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Spectral Theory For Strongly Continuous Cosine

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Abstract: Let $(C(t))_{t\in\mathbb{R}}$ be a strongly continuous cosine family and A be its infinitesimal generator. In this work, we prove that, if $C(t) - \cosh \lambda t$ is semi-Fredholm (resp. semi-Browder, Drazin inversible, left essentially Drazin and right essentially Drazin invertible) operator and $\lambda t \notin i\pi\mathbb{Z}$, then $A - \lambda^2$ is also. We show by counter-example that the converse is false in general.

Keywords: Cosine, semi-Fredholm, Drazin invertible, semi-Browder, left essentially Drazin invertible, right essentially Drazin invertible.

MSC: 47D09; 47A11

1 Introduction

Let X be a complex Banach space, $\mathcal{B}(X)$ denote the algebra of all bounded linear operators on X and $\mathcal{C}(X)$ the set of all linear closed operators from X to X. We write $\mathcal{D}(T)$, $\mathcal{R}(T)$, $\mathcal{N}(T)$, $\rho(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$ and $\sigma_r(T)$ respectively for the domain, the range, the kernel, the resolvent, the spectrum, the point spectrum, the approximate point spectrum and residual spectrum of an operator $T \in \mathcal{C}(X)$. The function resolvent of $T \in \mathcal{C}(X)$ is defined for all $\lambda \in \rho(T)$ by $R(\lambda, T) = (\lambda - T)^{-1}$. The ascent $a(T)$, the descent $d(T)$, the essential ascent $a_e(T)$ and the essential descent $d_e(T)$ of an operator $T \in \mathcal{C}(X)$ are defined respectively by

$$a(T) = \inf\{k \in \mathbb{N} : \mathcal{N}(T^k) = \mathcal{N}(T^{k+1})\},$$

$$d(T) = \inf\{k \in \mathbb{N} : \mathcal{R}(T^k) = \mathcal{R}(T^{k+1})\},$$

$$a_e(T) = \min\{k \in \mathbb{N} : \dim \mathcal{N}(T^{k+1})/_{\mathcal{N}(T^k)} < \infty\},$$

$$d_e(T) = \min\{k \in \mathbb{N} : \dim \mathcal{R}(T^k)/_{\mathcal{R}(T^{k+1})} < \infty\},$$

with the convention $\inf(\varnothing) = \infty$, see ([5, 13]). For $T \in \mathcal{C}(X)$, if there is an operator $S \in \mathcal{B}(X)$ with $\mathcal{R}(S) \subseteq \mathcal{D}(T)$ such that $STS = S$, $TSx = STx$ for all $x \in \mathcal{D}(T)$, and $T^k(I - TS) = 0$ for some $k \in \mathbb{N}$, then S is called a Drazin inverse of T. Note that an operator $T \in \mathcal{C}(X)$ has a Drazin inverse if and only if there exists $k \in \mathbb{N}$ such that $a(T) = d(T) = k$ and $X = \mathcal{R}(T^k) \oplus \mathcal{N}(T^k)$, (see [12]). An operator $T \in \mathcal{C}(X)$ is a left essentially Drazin invertible operator if $a_e(T) < \infty$ and $\mathcal{R}(T^{a_e(T)+1})$ is closed. If $d_e(T) < \infty$ and $\mathcal{R}(T^{a_e(T)})$ is closed, then T is called right essentially Drazin invertible. The Drazin invesible, left essentially Drazin and right essentially Drazin invertible spectra are defined by

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Drazin inversible }\},$$

$$\sigma_{ld}^e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not left essentially Drazin inversible }\},$$

$$\sigma_{rd}^e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not right essentially Drazin inversible }\}.$$

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An operator $T \in \mathcal{C}(X)$ is called upper semi-Fredholm (resp. lower semi-Fredholm) if the range $\mathcal{R}(T)$ is closed and $\dim \mathcal{N}(T) < \infty$ (resp. $\operatorname{codim}\mathcal{R}(T) < \infty$). If T is either upper or lower semi-Fredholm, then T is called a semi-Fredholm operator. If T is both upper and lower semi-Fredholm, then T is called a Fredholm operator, see ([11]). The upper semi-Fredholm spectrum $\sigma_{uf}(T)$, the lower semi-Fredholm spectrum $\sigma_{lf}(T)$, the spectrum semi-Fredholm $\sigma_{sf}(T)$ and the Fredholm spectrum $\sigma_f(T)$ of T are defined by

$$\sigma_{uf}(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not upper semi-Fredholm}\},$$
$$\sigma_{lf}(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not lower semi-Fredholm}\},$$
$$\sigma_{sf}(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not semi-Fredholm}\},$$
$$\sigma_f(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not Fredholm}\}.$$

We say that an operator $T \in \mathcal{C}(X)$ is upper semi-Browder if it is upper semi-Fredholm and has finite ascent. Similarly, T is lower semi-Browder if it is lower semi-Fredholm and has finite descent. An operator T is Browder if it is both lower and upper semi-Browder, see ([13]). The upper semi-Browder spectrum $\sigma_{ub}(T)$, the lower semi-Browder spectrum $\sigma_{lb}(T)$, the spectrum semi-Browder $\sigma_{sb}(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined by

$$\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not upper semi-Browder}\},$$
$$\sigma_{lb}(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not lower semi-Browder}\},$$
$$\sigma_{sb}(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not semi-Browder}\},$$
$$\sigma_b(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not Browder}\}.$$

Consider in X the well-posed Cauchy problem

$$(\star)\begin{cases} u''(t) = Au(t), & t \in \mathbb{R} \\ u(0) = u_0 \\ u'(0) = u_1 & . \end{cases}$$

Where $A : X \longrightarrow X$ is a densely defined closed operator with nonempty resolvent set $\rho(A)$. The problem $(\star)$ is (see [4, 10]) well-posed if and only if A generates a strongly continuous cosine operator function $(C(t))_{t\in\mathbb{R}}$, i.e., a family of operators satisfying the following conditions:

1. $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$.
2. $C(0) = I$ (the identity operator).
3. $C(t)x$ is strongly continuous with respect to t for any fixed $x \in X$.

There exist some $M \geq 1$, $\omega \in \mathbb{R}$ such that $\|C(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

If $(C(t))_{t\in\mathbb{R}}$ is a strongly continuous cosine operator function, then the infinitesimal generating operator A is defined by

$$\mathcal{D}(A) = \left\{ x \in X : \lim_{s\to 0} \frac{2(C(s)x - x)}{s^2} \text{ exists} \right\}$$

and

$$Ax = \lim_{s\to 0} \frac{2(C(s)x - x)}{s^2}.$$

A solution of problem $(\star)$ is given with the help of a strongly continuous cosine operator function by the formula $u(t) = C(t)u_0 + S(t)u_1$ for $t \in \mathbb{R}$, where $S(t)$ is the sine operator function associated with the $(C(t))_{t\in\mathbb{R}}$ and is defined as $S(t)x := \int_0^t C(s)x\,ds,\ t \in \mathbb{R},\ x \in X$. In this work we will use the theory of integration in the sense of Bochner.

If $(C(t))_{t\in\mathbb{R}}$ is a uniformly continuous operator cosine function then there is an $A \in \mathcal{B}(X)$ with $C(t) = \cosh t\sqrt{A}$, $t \in \mathbb{R}$. We have $A = \lim_{s\to 0} \dfrac{2(C(s) - I)}{s^2}$ in the uniform operator topology, see [6, Theorem.2.18].

For $t \in \mathbb{R}$, the function $f : z \in \mathbb{C} \mapsto \cosh t\sqrt{z}$ defines an entire function. Thus, according to the spectral mapping theorem, we have $\cosh t\sqrt{\sigma_\star(A)} = \sigma_\star(C(t))$, for all $t \in \mathbb{R}$, with $\sigma_\star$ the spectrum corresponding to regularity in the sense of V. Müller [7, Definition 6.1].

In the context of a strongly continuous cosine the following spectral inclusion $\cosh t\sqrt{\sigma(A)} \subseteq \sigma(C(t))$, $t \in \mathbb{R}$ was obtained by B. Nagy; he also gave an example where the reverse inclusion fails [8] and he showed that $\sigma_\star(C(t)) = \cosh t\sqrt{\sigma_\star(A)}$, $t \in \mathbb{R}$, with $\star \in \{p, r\}$.
However there are several large classes of generators A for which the spectrum of $C(t)$ can be expressed in terms of $\sigma(A)$, namely $\sigma(C(t)) = \overline{\cosh t\sqrt{\sigma(A)}}$, $t \in \mathbb{R}$, if A is the generator of a uniformly bounded cosine function on a Hilbert space [14] or of a cosine function of normal operators [15].

In this paper, we continue to study the spectral theory of strongly continuous cosine operator function. We investigate the relationships between the different spectra of a strongly continuous cosine operator function and their generators, precisely we prouve that

$$\cosh t\sqrt{\sigma_\star(A)} \cup \{-1, 1\} \subseteq \sigma_\star(C(t)) \cup \{-1, 1\},$$

where $\sigma_\star$ denotes the upper and lower semi-Fredholm, semi-Fredholm, Fredholm, Drazin, upper and lower semi-Browder, semi-Browder, Browder, essential ascent and descent spectra. We show by counter-example that these inclusions are strict in general.

2 Main results

The following lemmas are among the most widely used results of this paper.

Lemma 2.1. [8, Lemma. 4] Let A be the generator of the cosine operator function C. For $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, let

$$S_\lambda(t)x := \int_0^t \sinh \lambda(t-s)C(s)x\,ds, \quad x \in X.$$

Then $S_\lambda(t) \in \mathcal{B}(X)$ is an operator that commutes with A, and

$$(A - \lambda^2)S_\lambda(t)x = \lambda(C(t) - \cosh \lambda t)x,$$

for all $x \in X$.

Lemma 2.2. *Let A be the generator of a cosine operator function $(C(t))_{t\in\mathbb{R}}$. Then for all $t \neq 0$ and $\lambda \in \mathbb{C}$ with $\lambda t \notin i\pi\mathbb{Z}$, there exist two operators $L_\lambda(t), G_\lambda(t) \in \mathcal{B}(X)$ such that*

$$(A - \lambda^2)L_\lambda(t) + G_\lambda(t)S_\lambda(t) = I.$$

Moreover, the operators $L_\lambda(t)$, $S_\lambda(t)$, $G_\lambda(t)$ and $A - \lambda^2$ are mutually commuting.

Proof. For all $t \neq 0$ and $\lambda \in \mathbb{C}$ with $\lambda t \notin i\pi\mathbb{Z}$, let $K_\lambda(t)x := \int_0^t \sinh \lambda(t-s)S_\lambda(s)x\,ds$. It is clear that $K_\lambda(t)$ is a bounded linear operator of X. We consider the function $f : s \in [0, t] \longrightarrow \sinh \lambda(t-s)S_\lambda(s)x$, then f is Bochner integrable and $f(s) \in \mathcal{D}(A)$ for all $s \in [0, t]$. Moreover, $(A-\lambda^2)f(s) = \lambda \sinh \lambda(t-s)(C(t)-(\cosh \lambda t)I)$ is Bochner integrable. From [2, Proposition 1.1.7], $K_\lambda(t)x \in \mathcal{D}(A)$ and

$$\begin{aligned}(A-\lambda^2)K_\lambda(t)x &= \int_0^t \lambda \sinh \lambda(t-s)(C(s)-\cosh \lambda s)xds \\ &= \lambda \int_0^t \lambda \sinh \lambda(t-s)C(s)xds - \lambda \int_0^t \sinh \lambda(t-s)\cosh \lambda s x ds \\ &= \lambda S_\lambda(t)x - t\lambda \sinh \lambda t x.\end{aligned}$$

We put $F_\lambda(t) := (-t\lambda \sinh \lambda t)^{-1}K_\lambda(t)$ and $G_\lambda(t) := (t \sinh \lambda t)^{-1} I$. Then we have $(A-\lambda^2)L_\lambda(t) + G_\lambda(t)S_\lambda(t) = I$. Furthermore, it is clear that the operators $L_\lambda(t)$, $S_\lambda(t)$, $G_\lambda(t)$ and $A-\lambda^2$ are mutually commuting. □

Lemma 2.3. *Let A be the generator of a cosine operator function $(C(t))_{t\in\mathbb{R}}$. Then for all $n \in \mathbb{N}^*$, $t \neq 0$ and $\lambda \in \mathbb{C}$ with $\lambda t \notin i\pi\mathbb{Z}$, there exist two operators $F_{\lambda,n}(t), H_{\lambda,n}(t) \in \mathcal{B}(X)$ such that,*

$$(A-\lambda^2)^n F_{\lambda,n}(t) + H_{\lambda,n}(t)S_\lambda^n(t) = I.$$

Moreover, the operators $F_{\lambda,n}(t)$, $H_{\lambda,n}(t)$, $S_\lambda^n(t)$ and $(A-\lambda^2)^n$ are mutually commuting.

Proof. By Lemma 2.2, there exist two operators $L_\lambda(t), G_\lambda(t) \in \mathcal{B}(X)$ such that

$$(A-\lambda^2)L_\lambda(t) + G_\lambda(t)S_\lambda(t) = I.$$

For all $n \geq 1$ and $x \in X$, we have $L_\lambda^n(t)x \in \mathcal{D}(A^n)$. In fact, the proof is by induction. For $n = 1$, from lemma 2.2 $L_\lambda(t)x \in \mathcal{D}(A)$. suppose that $L_\lambda^{n-1}(t)x \in \mathcal{D}(A^{n-1})$, so $L_\lambda^n(t)x \in \mathcal{D}(A^{n-1})$ and

$$\begin{aligned}(A-\lambda^2)^{n-1}L_\lambda^n(t)x &= [(A-\lambda^2)L_\lambda(t)]^{n-1}L_\lambda(t)x \\ &= L_\lambda(t)[(A-\lambda^2)L_\lambda(t)]^{n-1}x \in \mathcal{D}(A),\end{aligned}$$

hence, $L_\lambda^n(t)x \in \mathcal{D}(A^n)$. Furthermore,

$$\begin{aligned}(A-\lambda^2)^n L_\lambda^n(t) &= [(A-\lambda^2)L_\lambda(t)]^n \\ &= [I - G_\lambda(t)S_\lambda(t)]^n \\ &= I - T_{\lambda,n}(t)S_\lambda(t),\end{aligned}$$

with $T_{\lambda,n}(t) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k} G_\lambda^k(t)S_\lambda^{k-1}(t)$. So $(A-\lambda^2)^n L_\lambda^n(t) + T_{\lambda,n}S_\lambda(t) = I$. Similarly, we have

$$\begin{aligned}T_{\lambda,n}^n(t)S_\lambda^n(t) &= [I - (A-\lambda^2)^n L_\lambda^n(t)]^n \\ &= I - (A-\lambda^2)^n \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(A-\lambda^2)^{n(k-1)}L_\lambda^{nk}(t).\end{aligned}$$

We define $F_{\lambda,n}t) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(A-\lambda^2)^{n(k-1)}L_\lambda^{nk}(t)$ and $H_{\lambda,n}(t) = T_{\lambda,n}^n(t)$. Then $(A-\lambda^2)^n F_{\lambda,n}(t) + H_{\lambda,n}(t)S_\lambda^n(t) = I$. Moreover the operators $(A-\lambda^2)^n$, $F_{\lambda,n}(t)$, $H_{\lambda,n}(t)$ and $S_\lambda^n(t)$ are pairwise commuting. □

Lemma 2.4. *Let A be the generator of a cosine operator function $(C(t))_{t\in\mathbb{R}}$. Then for all $q \in \mathbb{N}$, $t \neq 0$ and $\lambda \in \mathbb{C}$ with $\lambda t \notin i\pi\mathbb{Z}$, if $\mathcal{R}\left(C(t) - \cosh \lambda t I\right)^q$ is closed, then $\mathcal{R}(A-\lambda^2)^q$ is also closed.*

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{R}(A-\lambda^2)^q$ converging to $y \in X$. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathcal{D}(A^q)$ satisfying $(A-\lambda^2)^q x_n = y_n$. By Lemma 2.2, we obtain

$$(A-\lambda^2)^q F_{\lambda,q}(t)y_n + H_{\lambda,q}(t)S_\lambda^q(t)y_n = y_n.$$

Hence, we conclude that

$$\begin{aligned}\lambda^q(C(t)-\cosh\lambda t)^q H^q_{\lambda,q}(t)x_n &= S^q_\lambda(t)(A-\lambda^2)^q H^q_{\lambda,q}(t)x_n\\ &= H^q_{\lambda,q}(t)S^q_\lambda(t)(A-\lambda^2)^q x_n\\ &= H^q_{\lambda,q}(t)S^q_\lambda(t)y_n\\ &= y_n-(A-\lambda^2)^q F_{\lambda,q}(t)y_n.\end{aligned}$$

Thus, $y_n-(A-\lambda^2)^q F_{\lambda,q}(t)y_n \in \mathcal{R}(C(t)-\cosh\lambda t)^q$. Since $\mathcal{R}(C(t)-\cosh\lambda t)^q$ is closed and $(A-\lambda^2)^q F_{\lambda,q}(t)$ is bounded linear operator, it follows that the sequence $y_n-(A-\lambda^2)^q F_{\lambda,q}(t)y_n$ converges to $y-(A-\lambda^2)^q F_{\lambda,q}(t)y$ as n tends to ∞ and

$$y-(A-\lambda^2)^q F_{\lambda,q}(t)y \in \mathcal{R}(C(t)-\cosh\lambda t)^q \subseteq \mathcal{R}(A-\lambda^2)^q.$$

We obtain $y \in \mathcal{R}(A-\lambda^2)^q$, which completes the proof. □

Theorem 2.1. *Let A be the generator of a cosine operator function $(C(t))_{t\in\mathbb{R}}$. Then for all $t \neq 0$,*

$$\cosh t\sqrt{\sigma_\star(A)} \cup \{-1,1\} \subseteq \sigma_\star(C(t)) \cup \{-1,1\},$$

with $\star \in \{uf, lf, sf, f\}$.

Proof. Suppose that $C(t)-\cosh t\lambda$ is upper semi-Fredholm, then $\mathcal{N}(C(t)-\cosh\lambda t)$ is finite dimensional and $\mathcal{R}(C(t)-\cosh\lambda t)$ is closed. By Lemma 2.4, we obtain $R(A-\lambda^2)$ is closed. Since $\mathcal{N}(A-\lambda^2) \subseteq \mathcal{N}(C(t)-\cosh\lambda t)$, then $\mathcal{N}(A-\lambda^2)$ is finite dimensional. Therefore $A-\lambda^2$ is upper semi-Fredholm.

2. Let $C(t)-\cosh t\lambda$ is lower semi-Fredholm, then $\mathcal{R}(C(t)-\cosh\lambda t)$ is a subspace of X of finite codimension. Since $\mathcal{R}(C(t)-\cosh\lambda t) \subseteq \mathcal{R}(A-\lambda^2)$, then $\mathcal{R}(A-\lambda^2)$ is a subspace of X of finite codimension. Therefore $A-\lambda^2$ is lower semi-Fredholm.
3. It is easy by the previous assertions of this theorem.
4. Obvious.

□

Theorem 2.2. *Let $(C(t))_{t\in\mathbb{R}}$ be a strongly continuous cosine function of operators with infinitesimal generator A. Then for all $t \neq 0$,*

$$\cosh t\sqrt{\sigma_D(A)} \cup \{-1,1\} \subseteq \sigma_D(C(t)) \cup \{-1,1\}$$

We need the following lemma, which will also be useful later.

Lemma 2.5. *Let A be the generator of a cosine operator function $C(t)_{t\in\mathbb{R}}$. Then for all $t \neq 0$ and $\lambda \in \mathbb{C}$ with $\lambda t \notin i\pi\mathbb{Z}$,*

1. *if $d(C(t)-\cosh\lambda t) = n$, then $d(A-\lambda^2) \leq n$.*
2. *If $a(C(t)-\cosh\lambda t) = n$, then $a(A-\lambda^2) \leq n$.*

Proof. If $d(C(t)-\cosh\lambda t) = n$, then $\mathcal{R}(C(t)-\cosh\lambda t)^n = \mathcal{R}\left(C(t)-\cosh\lambda t\right)^{n+1}$. Let $y \in \mathcal{R}(A-\lambda^2)^n$, then there exists $x \in \mathcal{D}(A^n)$ such that $(A-\lambda^2)^n x = y$. Then there exists $z \in X$ such that $(C(t)-\cosh\lambda t)^n x = (C(t)-\cosh\lambda t)^{n+1}z$. By Lemma 2.2 we have $(A-\lambda^2)^n F_{\lambda,n}(t) + H_{\lambda,n}(t)S^n_\lambda(t) = I$. Thus,

$$\begin{aligned} y &= (A-\lambda^2)^n\left((\lambda-A)^n F_{\lambda,n}(t) + H_{\lambda,n}(t)S^n_\lambda(t)\right)x,\\ &= (A-\lambda^2)^{2n}F_{\lambda,n}(t)x + \lambda^n H_{\lambda,n}(t)(C(t)-\cosh\lambda t)^n x,\\ &= (A-\lambda^2)^{2n}F_{\lambda,n}(t)x + \lambda^n H_{\lambda,n}(t)(C(t)-\cosh\lambda t)^{n+1}z,\\ &= (A-\lambda^2)^{2n}F_{\lambda,n}(t)x + \lambda^n H_{\lambda,n}(t)(A-\lambda^2)^{n+1}S^{n+1}_\lambda(t)z,\\ &= (A-\lambda^2)^{n+1}\left((A-\lambda^2)^{n-1}F_{\lambda,n}(t)x + \lambda^n H_{\lambda,n}(t)S^{n+1}_\lambda(t)z\right).\end{aligned}$$

So $y \in \mathcal{R}(A-\lambda^2)^{n+1}$, hence $\mathcal{R}(A-\lambda^2)^n = \mathcal{R}(A-\lambda^2)^{n+1}$. Finally $d(A-\lambda^2) \leq n$.

2. If $a(C(t) - \cosh\lambda t) = n$ then $\mathcal{N}(C(t) - \cosh\lambda t)^n = \mathcal{N}(C(t) - \cosh\lambda t)^{n+1}$. Let $x \in \mathcal{N}(\lambda - A)^{n+1}$. From Lemma 2.1, $x \in \mathcal{N}(C(t) - \cosh\lambda t)^n$. Then,

$$\begin{aligned}(A-\lambda^2)^n x &= (A-\lambda^2)^n[(A-\lambda^2)^n F_{\lambda,n}(t) + H_{\lambda,n}(t)S_\lambda^n(t)]x,\\ &= (A-\lambda^2)^{2n}F_{\lambda,n}(t)x + (A-\lambda^2)^n H_{\lambda,n}(t)S_\lambda^n(t)x,\\ &= (A-\lambda^2)^{n+1}F_{\lambda,n}(t)x + \lambda^n H_{\lambda,n}(t)(C(t)-\cosh\lambda t)^n x,\\ &= (A-\lambda^2)^{n-1}F_{\lambda,n}(t)(A-\lambda^2)^{n+1}x,\\ &= 0.\end{aligned}$$

Therefore, $x \in \mathcal{N}(A - \lambda^2)^n$ and hence $a(A - \lambda^2) \leq n$.

□

Theorem 2.1 and Lemma 2.5 imply the following corollary:

Corollary 2.1. *Let $(C(t))_{t\in\mathbb{R}}$ be a strongly continuous cosine function of operators with infinitesimal generator A. Then for all $t \neq 0$,*

$$\cosh t\sqrt{\sigma_\star(A)} \cup \{-1, 1\} \subseteq \sigma_\star(C(t)) \cup \{-1, 1\},$$

with $\star \in \{ub, lb, sb, b\}$.

Proof of Theorem 2.2. If $C(t) - \cosh\lambda t$ is of invertible Drazin, then the descent and the ascent of $C(t) - \cosh\lambda t$ are finite and equal to n. From [7, Theorem.20. 4], we have $\mathcal{R}(C(t) - \cosh\lambda t)^n$ is closed. By Lemma 2.4, $\mathcal{R}(C(t) - \cosh\lambda t)^n$ is closed. According to Lemma 2.5, the descent and the ascent $A - \lambda^2$ are finite and from [12, Theorem 2.1], we have $a(A - \lambda^2) \leq d(A - \lambda^2) \leq n$. It is easy to see that $\mathcal{N}(A - \lambda^2)^n \cap \mathcal{R}(A - \lambda^2)^n = \{0\}$. Indeed if $u \in \mathcal{N}(A - \lambda^2)^n \cap \mathcal{R}(A - \lambda^2)^n$, then there exists $v \in \mathcal{D}(A - \lambda^2)^n$ such that $u = (A - \lambda^2)^n v$. And as $u \in \mathcal{N}(A - \lambda^2)^n$, we see that $v \in \mathcal{N}(A - \lambda^2)^{2n} = \mathcal{N}(A - \lambda^2)^n$ and therefore $u = (A - \lambda^2)^n v = 0$. Let us show that $X = \mathcal{R}(A - \lambda^2)^n + \mathcal{N}(A - \lambda^2)^n$. As $d(A - \lambda^2) \leq n$, then in particular we have $\mathcal{R}(A - \lambda^2)^n = \mathcal{R}(A - \lambda^2)^{2n}$. So if $u \in \mathcal{D}(A - \lambda^2)^n$, then $(A - \lambda^2)^n u \in \mathcal{R}(A - \lambda^2)^n = \mathcal{R}(A - \lambda^2)^{2n}$. So there exists $v \in \mathcal{D}(A - \lambda^2)^{2n}$ such that $A^n u = (A - \lambda^2)^{2n} v$. Hence $u - (A - \lambda^2)^n v \in \mathcal{N}(A - \lambda^2)^n$. Consequently $u \in \mathcal{R}(A - \lambda^2)^n + \mathcal{N}(A - \lambda^2)^n$ and therefore $\mathcal{D}(A - \lambda^2)^n \subseteq \mathcal{R}(A - \lambda^2)^n + \mathcal{N}(A - \lambda^2)^n$. According [2, Theorem 3.14.17], A generates a strongly continuous semigroup and from [3, 1.8 Proposition], $\mathcal{D}(A^n)$ is dense in X, since $\mathcal{R}(A - \lambda^2)^n + \mathcal{N}(A - \lambda^2)^n$ is closed, then $X = \mathcal{R}(A - \lambda^2)^n + \mathcal{N}(A - \lambda^2)^n$. From [1, Theorem 1.35], we have $A - \lambda^2$ is of invertible Drazin, which finishes the proof.

Lemma 2.6. *Let $(C(t))_{t\in\mathbb{R}}$ be a strongly continuous cosine function of operators with infinitesimal generator A. Then for all $t \neq 0$ and $\lambda \in \mathbb{C}$ with $\lambda t \notin i\pi\mathbb{Z}$,*

1. *if $d_e(C(t) - \cosh\lambda t) = n$, then $d_e(A - \lambda^2) \leq n$.*
2. *If $a_e(C(t) - \cosh\lambda t) = n$, then $a_e(A - \lambda^2) \leq n$.*

Proof. Suppose that $C(t) - \cosh\lambda t$ has finite essential descent, then there exists $n \in \mathbb{N}$ such that $\mathcal{R}(C(t) - \cosh\lambda t)^n / \mathcal{R}(C(t) - \cosh\lambda t)^{n+1}$ is finite dimensional. Let

$$\phi : \mathcal{R}(A - \lambda^2)^n \longrightarrow \mathcal{R}(C(t) - \cosh\lambda t)^n / \mathcal{R}(C(t) - \cosh\lambda t)^{n+1}$$

the mapping defined by

$$\phi((\lambda - A)^n x) = (C(t) - \cosh\lambda t)^n x + \mathcal{R}(C(t) - \cosh\lambda t)^{n+1}.$$

Thus, by isomorphism Theorem, we obtain $\mathcal{R}(A - \lambda^2)^n / \mathcal{N}(\phi)$ is isomorphic to $\mathcal{R}(C(t) - \cosh\lambda t)^n / R(C(t) - \cosh\lambda t)^{n+1}$. Therefore $\mathcal{R}(\lambda - A)^n / N(\phi)$ is finite dimensional. Since $\mathcal{N}(\phi) \subseteq \mathcal{R}(C(t) - \cosh\lambda t)^{n+1} \subseteq \mathcal{R}(A - \lambda^2)^{n+1}$, then $\mathcal{R}(\lambda - A)^n / \mathcal{R}(\lambda - A)^{n+1} \subseteq \mathcal{R}(\lambda - A)^n / \mathcal{N}(\phi)$. Finally, $A - \lambda^2$ has finite essential descent.

2. Suppose that $C(t) - \cosh \lambda t$ has finite essential ascent. Then $\mathcal{N}(C(t) - \cosh \lambda t)^{n+1} / \mathcal{N}(C(t) - \cosh \lambda t)^n$ is finite dimensional. Let

$$\psi : \mathcal{N}(\lambda - A)^{n+1} \longrightarrow \mathcal{N}(C(t) - \cosh \lambda t)^{n+1} / \mathcal{N}(C(t) - \cosh \lambda t)^n$$

the mapping defined by

$$\psi(x) = x + \mathcal{N}(C(t) - \cosh \lambda t)^n.$$

By isomorphism Theorem, $\mathcal{N}(\lambda - A)^{n+1} / N(\psi)$ is isomorphic to $\mathcal{R}(\psi)$, since $\mathcal{R}(\psi) \subseteq \mathcal{N}(C(t) - \cosh \lambda t)^{n+1} / \mathcal{N}(C(t) - \cosh \lambda t)^n$, Then $\mathcal{N}(\lambda - A)^{n+1} / \mathcal{N}(\psi)$ is finite dimensional. From lemma 2.3, we have,

$$\mathcal{N}(\psi) \subseteq \mathcal{N}(\lambda - A)^{n+} \cap \mathcal{N}(C(t) - \cosh \lambda t)^n \subseteq \mathcal{N}(\lambda - A)^n,$$

hence,

$$\mathcal{N}(\lambda - A)^{n+1} / \mathcal{N}(\lambda - A)^n \subseteq \mathcal{N}(\lambda - A)^{n+1} / \mathcal{N}(\psi).$$

Finally, $\mathcal{N}(\lambda - A)^{n+1} / \mathcal{N}(\lambda - A)^n)$ is finite dimensional.

□

Theorem 2.3. *Let $C(t)_{t \in \mathbb{R}}$ be a strongly continuous cosine function of operators with infinitesimal generator A. Then for all $t \neq 0$,*

$$\cosh t \sqrt{\sigma_\star(A)} \cup \{-1, 1\} \subseteq \sigma_\star(C(t)) \cup \{-1, 1\},$$

with $\sigma_\star \in \{\sigma_{ld}^e, \sigma_{rd}^e\}$.

Proof. The conclusion follows from Lemma 2.4 and Lemma 2.6 □

Remark 2.1. *Let X be the complex l_2 space, and for $(z_n)_{n \in \mathbb{N}} \in l_2$, $s \in \mathbb{R}$ put $C(s)(z_n)_n = (\cos(ns) z_n)_n$. Then $A(z_n)_n = (-n^2 z_n)_n$ with $\mathcal{D}(A) = \{(z_n)_n \in l_2 : \sum_{n=1}^{\infty} n^4 |z_n|^2 < \infty\}$ and $\sigma(A) = \sigma_p(A) = \{-n^2 : n \in \mathbb{N}^*\}$. Then* $\cosh \sqrt{\sigma_\star(A)}$ *is countable, with $\sigma_\star \in \{\sigma_{uf}, \sigma_{lf}, \sigma_{sf}, \sigma_f, \sigma_D, \sigma_{ub}, \sigma_{lb}, \sigma_{sb}, \sigma_b, \sigma_{ld}^e, \sigma_{rd}^e\}$. From [8], $\sigma(C(1))$ contains the set* $[-1, 1] \backslash \{\cos n : n \in \mathbb{N}^*\}$ *which is uncountable set. Then $\sigma(C(1))$ is uncountable. By* [9, Corollary 2.10], we have $\sigma_\star(C(1))$ is uncountable, with $\sigma_\star \in \{\sigma_{uf}, \sigma_{lf}, \sigma_{sf}, \sigma_f, \sigma_D, \sigma_{ub}, \sigma_{lb}, \sigma_{sb}, \sigma_b, \sigma_{ld}^e, \sigma_{rd}^e\}$. This shows that all of the above inclusions are strict.

Question 2.1. *Under what conditions were equal in the previous incusions ?*

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