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The dual of the space of bounded operators on a Banach space

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Abstract: Given Banach spaces X and Y, we ask about the dual space of the $\mathcal{L}(X, Y)$. This paper surveys results on tensor products of Banach spaces with the main objective of describing the dual of spaces of bounded operators. In several cases and under a variety of assumptions on X and Y, the answer can best be given as the projective tensor product of X^{**} and Y^*.

Keywords: Tensor products; dual spaces; dual of spaces of operators

MSC: Primary 46B78, secondary 46B10

Dedicated to the memory of our friend Professor James E. Jamison.

1 Introduction

The space $\mathcal{L}(X, Y)$ of bounded linear operators from the Banach space X to the Banach space Y is one of the fundamental spaces of study in Banach space theory, and it seems natural to inquire about its dual space. A preliminary search of articles on this subject gave a rather sparse result. Hence we thought it might be of use to investigate an answer to this question. Let us begin with a very naive and elementary approach. In the case where Y is one dimensional, the answer is just X^*, so what is the answer in the two dimensional case? It depends, of course, on the norm of the space. Let us consider $Y = \ell^\infty(2)$, where X is any Banach space. Clearly $T \in \mathcal{L}(X, \ell^\infty(2))$ can be written as $Tx = (\varphi_1(x), \varphi_2(x))$ for $x \in X$ and where φ_1, φ_2 are elements of X^*. Furthermore, $\|T\| = \max\{\|\varphi_1\|, \|\varphi_2\|\}$. It follows that if $\psi \in \mathcal{L}(X, Y)^*$, then $\psi = (\psi_1, \psi_2)$ where $\psi_1, \psi_2 \in X^{**}$ and $\|\psi\| = \|\psi_1\| + \|\psi_2\|$. Thus, the dual space in this case is the space $\ell^1(2, X^{**})$. This can be extended inductively to any finite dimension. Corresponding results for other common norms can also be obtained.

Notation is fairly standard. Because of a variety of uses of the letter B, we will use $\mathcal{L}(X, Y)$ to denote the bounded operators from X to Y and similarly, $\mathcal{L}_A(X, Y)$ for the approximable operators (those for which the finite rank operators are dense), and $\mathcal{L}_K(X, Y)$ for the compact operators from X to Y. The closed unit ball of a Banach space X is denoted by B_X.

In years gone by when researchers were discussing a particular result, someone would often say, “Well, it’s probably somewhere in Grothendieck!" That is the case here as well. A more elegant approach to the description of $\mathcal{L}(X, Y)^*$ involves tensor products, and the result that $\mathcal{L}(X, \ell^\infty(n))^* = \ell^1(n, X^{**})$ which we described above could be expressed by $\mathcal{L}(X, \ell^\infty(n))^* = X^{**}\hat{\otimes}_\pi(\ell^\infty(n))^*$. Before showing how that works, let us take a little refresher course on tensor products.

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2 Tensor products

We mention here some basic and useful facts about tensor products for the benefit of readers who may not be familiar with the subject. What is reported here is material taken from the wonderful book of Raymond Ryan [7]. Let $B(X \times Y)$ denote the space of bilinear functions from $X \times Y$, where X, Y are vector spaces, to the scalar field $\mathbb{K}$ (which is always the real or complex numbers). In its most elementary form, the tensor product $X \otimes Y$ can be thought of as a subspace of the space of linear functionals on $B(X \times Y)$. Given $x \in X$ and $y \in Y$, let $x \otimes y$ denote the linear functional on $B(X \times Y)$ defined by

$$(x \otimes y)(A) = \langle A, x \otimes y \rangle = A(x, y),$$

where A is a bilinear form on $X \times Y$. The tensor product $X \otimes Y$ is the space spanned by these elements. Thus an element of this tensor product can be written

$$u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i$$

and since such representations are not unique, we may always write an element in the form

$$u = \sum_{i=1}^{n} x_i \otimes y_i.$$

The *projective norm*, π, on the tensor product of two Banach spaces $X \otimes Y$ is defined by

$$\pi(u) = \inf \left\{ \sum_{i=1}^{m} \|x_i\| \|y_i\| : u = \sum_{i=1}^{m} x_i \otimes y_i \right\}.$$

The infimum is taken, of course, over all representations of u. Then π is a norm with the property that $\pi(x \otimes y) = \|x\|\|y\|$ for $x \in X$ and $y \in Y$. The tensor product $X \otimes Y$ with the projective norm is denoted by $X \otimes_\pi Y$ and its completion is given by $X \hat{\otimes}_\pi Y$. Some useful facts for future reference are that $\ell^1 \hat{\otimes}_\pi Y = \ell^1(Y)$ and this holds also for the space $\ell^1(J)$ where J is an arbitrary indexing set.

An interesting and important result about the projective tensor product is given in the following theorem. We think it instructive to offer a proof, although we give an alternative to the interesting one given in [7].

Theorem 1. *Let X and Y be Banach spaces, $u \in X \hat{\otimes}_\pi Y$, and $\epsilon > 0$. Then there exist bounded sequences $\{x_n\}, \{y_n\}$ in X, Y respectively, such that the series $\sum_{n=1}^{\infty} x_n \otimes y_n$ converges to u and*

$$\sum_{n=1}^{\infty} \|x_n\| \|y_n\| < \pi(u) + \epsilon.$$

Proof. Given $u \in X \hat{\otimes}_\pi Y$ and $\epsilon > 0$ there exists $u_1 = \sum_{i=1}^{k_1} x_i \otimes y_i$ such that $\pi(u - u_1) < \epsilon/8$ and $\pi(u_1) \le \sum_{i=1}^{k_1} \|x_i\| \|y_i\| < \pi(u_1) + \epsilon/4$. Thus $\pi(u_1) < \pi(u) + \epsilon/8$ so that $\sum_{i=1}^{k_1} \|x_i\| \|y_i\| < \pi(u) + \epsilon/8 + \epsilon/4 < \pi(u) + \epsilon/2$. Without loss of generality, we may assume that, for all $i \in \{1, 2, \dots k_1\}$, $\|x_i\| = \|y_i\|$. We assume this as well for all x_i, y_i chosen below. Since $u - u_1 \in X \hat{\otimes}_\pi Y$ there exists $u_2 = \sum_{i=k_1+1}^{k_2} x_i \otimes y_i$ such that $\pi(u - u_1 - u_2) < \epsilon/16$ and $\pi(u_2) \le \sum_{i=k_1+1}^{k_2} \|x_i\| \|y_i\| < \pi(u_2) + \epsilon/8$. Then $\pi(u_2) \le \pi(u - u_1) + \epsilon/16 < \epsilon/8 + \epsilon/16$, and therefore $\pi(u_2) < \epsilon/4$.

We continue this process to define a sequence $\{u_n\}$ such that $\pi(u - u_1 - u_2 - \cdots - u_n) < \epsilon/2^{n+2}$, where $u_n = \sum_{i=k_{n-1}+1}^{k_n} x_i \otimes y_i$, and $\pi(u_n) \le \sum_{i=k_{n-1}+1}^{k_n} \|x_i\| \|y_i\| \le \pi(u_n) + \epsilon/2^{n+1}$. Therefore $\pi(u_n) \le \pi(u - u_1 - \cdots u_{n-1}) + \epsilon/2^{n+1} < \epsilon/2^{n+2} + \epsilon/2^{n+1} < \epsilon/2^n$. It follows that

$$u = \sum_{n=1}^{\infty} u_n = \sum_{i=1}^{\infty} x_i \otimes y_i,$$

and

$$\sum_{i=1}^{\infty} \|x_i\| \, \|y_i\| < \pi(u) + \sum_{n=1}^{\infty} \epsilon/2^n = \pi(u) + \epsilon.$$

We observe that sequences $\{\|x_i\|\}$ and $\{\|y_i\|\}$ are bounded by $\sqrt{\pi(u)+\epsilon}$. This completes the proof. □

It follows from the above result that

$$\pi(u) = \inf \left\{ \sum_{n=1}^{\infty} \|x_n\|\|y_n\| : \sum_{n=1}^{\infty} \|x_n\|\|y_n\| < \infty, u = \sum_{n=1}^{\infty} x_n \otimes y_n \right\},$$

where the infimum is taken over all such representations of u. This is another of several ways of calculating the projective norm.

Dual spaces are of particular interest to us, so let us describe the dual space of the projective tensor product of two Banach spaces, X, Y. A bilinear form B on $X \times Y$ is bounded if there exists $C > 0$ such that $|B(x,y)| \le C\|x\|\|y\|$ for all $x \in X$ and $y \in Y$. Let $\mathcal{B}(X,Y)$ denote the space of bounded bilinear forms on $X \times Y$, which is a Banach space with the norm

$$\|B\| = \sup\{|B(x,y)| : x \in B_X, y \in B_Y\}.$$

Given $B \in \mathcal{B}(X \times Y)$ there exists a unique linear functional $\tilde{B}$ on $X\hat{\otimes}_\pi Y$ satisfying $\tilde{B}(x \otimes y) = B(x,y)$, and it is straitforward to show that the correspondence $B \leftrightarrow \tilde{B}$ is an isometric isomorphism between $\mathcal{B}(X \times Y)$ and $(X\hat{\otimes}_\pi Y)^*$. We note also, that given $B \in \mathcal{B}(X \times Y)$, we can define an operator $L_B : X \to Y^*$ by $\langle y, L_B(x)\rangle = B(x,y)$ and the correspondence $B \leftrightarrow L_B$ describes an isometric isomorphism between $\mathcal{B}(X \times Y)$ and $\mathcal{L}(X, Y^*)$. In a similar way, the operator $R_B : Y \to X^*$ which satisfies $\langle x, R_B(y)\rangle = B(x,y)$ provides an isometric isomorphism between $\mathcal{B}(X \times Y)$ and $\mathcal{L}(Y, X^*)$ via $B \leftrightarrow R_B$. Hence we have

$$(X\hat{\otimes}_\pi Y)^* \cong \mathcal{B}(X \times Y) \cong \mathcal{L}(X, Y^*) \cong \mathcal{L}(Y, X^*). \tag{1}$$

An element $u \in X \otimes Y$ can be considered as a bilinear form B_u on $X^* \times Y^*$ by

$$B_u(\varphi, \psi) = \sum_{i=1}^{m} \varphi(x_i)\psi(y_i),$$

where $u = \sum_{i=1}^m x_i \otimes y_i$ is any representation of u. This provides a canonical embedding of $X \otimes Y$ into $\mathcal{B}(X^* \times Y^*)$, and we use it to define what is called the *injective norm* on $X \otimes Y$, which is the norm induced by this embedding. We denote by $\varepsilon(u)$ the injective norm of $u \in X \otimes Y$. Hence, we have

$$\varepsilon(u) = \sup \left\{ |\sum_{i=1}^{m} \varphi(x_i)\psi(y_i)| : \varphi \in B_{X^*}, \psi \in B_{Y^*} \right\},$$

where the supremum is taken over all representations of u. By consideration of the operators $L_u : X^* \to Y$ and $R_u : Y^* \to X$, we have alternate forms for $\varepsilon(u)$ given by

$$\begin{aligned} \varepsilon(u) &= \sup\{\|\sum_{i=1}^{m} \varphi(x_i)y_i\| : \varphi \in B_{X^*}\} \\ &= \sup\{\|\sum_{i=1}^{m} \psi(y_i)x_i\| : \psi \in B_{Y^*}\}. \end{aligned}$$

We will denote by $X \otimes_\varepsilon Y$ the tensor product $X \otimes Y$ with the injective norm and its completion (which is called the injective tensor product) by $X\hat{\otimes}_\varepsilon Y$. This injective tensor product may be considered as a subspace of $\mathcal{B}(X^* \times Y^*)$ or of $\mathcal{L}(X^*, Y)$ or $\mathcal{L}(Y^*, X)$. Of particular interest to us is that

$$X^*\hat{\otimes}_\varepsilon Y \subset \mathcal{B}(X \times Y^*), \quad \mathcal{L}(Y^*, X^*), \quad or \quad \mathcal{L}(X, Y). \tag{2}$$

It is the case that $\varepsilon(u) \leq \pi(u)$ for every $u \in X \otimes Y$ and $\varepsilon(x \otimes y) = \|x\|\|y\|$ for every $x \in X$ and $y \in Y$. It can be shown that $c_0 \hat{\otimes}_\varepsilon X = c_0(X)$ and $\ell^1 \hat{\otimes}_\varepsilon X = \ell^1[X]$, where $\ell^1[X]$ is the space of all unconditionally summable sequences in X. Furthermore, the injective tensor product of $C(K)$ with a Banach space X, where K is a compact Hausdorff space, can be identified with the space $C(K, X)$ of continuous functions from K into X.

If we consider u in $X \otimes_\varepsilon Y$ as a continuous function on the product space $B_{X^*} \times B_{Y^*}$, which is compact when endowed with the weak* topology on each unit ball, then we can think of $X \hat{\otimes}_\varepsilon Y$ as embedded into $C(B_{X^*} \times B_{Y^*})$. A linear functional on $X \hat{\otimes}_\varepsilon Y$ can be extended to all of $C(B_{X^*} \times B_{Y^*})$. Thus, if $B \in \mathcal{B}(X \times Y)$, its linearization $\tilde{B}$ defined by $\tilde{B}(x \otimes y) = B(x, y)$ can extend to a bounded linear functional on $C(B_{X^*} \times B_{Y^*})$. Since we know the description of such functionals, we have the following theorem, which describes the dual space of $X \hat{\otimes}_\varepsilon Y$.

Theorem 2. *Let $B \in \mathcal{B}(X \times Y)$. Then $\tilde{B}$, as defined above, is a bounded linear functional on $X \hat{\otimes}_\varepsilon Y$ if and only if there exists a regular Borel measure μ on the compact space $B_{X^*} \times B_{Y^*}$ such that*

$$B(x, y) = \int_{B_{X^*} \times B_{Y^*}} \varphi(x)\psi(y) d\mu(\varphi, \psi) \tag{3}$$

for every $x \in X$, $y \in Y$. Furthermore, the norm of $\tilde{B}$ is given by

$$\|\tilde{B}\| = \inf \|\mu\|,$$

where μ ranges over the set of all measures that correspond to B in this way, and this infimum is attained.

A bilinear form B on $X \times Y$ is called an *integral bilinear form* if $\tilde{B}$, as in the theorem, is a bounded linear functional on $X \hat{\otimes}_\varepsilon Y$, with integral norm defined by

$$\|B\|_I = \inf \|\mu\|,$$

where the infimum is taken over all the regular Borel measures on $B_{X^*} \times B_{Y^*}$ that satisfy (3). The Banach space of integral bilinear forms with integral norm is denoted by $\mathcal{B}_I(X \times Y)$. Hence we have

$$(X \hat{\otimes}_\varepsilon Y)^* = \mathcal{B}_I(X \times Y).$$

We mention here that a bounded bilinear form B on $X \times Y$ is said to be *nuclear* if and only if there exist bounded sequences $\{\varphi_n\}$ in X^* and $\{\psi_n\}$ in Y^* which satisfy $\sum_{n=1}^\infty \|\varphi_n\|\|\psi_n\| < \infty$, such that

$$B(x, y) = \sum_{n=1}^{\infty} \varphi_n(x)\psi_n(y)$$

for every $x \in X$, $y \in Y$. An expression $\sum_{n=1}^\infty \varphi_n \otimes \psi_n$ is called a nuclear representation of B. The nuclear norm of B is defined by

$$\|B\|_N = \inf \left\{ \sum_{n=1}^{\infty} \|\varphi_n\|\|\psi_n\| : B(x, y) = \sum_{n=1}^{\infty} \varphi_n(x)\psi_n(y) \right\},$$

the infimum taken over all the nuclear representations of B. It is clear that $\|B\| \leq \|B\|_N$.

It is worth noting that every nuclear bilinear form is also integral. If $B(x, y) = \sum_{n=1}^\infty \lambda_n \varphi_n(x)\psi_n(y)$, where $\{\lambda_n\}$ is a summable sequence of scalars, and $\{\varphi_n\}$, $\{\psi_n\}$ are sequences of distinct unit vectors in X^*, Y^* respectively, the sum may be interpreted as an integral like that in (3), where μ is the measure with point masses λ_n at the points (φ_n, ψ_n).

The notions of integral and nuclear can also be applied to operators. An operator $T : X \to Y$ corresponds to a bilinear form B_T defined on $X \times Y^*$ by $B_T(x, \psi) = \langle Tx, \psi \rangle$. Then T is said to be integral if B_T is integral, and the integral norm $\|T\|_I$ is that of B_T. The space of integral operators with that norm is denoted by $\mathcal{I}(X, Y)$. An operator from X to Y is said to be nuclear if it is in the range of the canonical operator $N : X^* \hat{\otimes}_\pi Y \to \mathcal{L}(X, Y)$ of unit norm that associates the tensor $u = \sum_{n=1}^\infty \varphi_n \otimes y_n$ with the operator L_u given by $L_u(x) = \sum_{n=1}^\infty \varphi_n(x) y_n$.

The collection of such operators is written as $N(X, Y)$, and the nuclear norm of such an operator T is defined as

$$\|T\|_N = \inf\left\{\sum_{n=1}^{\infty}\|\varphi_n\|\|y_n\| : T(x) = \sum_{n=1}^{\infty}\varphi_n(x)y_n\right\},$$

where the infimum is taken over all such representations of T where $\{\varphi_n\}, \{y_n\}$ are sequences in X^*, Y respectively such that $\sum_{n=1}^{\infty}\|\varphi_n\|\|y_n\| < \infty$. We note for future reference that the map N is not necessarily one-to-one.

From (1) we remember that $\mathcal{B}(X \times Y)$ is isometrically isomorphic to the space $\mathcal{L}(X, Y^*)$, where a bilinear form B is associated with the operator T by $B(x, y) = \langle y, Tx\rangle$. Such a form B is integral if and only if the associated T is integral and the two share their integral norms. This pairing establishes an isometric isomorphism of $\mathcal{B}_I(X \times Y)$ with $\mathcal{I}(X, Y^*)$. This allows us to say that

$$(X\hat{\otimes}_\varepsilon Y)^* = \mathcal{B}_I(X \times Y) = \mathcal{I}(X, Y^*) \tag{4}$$

where we use “equal" signs rather than the more precise congruences.

3 Dual spaces

In thinking about the dual space of $\mathcal{L}(X, Y)$, we are drawn to the fact that the injective tensor product $X^*\hat{\otimes}_\varepsilon Y$ can be regarded as a subspace of $\mathcal{L}(X, Y)$ as in (2). In fact, we can write $X^*\hat{\otimes}_\varepsilon Y = \mathcal{L}_A(X, Y)$, the space of approximable operators. Our main theorem will be a theorem recorded by Ryan [7, Theorem 5.33] that gives a special characterization of the dual of $X\hat{\otimes}_\varepsilon Y$ under certain conditions on X and Y. Our proof, though put together a bit differently, is certainly influenced by that of Ryan. We recall that a Banach space X is said to have the *approximation property* if for every Banach space Y, every operator $T : X \to Y$ may be approximated on compact sets by finite rank operators. Equivalently, X has the approximation property if and only if for every Banach space Y, every compact operator from Y into X is approximable. In addition, we need to use the notion of a Banach space having the Radon-Nikodým property. A vector measure μ on a σ-algebra Σ with values in a Banach space X satisfies the Radon-Nikodým Theorem if μ has bounded variation and is absolutely continuous with respect to a finite positive measure λ, then μ is an indefinite Bochner integral with respect to λ. We say X has the Radon-Nikodým property if every such measure as above with values in X has the Radon-Nikodým property.

Before stating and proving the theorem just mentioned, we need two lemmas, also given by Ryan [7].

Lemma 3. *Let Σ be a σ-algebra of subsets of Ω, $\mathcal{M}(\Sigma)$ the Banach space of scalar measures on Σ with the variation norm, and X a Banach space. Then $\mathcal{M}(\Sigma)\hat{\otimes}_\pi X$ is isometrically isomorphic to the space $\mathcal{M}_1(\Sigma, X)$ of vector measures with the Radon-Nikodým property, with the variation norm $\|\mu\|_1 = |\mu|_1(\Omega)$.*

Proof. If $u = \sum_{i=1}^{m}\mu_i \otimes x_i$ in $\mathcal{M}(\Sigma) \otimes X$, it corresponds to $\mu \in \mathcal{M}(\Sigma, X)$ by $\mu(E) = \sum_{i=1}^{m}\mu_i(E)x_i$. It is known that $\pi(u) = \|\mu\|_1$ and that the correspondence is an isometry. It remains to show that $\mathcal{M}(\Sigma) \otimes X$ is dense in $\mathcal{M}_1(\Sigma, X)$. Let $\mu \in \mathcal{M}_1(\Sigma, X)$. There exists $f \in L_1(|\mu|_1, X)$ such that $\mu(E) = \int_E fd|\mu|_1$ for all $E \in \Sigma$. Since $L_1(|\mu|_1, X) = L_1(|\mu|_1)\hat{\otimes}_\pi X$, there exist bounded sequences $\{g_n\}, \{x_n\}$ in $L_1(|\mu|_1), X$ respectively such that $\sum_{n=1}^{\infty}\|g_n\|\|x_n\| < \infty$ and $f = \sum_{n=1}^{\infty} g_n \otimes x_n$. Let μ_n be defined by $\mu_n(E) = \int_E g_n d|\mu|_1$ for each n. Then μ corresponds to $\sum_{n=1}^{\infty}\mu_n \otimes x_n \in \mathcal{M}_1(\Sigma, X)$. ☐

Before stating the next lemma, we want to define what is called a representing measure. If T is a bounded operator on the space $C(K)$ of continuous functions on a compact, Hausdorf space K to X, we may define a function μ_T on the σ-algebra of Borel subsets of K with values in the bidual X^{**} by $\mu_T(E) = T^{**}(\chi(E))$. Then for each $\varphi \in X^*$ we have

$$\langle\varphi, \mu_T(E)\rangle = T^*\varphi(E),$$

where $T^{\star}\varphi \in C(K)^{\star}$ is a regular Borel measure on K. The measure μ_T is called the *representing measure* for T. On the other hand, given a regular vector measure μ on the Borel sets of K with values in $X^{\star\star}$, we have an associated operator T_μ defined on $X^{\star}$ by $T_\mu\varphi = \varphi\mu$. The scalar measure $\varphi\mu$ is regular whenever μ is regular.

Lemma 4. *Let $T : C(K) \to X$ be an operator with representing measure μ. Then T is a nuclear operator if and only if μ has the Radon-Nikodým property. If T is nuclear, then $\|T\|_N = \|\mu\|_1$.*

Proof. It is known that if T is an operator on a space whose dual space has the approximation property, then $T^{\star}$ is nuclear if and only if T is nuclear. Since $T : C(K) \to X$ in this case, we know that $C(K)^{\star}$ is the space of regular Borel measures on the σ-algebra $\mathcal{B}_K$ of Borel sets in K, and since this space has the approximation property, we have T is nuclear if and only if $T^{\star}$ is nuclear. Furthermore, it is easily shown that $C(K)^{\star}$ is complemented in $\mathcal{M}(\mathcal{B}_K)$ by a projection of norm one. If I denotes the embedding of $C(K)^{\star}$ into $\mathcal{M}(\mathcal{B}_K)$, then we see that $IT^{\star}$ is the operator associated with μ, so that $IT^{\star}(\varphi) = \varphi\mu$ for all $\varphi \in X^{\star}$.

Supppose μ has the Radon-Nikodým property. We have from Lemma 3 that $\mathcal{M}(\mathcal{B}_K)\hat{\otimes}_\pi X = \mathcal{M}_1(\mathcal{B}_K, X)$. Therefore, μ can be thought of as a member of the projective tensor product, so we can write

$$\mu = \sum_{n=1}^{\infty} \mu_n \otimes x_n.$$

where $\sum_{n=1}^{\infty} \|\mu_n\|\|x_n\| < \infty$. If we write $S = IT^{\star}$, then $S(\varphi) = \varphi\mu$ and

$$\begin{aligned} S(\varphi) = \varphi(\sum_{n=1}^{\infty} \mu_n \otimes x_n) &= \sum_{n=1}^{\infty} \varphi(\mu_n \otimes x_n) \\ = \sum_{n=1}^{\infty} \varphi(x_n)\mu_n &= \sum_{n=1}^{\infty} Jx_n(\varphi)\mu_n, \end{aligned}$$

where J is the canonical mapping from X into $X^{\star\star}$. Let $u \in X^{\star\star}\hat{\otimes}_\pi\mathcal{M}(\mathcal{B}_K)$ be given by $u = \sum_{n=1}^{\infty} Jx_n \otimes \mu_n$. We see that the canonical map N from $X^{\star\star}\hat{\otimes}_\pi\mathcal{M}(\mathcal{B}_K)$ to $\mathcal{L}(X^{\star}, \mathcal{M}(\mathcal{B}_K))$ defined by $Nu(\varphi) = \sum_{n=1}^{\infty} Jx_n(\varphi)\mu_n$ agrees with S so that S (and therefore $T^{\star}$) is nuclear.

On the other hand, suppose T and therefore $T^{\star}$ is nuclear. Then $S = IT^{\star}$ is nuclear and must be defined as above. It follows that μ must be in $\mathcal{M}_1(\mathcal{B}_K), X^{\star\star})$ and so satisfies the Radon-Nikodým property. □

At this point we are ready to state and prove the theorem we mentioned at the beginning of this section.

Theorem 5. *Let X and Y be Banach spaces for which $X^{\star}$ has the Radon-Nikodým property and either $X^{\star}$ or $Y^{\star}$ has the approximation property. Then*

$$(X\hat{\otimes}_\varepsilon Y)^{\star} = X^{\star}\hat{\otimes}_\pi Y^{\star}.$$

Proof. In this proof, we will assume that $Y^{\star}$ has the approximation property. We begin by considering the canonical map N from $X^{\star}\hat{\otimes}_\pi Y^{\star}$ to $\mathcal{L}(Y, X^{\star})$ which we have considered in the discussion of nuclear operators. If we can show that N is one-to-one, then we will establish that the space $N(Y, X^{\star})$ is isometrically isomorphic to $X^{\star}\hat{\otimes}_\pi Y^{\star}$. Let $u = \sum_{n=1}^{\infty} \varphi_n \otimes \psi_n \in X^{\star}\hat{\otimes}_\pi Y^{\star}$, where $\varphi_n \in X^{\star}$, $\psi_n \in Y^{\star}$, and $\sum_{n=1}^{\infty} \|\varphi_n\|\|\psi_n\| < \infty$. Then $Nu(y) = \sum_{n=1}^{\infty} \psi_n(y)\varphi_n$. Suppose $Nu = 0$. We may assume that $\psi_n \to 0$ and $\sum_{n=1}^{\infty} \|\varphi_n\| < \infty$, for if not, we can alter the representation of u so that both hold. Let $T \in \mathcal{L}(Y^{\star}, X^{\star\star})$ and $\epsilon > 0$ be given. Since $\{\psi_n\}$ is a zero convergent sequence in $Y^{\star}$, it is relatively compact and because $Y^{\star}$ has the approximation property, there is a finite rank operator S such

$$\|T(\psi_n) - S(\psi_n)\| < \epsilon$$

for every n. The operator S has the form $S(y^\star) = \sum_{i=1}^m \psi_i^\star(y^\star)\varphi_i^\star$, where $\psi_i^\star \in Y^{\star\star}$, $\varphi_i^\star \in X^{\star\star}$ for $i = 1, \dots, m$. We have $S \in (X^\star \hat{\otimes}_\pi Y^\star)^\star = \mathcal{L}(Y^\star, X^{\star\star})$, so that

$$\begin{aligned}\langle u, S\rangle = \langle \sum_{n=1}^\infty \varphi_n \otimes \psi_n, S\rangle &= \sum_{n=1}^\infty \langle \varphi_n \otimes \psi_n, S\rangle \\ &= \sum_{n=1}^\infty \langle \varphi_n, S\psi_n\rangle \\ &= \sum_{n=1}^\infty \sum_{i=1}^m \psi_i^\star(\psi_n)\varphi_i^\star(\varphi_n) \\ &= \sum_{i=1}^m (\sum_{n=1}^\infty \psi_i^\star(\psi_n)\varphi_i^\star(\varphi_n)) \\ &= \sum_{i=1}^m \varphi_i^\star(\sum_{n=1}^\infty \psi_i^\star(\psi_n)\varphi_n) \\ &= 0.\end{aligned}$$

The zero occurs in the last line above since we know that $\sum_{n=1}^\infty \psi(y)\varphi_n = 0$ for every $y \in Y$. This is enough to show that $\sum_{n=1}^\infty y^{\star\star}(\psi_i)\varphi_n = 0$ for every $y^{\star\star} \in Y^{\star\star}$ since $\{Jy : y \in Y\}$ is $w^\star$-dense in $Y^{\star\star}$. We conclude that $\langle u, S\rangle = 0$. This allows us to write

$$|\langle u, T\rangle| = |\langle u, T - S\rangle + \langle u, S\rangle| \le \sum_{n=1}^\infty \|T\psi_n - S\psi_n\|\|\varphi_n\| + 0 < \epsilon \sum_{n=1}^\infty \|\varphi_n\|.$$

Since $\sum_{n=1}^\infty \|\varphi_n\| < \infty$ and ϵ is arbitrary, we must have $\langle u, T\rangle = 0$ for all T from which we conclude that $u = 0$ and N is injective. We remark that if the assumption is that $X^\star$ has the approximation property, in the above argument let $T \in \mathcal{L}(X^\star, Y^{\star\star})$ and proceed in the same way.

From (4), we see that the proof will be complete if we can show that any $T \in \mathcal{I}(Y, X^\star)$ is also in $N(Y, X^\star)$. Given such a T, we know it corresponds to a bilinear form $B \in \mathcal{B}_I(X \otimes Y)$ by $\langle x, Ty\rangle = B(x, y)$. Because B is integral, there is a regular Borel measure ν on $K = B_{X^\star} \times B_{Y^\star}$ such that

$$B(x, y) = \int_K \varphi(x)\psi(y)d\nu.$$

By the Radon-Nikodým Theorem there is a Borel measurable function h such that $|h(t)| = 1$ for all $t \in K$ such that $d\nu = hd|\nu|$, where $|\nu|$ is the total variation of ν. Let $\mu = |\nu|$, $S : Y \to C(K)$, defined by $Sy(\varphi, \psi) = \psi(y)$ and $R : X \to L^\infty(\mu)$ given by $Rx(\varphi, \psi) = \varphi(x)$. This gives

$$B(x, y) = \int_K (Sy)(Rx)d\mu.$$

Let V denote the restriction of $R^\star$ to $L^1(\mu)$ (really $JL^1(\mu)$), and let I denote the canonical embedding of $C(K)$ into $L^1(\mu)$. By the duality between $L^1(\mu)$ and $L^\infty(\mu)$, we can see that

$$\langle Rx, ISy\rangle = \langle x, VISy\rangle = \langle x, Ty\rangle.$$

Hence we can write $T = VIS$. Let $U : C(K) \to X^\star$ denote the map $U = VI$. Since $X^\star$ has the Radon-Nikodým property, it follows from Lemma 4 that $U \in N(C(K), X^\star)$. From this we know there is some $u = \sum_{n=1}^\infty \mu_n \otimes \varphi_n \in C(K)^\star \hat{\otimes}_\pi X^\star$ with $\mu_n \in C(K)^\star$, $\varphi_n \in X^\star$ for each n, and $\sum_{n=1}^\infty \|\mu_n\|\|\varphi_n\| < \infty$ so that

$$Nu(f) = \sum_{n=1}^\infty \mu_n(f) = Uf \text{ for all } f \in C(K).$$

Let $\psi_n \in Y^\star$ be defined by $\psi_n(y) = \mu_n(Sy)$. Then

$$|\psi_n(y)| = |\mu_n(Sy)| \le \|\mu_n\|\|Sy\| = \|\mu_n\| \sup\{|Sy(\varphi, \psi)| : (\varphi, \psi) \in K\} \le \|\mu_n\|\|y\|.$$

Thus $\|\psi_n\| \le \|\mu_n\|$, so $\sum_{n=1}^{\infty} \|\psi_n\|\|\varphi_n\| < \infty$. Let $w = \sum_{n=1}^{\infty} \psi_n \otimes \varphi_n \in Y^* \hat{\otimes}_\pi X^*$. We have

$$\begin{aligned} Nw(y) = \sum_{n=1}^{\infty} \psi_n(y)\varphi_n &= \sum_{n=1}^{\infty} \mu_n(Sy)\varphi_n \\ &= U(Sy) = Ty. \end{aligned}$$

Hence, $Nw = T$ and $T \in N(Y, X^*)$. We have noted before that nuclear bilinear forms are integral, and the same holds for nuclear operators. This completes the proof. □

Since $X\hat{\otimes}_\varepsilon Y = Y\hat{\otimes}_\varepsilon X$ and $X^*\hat{\otimes}_\pi Y^* = Y^*\hat{\otimes}_\pi X^*$, we have the following corollary.

Corollary 6. *If either X^* or Y^* has the Radon-Nikodým property and either X^* or Y^* has the approximation property, then*

$$(X\hat{\otimes}_\varepsilon Y)^* = X^*\hat{\otimes}_\pi Y^*.$$

If both X^* and Y^* have the Radon-Nikodým property and either X^* or Y^* has the approximation property, then $X^*\hat{\otimes}_\pi Y^*$ has the Radon-Nikodým property. (See [2, p.29].)

We now turn to the original question. What can we say about the dual of $\mathcal{L}(X, Y)$? We begin with the finite dimensional case. We recall from (2) that $X^*\hat{\otimes}_\varepsilon Y$ is contained in $\mathcal{L}(X, Y)$ and equal to $\mathcal{L}_A(X, Y)$. Since this space is equal to $\mathcal{L}(X, Y)$ if either X or Y is finite dimensional, we have the following results.

Theorem 7. *Suppose X and Y are Banach spaces.*
(i) If Y is finite dimensional, then

$$\mathcal{L}(X, Y)^* = (X^*\hat{\otimes}_\varepsilon Y)^* = X^{**}\hat{\otimes}_\pi Y^* \text{ for all Banach spaces } X.$$

(ii) If X is finite dimensional, then

$$\mathcal{L}(X, Y)^* = (X^*\hat{\otimes}_\varepsilon Y)^* = X^{**}\hat{\otimes}_\pi Y^* = X\hat{\otimes}_\pi Y^* \text{ for all Banach spaces } Y.$$

Proof. Since finite dimensional spaces possess both the Radon-Nikodým property and the approximation property, the results follow immediately from Corollary 6. □

For example, we can get results of the kind mentioned in the very beginning, such as $\mathcal{L}(X, \ell^\infty(n))^* = (X^{**}\hat{\otimes}_\pi \ell^1(n) = \ell^1(n, X^{**})$ or $\mathcal{L}(X, \ell^1(n))^* = X^{**}\hat{\otimes}_\pi \ell^\infty(n) = \ell^\infty(n, X^{**})$.

Theorem 8. *If $1 < p < q < \infty$, then $\mathcal{L}(\ell^q, \ell^p)^* = \ell^q\hat{\otimes}_\pi \ell^{p'}$, where p' is conjugate to p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.*

Proof. Since both spaces involved satisfy both the Radon-Nikodým property and the approximation property, and since every operator from ℓ^q to ℓ^p is compact (by Pitt's Theorem), the result follows from Corollary 6. □

Since every operator from c_0 to ℓ^1 is compact and ℓ^1 has the Radon-Nikodým property, we also have

Theorem 9. $\mathcal{L}(c_0, \ell^1)^* = (\ell^1\hat{\otimes}_\varepsilon \ell^1)^* = \ell^\infty\hat{\otimes}_\pi \ell^\infty$.

We note here that if either X^* or Y has the approximation property, then $X^*\hat{\otimes}_\varepsilon Y = \mathcal{L}_K(X, Y)$, the compact operators from X to Y. Hence the following theorem follows from Corollary 6.

Theorem 10. *(Grothendieck [3]. (See also [6].) If either X^{**} or Y^* has the Radon-Nikodým property and either has the approximation property, then*

$$\mathcal{L}_K(X, Y)^* = X^{**}\hat{\otimes}_\pi Y^*.$$

The three theorems previous to Theorem 10 were, of course, actually included in this last one, since in each such case, the bounded operators from X to Y were all compact, hence as we said in the beginning, "probably in Grothendieck."

The most general statement we can make concerning duals of bounded operators is that

$$\mathcal{L}(X, Y^*)^* = (X\hat{\otimes}_\pi Y)^{**}.$$

This is most useful when we can describe the projective tensor product of X and Y. Since we know that $\ell^1\hat{\otimes}_\pi X = \ell^1(X)$ and $\ell^1\hat{\otimes}_\pi \ell^1 = \ell^1$, we can state the following.

Theorem 11. *For any Banach space X,*
(i)

$$\mathcal{L}(\ell^1, X^*) = (\ell^1\hat{\otimes}_\pi X)^* = \ell^1(X)^* = \ell^\infty(X^*).$$

(ii)

$$\mathcal{L}(\ell^1, X^*)^* = (\ell^\infty(X^*))^* = \ell^1(X)^{**}.$$

(iii)

$$\mathcal{L}(\ell^1, \ell^\infty)^* = (\ell^1\hat{\otimes}_\pi \ell^1)^{**} = (\ell^1)^{**} = (\ell^\infty)^*.$$

Furthermore, if $L^1(\mu)$ denotes the usual integration space, then

$$(L^1(\mu)\hat{\otimes}_\pi X)^* = (L^1(\mu, X))^* = \mathcal{L}(L^1(\mu), X^*),$$

and for compact K,

$$C(K)\hat{\otimes}_\varepsilon X = C(K, X),$$

so we have

Theorem 12. *For a Banach space X ,*
(i)

$$\mathcal{L}(L^1(\mu), X^*)^* = L^1(\mu, X)^{**}.$$

(ii) If X^* has the Radon-Nikodým property, then

$$C(K, X)^* = (C(K)\hat{\otimes}_\varepsilon X)^* = C(K)^*\hat{\otimes}_\pi X^*, \text{ so that } \mathcal{L}(C(K)^*, X^{**})^* = C(K, X)^{***}.$$

4 More on Tensor products

It was shown in Ryan's book, [7, p. 23], that the tensor diagonal subspace of $\ell^2\hat{\otimes}_\pi \ell^2$ is isometrically isomorphic to ℓ^1. Actually, it is a 1-complemented subspace. The following is also true.

Theorem 13. *Let $1 < p, q < \infty$ with p and q conjugate. Then ℓ^1 is isometrically embedded in $\ell^p\hat{\otimes}_\pi \ell^q$.*

Proof. The argument follows as in Ryan's book. We denote by $\{e_n\}$ and $\{f_n\}$ the standard Schauder bases for ℓ^p and ℓ^q respectively. We consider the subspace W spanned by $e_n \otimes f_n$. An arbitrary element u in W is of the form $u = \sum_{i=1}^\infty \alpha_n e_n \otimes f_n$. First, we have $\pi(u) \le \sum_{n=1}^\infty |\alpha_n|$. Towards the reversed inequality we define $B(x, y) = \sum_{n=1}^\infty sgn(\alpha_n) x_n y_n$ which implies that $|B(x, y)| \le \|x\|_p \|y\|_q$, from an application of the Hölder inequality. We recall that $sgn(\alpha_n)$ is a modulus 1 complex number such that $sgn(\alpha_n)\alpha_n = |\alpha_n|$ if $\alpha_n \neq 0$, otherwise, it is equal to zero. Since $\|B\| \le 1$ we have $\pi(u) \ge \langle u, B\rangle = \sum_{n=1}^\infty |\alpha_n|$. This implies that $\pi(u) = \sum_{n=1}^\infty |\alpha_n|$. Therefore W is isometrically isomorphic to ℓ^1, implying that ℓ^1 is isometrically embedded in $\ell^p\hat{\otimes}_\pi \ell^q$. As in the ℓ^2 case, ℓ^1 is a 1-complemented subspace in $\ell^p\hat{\otimes}_\pi \ell^q$. □

The only linear mapping determined by the bilinear map

$$P(x, y) = \sum_{n=1}^\infty x_n y_n e_n \otimes f_n$$

is a norm 1 projection. Several observations can be derived from this fact:

(i) ℓ^∞ is a 1-complemented subspace of $(\ell^p \hat{\otimes}_\pi \ell^q)^* = \mathcal{L}(\ell^p, \ell^p)$.
(ii) L_1 is isomorphic to a subspace of $\mathcal{L}(\ell^p, \ell^p)$. This follows from a result by Pelczynski, as mentioned in [5].
(iii) $\mathcal{L}(\ell^p, \ell^p)$ contains a subspace isomorphic to $C[0,1]^*$.
(iv) The unit balls in $\mathcal{L}(\ell^p, \ell^p)$ and $\ell^p \hat{\otimes}_\pi \ell^q$ are not weakly compact. This follows straightforwardly from Theorem 65 in [4].

The space of sequences (x_n) in a Banach space X for which the scalar sequence $(\varphi(x_n))$ belongs to ℓ^p for all $\varphi \in X^*$ is called the space of weakly p-summable sequences in X and is denoted by $\ell_p^w(X)$. If q is conjugate to p, the norm is given by

$$\begin{aligned}\|(x_n)\|_p^w &= \sup\{(\sum_{n=1}^\infty |\varphi(x_n)|^p)^{1/p} : \varphi \in B_{X^*}\}\\ &= \sup\{(\sum_{n=1}^\infty \|\lambda_n x_n\|_X : \lambda \in B_{\ell^q}\}, 1 < p < \infty.\end{aligned}$$

The norm for $p = 1$ is the same as above, except p is replaced by 1 and ℓ^q by c_0. The following fact is stated by Ryan [7, p.134], and we sketch a proof.

Theorem 14. *For any Banach space X,*
(i) if q is conjugate to p, then $\ell_p^w(X) = \mathcal{L}(\ell^q, X)$ for $1 < p < \infty$, and
(ii) for $p = 1$, $\ell_1^w(X) = \mathcal{L}(c_0, X)$.

Proof. Let $1 < p < \infty$ be given and q its conjugate. For $(x_n) \in \ell_p^w(X)$, let $S((x_n)) = T$, where T is the operator defined from ℓ^q to X by $\langle T\psi, \varphi\rangle = \sum_{n=1}^\infty \varphi(x_n)\alpha_n$ for $\varphi \in X^*$ and $\psi = (\alpha_n) \in \ell^q$. Let $R : X^* \to \ell^p$ be defined by $R\varphi = (\varphi(x_n))$. Then it is easy to see that

$$\|R\varphi\| = (\sum_{n=1}^\infty |\varphi(x_n)|^p)^{1/p} \le \|\varphi\| \|(x_n)\|_p^w,$$

so R is bounded by $\|(x_n)\|$. Furthermore, given $\epsilon > 0$, we may choose $\varphi \in B_{X^*}$ so that $\|(x_n)\|_p^w < (\sum_{n=1}^\infty |\varphi(x_n)|^p)^{1/p} + \epsilon = \|R(\varphi)\| + \epsilon \le \|R\| + \epsilon$. Since R is defined to be the adjoint of T, we conclude that $\|T\| = \|R\| = \|(x_n)\|_p^w$ and S is an isometry. If $T \in \mathcal{L}(\ell^q, X)$, then let $x_n = T(e_n)$, where $\{e_n\}$ denotes the usual Schauder basis for ℓ^q. Then it is easy to see that $S((x_n)) = T$. This completes the proof for part (i). The proof of the second part is similar. □

Thus $\mathcal{L}(\ell^q, X)^* = (\ell_p^w(X))^*$ and $\mathcal{L}(c_0, X)^* = (\ell_1^w(X))^*$. We do not have a good characterization of the duals of the weakly p-summable spaces, although if X is a dual space with predual Y, we could write

$$\ell_p^w(X)^* = \mathcal{L}(\ell^q, X)^* = (\ell^q \hat{\otimes}_\pi Y)^{**},$$

and

$$\ell_1^w(X)^* = \mathcal{L}(c_0, X)^* = (c_0 \hat{\otimes}_\pi Y)^{**}.$$

In particular, if $q > r$, and r' is conjugate to r, it follows from Corollary 4.24 in [7] that

$$\ell_p^w(\ell^r)^* = \mathcal{L}(\ell^q, \ell^r)^* = (\ell^q \hat{\otimes}_\pi \ell^{r'})^{**} = \ell^q \hat{\otimes}_\pi \ell^{r'}.$$

Of course, this also follows from Theorem 8.

Choi and Kim [1] introduced a subspace of $\ell_p^w(X)$, denoted by $\check{\ell}_p^w(X)$, which consists of sequences (x_n) from X for which

$$\sup_{\varphi \in B_{X^*}} (\sum_{n \ge m} |\varphi(x_n)|^p)^{1/p} \to 0 \text{ as } m \to \infty.$$

This definition is just what is needed to show that $\sum_{n=1}^\infty e_n \otimes x_n$ converges in $\ell^p \hat{\otimes}_\varepsilon X$, where $\{e_n\}$ is the usual Schauder basis for ℓ^p and $(x_n) \in \check{\ell}_p^w(X)$. We state the following interesting fact as a lemma.

Lemma 15. *If* $1 \le p < \infty$, *then*

$$\check{\ell}_p^w(X) = \ell^p \hat{\otimes}_\varepsilon X.$$

Proof. Given $(x_n) \in \check{\ell}_p^w(X)$, let $S((x_n)) = \sum_{n=1}^\infty e_n \otimes x_n$. Then

$$\varepsilon(S((x_n)) = \sup\{\|\sum_{n=1}^\infty \varphi(x_n)e_n\|_p : \varphi \in B_{X^*}\} \le \sup_{\varphi \in B_{X^*}} (\sum_{n=1}^\infty |\varphi(x_n)|^p)^{1/p} = \|(x_n)\|_p^w.$$

On the other hand, given $\epsilon > 0$ we may choose $\varphi \in B_{X^*}$ such that

$$\|(x_n)\|_p^w < (\sum_n |\varphi(x_n)|^p)^{1/p} + \epsilon \le \varepsilon(\sum_n e_n \otimes x_n) + \epsilon.$$

We conclude that S is an isometry. For surjectivity, suppose $u = \sum_{k=1}^m z_k \otimes x_k$ is an element of $\ell^p \otimes_\varepsilon X$, when $z_k = (z_{kj})_j \in \ell^p$ and $x_k \in X$ for each k. For each n let $y_n = \sum_{k=1}^m z_{kn} x_k$. Then $u = \sum_{n=1}^\infty e_n \otimes y_n$ and we show that $(y_n) \in \check{\ell}_p^w(X)$. If $\epsilon > 0$ is given, there exists for each $j = 1, \dots, m$ a positive integer n_j such that

$$(\sum_{k \ge n_j} |z_{jn}|^p)^{1/p} < \frac{\epsilon}{m\|x_j\|}.$$

For $\varphi \in B_{X^*}$ and $n_0 = \max\{n_j : j = 1, \dots, m\}$, we have

$$\begin{aligned}(\sum_{n\ge n_0} |\varphi(y_n)|^p)^{1/p} &= (\sum_{n\ge n_0} |\varphi(\sum_{j=1}^m z_{jn} x_j|^p)^{1/p} \le (\sum_{n \ge n_0}\sum_{j=1}^m |\varphi(x_j|)z_{jn}|^p)^{1/p} \\ &= (\sum_{j=1}^m \sum_{n\ge n_0} |\varphi(x_j)z_{jn}|^p)^{1/p} \le \sum_{j=1}^m (\sum_{n \ge n_0} |\varphi(x_j)z_{jn}|^p)^{1/p} \\ &\le \sum_{j=1}^m \|\varphi\|\|x_j\|(\sum_{n\ge n_0} |z_{jn}|^p)^{1/p} < \epsilon.\end{aligned}$$

Since this holds for every $\varphi \in B_{X^*}$, we see that $(y_n) \in \check{\ell}_p^w$ and $S((y_n)) = u$ so that the range of S contains a dense subspace of $\ell^p \hat{\otimes}_\varepsilon X$ and S is surjective. □

With this we can describe the dual space of $\check{\ell}_p^w(X)$.

Theorem 16. *(Choi and Kim) Let* $1 < p < \infty$ *and* q *its conjugate. Then* $\check{\ell}_p^w(X)^*$ *consists of all linear functionals* f *of the form*

$$f((x_n)) = \langle (x_n), f\rangle = \sum_{j=1}^\infty \sum_{n=1}^\infty \alpha_{jn} x_j^*(x_n),$$

where $z_j = (\alpha_{jn}) \in \ell^q$ *and* $x_j^* \in X^*$ *for each* j *such that*

$$\sum_{j=1}^\infty \|z_j\|_q \|x_j^*\| < \infty.$$

Proof. It is clear that any f of the form given above is a continuous linear functional on $\check{\ell}_p^w(X)$ with

$$\|f\| \le \sum_{j=1}^\infty \|z_j\|_q \|x_j^*\|.$$

By Lemma 15 and Theorem 5, we know that $\check{\ell}_p^w(X)^* = (\ell^p \hat{\otimes}_\varepsilon X)^* = \ell^q \hat{\otimes}_\pi X^*$, where $\langle e_n \otimes x_n, z_j \otimes x_j^*\rangle = \langle e_n, z_j\rangle\langle x_n, x_j^*\rangle$. Hence, if $f \in \check{\ell}_p^w(X)^*$, there exist $z_j = (\alpha_{jn}) \in \ell^q$ and $\{x_j^*\} \in X^*$ with $\sum_{j=1}^\infty \|z_j\|_q \|x_n^*\| < \infty$ such that f corresponds to $\sum_{j=1}^\infty z_j \otimes x_j^* \in \ell^q \hat{\otimes}_\pi X^*$, and

$$\langle (x_n), f\rangle = f(\sum_{n=1}^\infty e_n \otimes x_n) = \sum_{j=1}^\infty \sum_{n=1}^\infty \langle e_n, z_j\rangle x_j^*(x_n) = \sum_{n=1}^\infty \sum_{j=1}^\infty \alpha_{jn} x_j^*(x_n).$$

□

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