided $m, n \ge 2$. The definition of τ_1 and the above estimate implies that $|\langle T_N f, f\rangle| \le \delta/2$.

□

Theorem 11 completely characterizes the numerical range of $C^{\star}_{be_m} C_{ae_n}$ when δ is small (as well as the numerical range of $C_{ae_n} C^{\star}_{be_m}$ when ν is small). It should be noted that even for small values of m and n (say $m = 2$ and $n = 3$), the numerical range of T_2 finally subsumes the numerical range of T_1 when $|a^m \bar{b}^n|$ is slightly larger than $7/10$. This implies that for a large collection of pairs a and b we can completely characterize the numerical range explicitly. The numerical range of this operator will always have the form $Co\left(r\overline{\mathbb{D}} \cup \{1\}\right)$ for $r = \omega(T_N)$ for some $N \ge 1$. As $\omega(T_N)$ is a continuously increasing convex function of δ which approaches $\cos(\frac{\pi}{N+1})$ as δ tends to one, we see that there will always be a value of δ, call it δ_N, for which $\omega(T_N(\delta)) > \omega(T_k(\delta))$ for $\delta \in [\delta_N, 1)$ and $k \le N-1$. In particular, this implies that as δ increases to one, the numerical range of $C^{\star}_{be_m} C_{ae_n}$ approaches $\mathbb{D} \cup \{1\}$, in agreement with Theorem 8.

The proof of the preceding theorem (along with numerical evidence in the case of small n and m) gives us reason to conjecture that there is a unique point of intersection between $\omega(T_k)$ and $\omega(T_n)$, call it $\tau(k, n)$, with the property that $\omega(T_n) > \omega(T_k)$ for all $\delta \in [\tau(k, n), 1)$. There is certainly one such value $\tau(k, n)$, and if this point is unique, then we can set $\delta_n = \tau(n, n+1)$ and conclude that this sequence is monotonic in n,

$$\delta_1 < \delta_2 < \cdots < \delta_n < \cdots$$

Similar reasoning will clearly apply to the operator $C_{ae_n} C^{\star}_{be_m}$ by replacing δ with ν and T_N with S_N. What we do have is the following theorem:

Theorem 12. *The numerical range of $C^{\star}_{b\varphi^m} C_{a\varphi^n}$ is the convex hull of the point $\{1\}$ and the closed disc with radius r, where $r = \max\{\omega(T_k) : k \in \mathbb{N}\}$. Similarly, the numerical range of $C_{a\varphi^n} C^{\star}_{b\varphi^m}$ is the convex hull of the point $\{1\}$ and the closed disc with radius ρ, where $\rho = \max\{\omega(S_k) : k \in \mathbb{N}\}$.*

Proof. By our reductions, we may assume that $\varphi(z) = z$ and $GCD(m, n) = 1$. We will only consider the operator $C^{\star}_{be_m} C_{ae_n}$. It follows from the same argument in the proof of Theorem 8 that the numerical range of T_N is a closed disc with radius $\omega(T_N)$. Our operator is a direct sum of operators of the form $T_{N,q}$. Since

$$|\langle T_N f, f\rangle| = \left|\sum_{k=0}^{N-1} a_k \overline{a_{k+1}} \delta^{m^{N-k-1} n^k}\right| \le N\delta^{(\min(m,n))^N - 1},$$

we see that $\omega(T_N(\delta))$ tends to zero as $N \to \infty$ for a fixed $\delta = |a^m b^n|$. From this it follows that the countable set $\{\omega(T_k) : k \in \mathbb{N}\}$ attains its maximum. Lemma 10 implies that the numerical radius of $C^\star_{be_m} C_{ae_n}$ will agree with the numerical radius of T_N for some $N \geq 1$. The result now follows from property 2 of Section 2.

□

Lemma 10 tells us that the only operators that we need to consider when computing the numerical range of $C^\star_{b\varphi^m} C_{a\varphi^n}$ and $C_{a\varphi^n} C^\star_{b\varphi^m}$ are those of form $T_N = T_{N,1}$ or $S_N = S_{N,1}$. As we have seen, computing the numerical range of T_N is equivalent to finding its numerical radius by circular symmetry. This problem is in turn equivalent to finding (up to a factor of 2) the spectral radius of $T_N + T_N^\star$, which is a tridiagonal matrix. The richness of the theory of tridiagonal matrices (see [14] for example) tells us that there is a sequence of polynomials, $p_{k,N}(x, \delta)$, that satisfy the recursion relation

$$p_{k+1,N+1}(x, \delta) = x p_{k,N+1}(x, \delta) - (\delta^{m^{N-k} n^{k-1}})^2 p_{k-1,N+1}(x, \delta), \quad k = 1, \ldots, N$$

with initial conditions $p_{-1,N+1}(x, \delta) = 0$ and $p_{0,N+1}(x, \delta) = 0$. The largest root of $p_{N+1,N+1}(x, \delta)$ is the spectral radius of $T_N + T_N^\star$. Furthermore, the roots of $p_{N+1,N+1}(x, \delta)$ interlace with those of $p_{N,N+1}(x, \delta)$, so the largest root of the later is dominated by the former. The structure of the matrices T_N show us that

$$p_{N,N}(x, \delta^m) = p_{N,N+1}(x, \delta),$$

but while this equality looks like a promising avenue for determining a range of δ for which the numerical radius of T_N dominates the numerical radius of T_M for any $M \neq N$, it has yet to reveal to us that this is the case. The problem for $S_N + S_N^\star$ gives rise to the same sequence of polynomials p_N with the parameter δ replaced with ν. We conjecture that the interval (0, 1) can be decomposed as

$$(0, 1) = \bigcup_{k=0}^{\infty} (\delta_k, \delta_{k+1}),$$

where δ_k is the unique intersection point of $\omega(T_k)$ and $\omega(T_{k+1})$ and on the interval (δ_k, δ_{k+1}) the maximum of the set $\{\omega(T_n)\}_{n \geq 1}$ is $\omega(T_{k+1})$.

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