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Fractional Cesàro Matrix and its Associated Sequence Space

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Abstract: In this research, we introduce a new fractional Cesàro matrix and investigate the topological properties of the sequence space associated with this matrix. We also introduce a fractional Gamma matrix as well and obtain some factorizations for the Hilbert operator based on Cesàro and Gamma matrices. The results of these factorizations are two new inequalities one of which is a generalized version of the well-known Hilbert's inequality. There are also some challenging problems that authors share at the end of the manuscript and invite the researcher for trying to solve them.

Keywords: Cesàro matrix, Hilbert matrix, Gamma matrix, Norm, Sequence space

MSC: 26D15, 40C05, 40G05, 47B37

Dedicated to Prof. Maryam Mirzakhani who in spite of a short lifetime, left a long standing impact on mathematics.

1 Introduction

Let ω be the space of all real-valued sequences. The space ℓ_p consists all real sequences $u = (u_k)_{k=0}^{\infty} \in \omega$ such that $\sum_{k=0}^{\infty} |u_k|^p < \infty$ which a Banach space with the norm

$$\|u\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |u_k|^p \right)^{1/p} < \infty,$$

where $1 \leq p < \infty$.

The spaces c, c_0 and ℓ_∞ consist of all convergent, null and bounded real sequences, respectively. These spaces are all Banach spaces with the norm $\|u\|_{\ell_\infty} = \sup_k |u_k|$. The supremum is taken over all $k \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Note that $\mathbb{N}$ is the set of all natural numbers $1, 2, 3, \ldots$.

A subspace of ω is called a sequence space. An infinite matrix is considered as a linear operator from a sequence space to another sequence space. Let U and V be sequence spaces and $A = (a_{i,j})$ be an infinite matrix. The mapping $A : U \to V$ defined as $Au = ((Au)_i) = \left(\sum_{j=0}^{\infty} a_{i,j} u_j \right)$ is a matrix transformation if the series is convergent for each $i \in \mathbb{N}_0$. By (U, V), we denote the family of all infinite matrices from U into V.

The sequence space

$$U_A = \{u \in \omega : Au \in U\}$$

is called the matrix domain of infinite matrix A in the sequence space U. By using matrix domains of special triangle matrices in classical spaces, many authors have introduced and studied new Banach spaces. For the relevant literature, we refer to the papers [1, 3, 4, 6, 16, 18–20, 22, 27, 28, 30] and textbooks [2] and [21].

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Recall the Hilbert matrix $H = (h_{j,k})$, which was introduced (1894) by David Hilbert to study a question in approximation theory:

$$h_{j,k} = \frac{1}{j+k+1} \qquad (j, k = 0, 1, \dots).$$

We know that ([14], Theorem 323) H is a bounded operator on ℓ_p with

$$\|H\|_{\ell_p} = \Gamma(1/p)\Gamma(1/p^\star) = \pi \csc(\pi/p).$$

Throughout this paper we suppose that n is a non-negative integer and we use the notation T^n to denote the matrix T of order n which does not mean the n-th power of T.

Consider the Hausdorff matrix $H^\mu = (h_{j,k})$, with entries of the form:

$$h_{j,k} = \begin{cases} \binom{j}{k} \int_0^1 \theta^k (1-\theta)^{j-k} d\mu(\theta) & 0 \le k \le j \\ 0 & k > j. \end{cases}$$

where μ is a probability measure on $[0, 1]$. The Hausdorff matrix contains some famous classes of matrices. Two of these classes are as follow:

(i) The choice $d\mu(\theta) = n(1-\theta)^{n-1} d\theta$ gives the Cesàro matrix of order n;

(ii) The choice $d\mu(\theta) = n\theta^{n-1} d\theta$ gives the Gamma matrix of order n.

For $1 < p < \infty$, Hardy's formula ([13], Theorem 216) states that the Hausdorff matrix is a bounded operator on ℓ_p if and only if $\int_0^1 \theta^{\frac{-1}{p}} d\mu(\theta) < \infty$ and

$$\|H^\mu\|_{\ell_p} = \int_0^1 \theta^{\frac{-1}{p}} d\mu(\theta). \tag{1.1}$$

By letting $d\mu(\theta) = n(1-\theta)^{n-1} d\theta$ and $d\mu(\theta) = n\theta^{n-1} d\theta$ in the definition of the Hausdorff matrix, the Cesàro matrix of order n, $C^n = (c^n_{j,k})$, and the Gamma matrix of order n, $\Gamma^n = (\gamma^n_{j,k})$, are

$$c^n_{j,k} = \begin{cases} \frac{\binom{n+j-k-1}{j-k}}{\binom{n+j}{j}} & 0 \le k \le j \\ 0 & otherwise, \end{cases} \tag{1.2}$$

and

$$\gamma^n_{j,k} = \begin{cases} \frac{\binom{n+k-1}{k}}{\binom{n+j}{j}} & 0 \le k \le j \\ 0 & otherwise, \end{cases}$$

respectively, which according to the Hardy's formula have the norms

$$\|C^n\|_{\ell_p} = \frac{\Gamma(n+1)\Gamma(1/p^\star)}{\Gamma(n+1/p^\star)},$$

and

$$\|\Gamma^n\|_{\ell_p} = \frac{np}{np-1},$$

where $p^\star$ is the conjugate of p i.e. $\frac{1}{p} + \frac{1}{p^\star} = 1$.

Note that, for $n = 0$, $C^0 = I$, where I is the identity matrix and for $n = 1$, $C^1 = \Gamma^1 = C$ is the well-known Cesàro matrix, which is defined by

$$c_{j,k} = \begin{cases} \frac{1}{j+1} & 0 \le k \le j \\ 0 & otherwise. \end{cases}$$

In 1986 Bennett [8], proved that the Hilbert matrix H, admits a factorization of the form $H = BC$, where C is the Cesàro matrix and the matrix $B = (b_{j,k})$ is defined by

$$b_{j,k} = \frac{k+1}{(j+k+1)(j+k+2)} \qquad (j,k = 0,1,\ldots),$$

and is a bounded operator on ℓ_p with $\|B\|_{\ell_p} = \frac{\pi}{p^\star}\csc(\pi/p)$, ([8], Proposition 2).

Later in 2019, Roopaei [24] generalized the Bennett's factorization of the forms $H = B^n C^n$ and $H = S^n \Gamma^n$, where C^n and Γ^n are the Cesàro and Gamma matrices of order n respectively. The result of these factorizations are the inequalities

$$\|Hu\|_{\ell_p} \le \frac{\Gamma(n+1/p^\star)\Gamma(1/p)}{\Gamma(n+1)}\|C^n u\|_{\ell_p}, \tag{1.3}$$

and

$$\|Hu\|_{\ell_p} \le \pi(1-\frac{1}{np})\csc(\pi/p)\|\Gamma^n u\|_{\ell_p},$$

where for $n = 0$ the first one is the Hilbert's inequality and for $n = 1$ the first and second inequalities become

$$\|Hu\|_{\ell_p} \le \frac{\pi}{p^\star}\csc(\pi/p)\|Cu\|_{\ell_p}, \tag{1.4}$$

which Bennett introduced it as Hardy's inequality versus Hilbert's.

In this study, we define fractional Cesàro and Gamma matrices and study the sequence space associated with fractional Cesàro matrix. Also, we tried to obtain several factorizations for the infinite Hilbert operator based on fractional Cesàro and Gamma matrices of the forms $H = B^{n/m}C^{n/m}$ and $H = S^{n/m}\Gamma^{n/m}$. For another purpose of this study we obtain the upper bounds for the factors presented in Hilbert factorizations. Note that we use the notation $\|.\|_{\ell_p}$ for the norm of operators from ℓ_p into itself.

Motivation.

The infinite Cesàro matrix was defined by Ernesto Cesàro (1859 – 1906) who was an Italian mathematician and worked in the field of differential geometry. After that, many mathematicians have done research on Cesàro sequence spaces and Cesàro function spaces, but all the researches have been around the Cesàro matrix of order one. Recently Roopaei et al. [25] have discussed the Köthe dual of Cesàro sequence spaces of order n, $C_p^n = \ell_p(C^n)$ with $1 \le p < \infty$ as well as the problem of finding the norm of operators on this sequence space. This manuscript investigates and characterizes spaces associated with a new matrix, who has a very close definition to the Cesàro matrix, their duals and their linear operators. The topic has been studied for more than half a century by now, and the author's aim is to fill in some empty spots in the theory. There is also a challenging problem about the norm of the new defined fractional Cesàro matrix that the authors invite the readers to involve and help to solve that.

2 Fractional Cesàro Banach spaces $C_p^{n/m}$ and $C_\infty^{n/m}$

Suppose that n, m are two non-negative integers that $n \ge m$. Let us define the fractional Cesàro matrix $C^{n/m} = (c_{j,k}^{n/m})$ by

$$c_{j,k}^{n/m} = \begin{cases} \frac{\binom{n+j-k-1}{j-k}}{\binom{m+j}{j}} & 0 \le k \le j \\ 0 & otherwise. \end{cases}$$

Note that for $n = m$, $C^{n/m} = C^n$, where C^n is the Cesàro matrix of order n defined by relation (1.2).

We say that $A = (a_{j,k})$ is a summability matrix if it is a lower-triangular matrix, i.e. $a_{j,k} = 0$ for $j < k$, and $\sum_{k=0}^{j} a_{j,k} = 1$ for all j. One can easily investigate that the Hausdorff matrices and consequently Cesàro matrix of order n are summability matrix, while for $n \ne m$, the fractional Cesàro matrix is not a summability matrix.

Lemma 2.1. *For* $j = 1, 2, \cdots$, *we have*

$$\sum_{k=0}^{j}(-1)^k \binom{n}{k}\binom{n+j-k-1}{j-k} = 0.$$

Proof. By applying the identity $(1-z)^{-n} = \sum_{j=0}^{\infty}\binom{n+j-1}{j}z^j$ we have

$$\begin{aligned} 1 &= (1-z)^n(1-z)^{-n} \\ &= \sum_{j=0}^{n}(-1)^j\binom{n}{j}z^j\sum_{j=0}^{\infty}\binom{n+j-1}{j}z^j \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^k\binom{n}{k}\binom{n+j-k-1}{j-k}z^j \\ &= 1+\sum_{j=1}^{\infty}\sum_{k=0}^{j}(-1)^k\binom{n}{k}\binom{n+j-k-1}{j-k}z^j, \end{aligned}$$

now the result is obvious. □

Lemma 2.2. *The fractional Cesàro matrix* $C^{n/m}$, *is invertible and it's inverse* $C^{-n/m} = (c_{j,k}^{-n/m})$ *is defined by*

$$c_{j,k}^{-n/m} = \begin{cases} (-1)^{(j-k)}\binom{n}{j-k}\binom{m+k}{k} & k \le j \le n+k, \\ 0 & otherwise. \end{cases}$$

Proof. By applying Lemma 2.1 we have

$$\begin{aligned} (C^{n/m}C^{-n/m})_{i,k} &= \frac{\binom{m+k}{k}}{\binom{m+i}{i}}\sum_{j=k}^{i}\binom{n+i-j-1}{i-j}(-1)^{(j-k)}\binom{n}{j-k} \\ &= \frac{\binom{m+k}{k}}{\binom{m+i}{i}}\sum_{j=0}^{i-k}(-1)^j\binom{n}{j}\binom{n+i-k-j-1}{i-k-j} \\ &= I_{i,k}, \end{aligned}$$

where I is the identity matrix. So the proof is complete. □

Now the sequence spaces $C_p^{n/m}$ $(1 \le p < \infty)$ and $C_\infty^{n/m}$ are introduced by using the fractional Cesàro matrix as the set of all sequences whose $C^{n/m}$-transforms are in the spaces ℓ_p and ℓ_∞, respectively; that is

$$C_p^{n/m} = \left\{u = (u_j) \in \omega : \sum_{j=0}^{\infty}\left|\frac{1}{\binom{m+j}{j}}\sum_{k=0}^{j}\binom{n+j-k-1}{j-k}u_k\right|^p < \infty\right\}$$

and

$$C_\infty^{n/m} = \left\{u = (u_j) \in \omega : \sup_j\left|\frac{1}{\binom{m+j}{j}}\sum_{k=0}^{j}\binom{n+j-k-1}{j-k}u_k\right| < \infty\right\}.$$

Throughout the study, $v = (v_j)$ will be the $C^{n/m}$-transform of a sequence $u = (u_j)$; that is,

$$v_j = (C^{n/m}u)_j = \frac{1}{\binom{m+j}{j}}\sum_{k=0}^{j}\binom{n+j-k-1}{j-k}u_k$$

for all $j \in \mathbb{N}_0$. Also, the relation

$$u_k = \sum_{i=0}^{k}(-1)^{k-i}\binom{m+i}{i}\binom{n}{k-i}v_i$$

holds for all $k \in \mathbb{N}_0$.

Theorem 2.3. *The spaces $C_p^{n/m}$ and $C_\infty^{n/m}$ are Banach spaces with the norms*

$$\|u\|_{C_p^{n/m}} = \left(\sum_{j=0}^{\infty} \left| \frac{1}{\binom{m+j}{j}} \sum_{k=0}^{j} \binom{n+j-k-1}{j-k} u_k \right|^p \right)^{1/p}$$

and

$$\|u\|_{C_\infty^{n/m}} = \sup_j \left| \frac{1}{\binom{m+j}{j}} \sum_{k=0}^{j} \binom{n+j-k-1}{j-k} u_k \right|,$$

respectively.

Proof. We omit the proof which is a routine verification. □

Remark 2.4. *If we choose $n = m$, then we obtain the Cesàro sequence spaces C_p^n and C_∞^n defined in [25] which are Banach spaces endowed with the norms*

$$\|u\|_{C_p^n} = \left(\sum_{j=0}^{\infty} \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+j-k-1}{j-k} u_k \right|^p \right)^{1/p}$$

and

$$\|u\|_{C_\infty^n} = \sup_j \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+j-k-1}{j-k} u_k \right|,$$

respectively.

Theorem 2.5. *The spaces $C_p^{n/m}$ and $C_\infty^{n/m}$ are linearly isomorphic to ℓ_p and ℓ_∞, respectively.*

Proof. Let define a mapping $T : C_p^{n/m} \to \ell_p$ with $T(u) = ((C^{n/m}u)_j)$ for any $u \in C_p^{n/m}$. It is clear that T is linear and one-to-one. Now consider the sequence $u = (u_k)$ given as

$$u_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{m+i}{i} \binom{n}{k-i} v_i \quad (k \in \mathbb{N}_0)$$

for any $v = (v_i) \in \ell_p$. We have that

$$\begin{aligned}
(C^{n/m}u)_j &= \frac{1}{\binom{m+j}{j}} \sum_{k=0}^{j} \binom{n+j-k-1}{j-k} u_k \\
&= \frac{1}{\binom{m+j}{j}} \sum_{k=0}^{j} \binom{n+j-k-1}{j-k} \sum_{i=0}^{k} (-1)^{k-i} \binom{m+i}{i} \binom{n}{k-i} v_i \\
&= \frac{1}{\binom{m+j}{j}} \sum_{i=0}^{j} \left(\sum_{k=0}^{j-i} (-1)^k \binom{n+j-k-i-1}{j-k-i} \binom{n}{k} \right) \binom{m+i}{i} v_i \\
&= \frac{1}{\binom{m+j}{j}} (-1)^0 \binom{n-1}{0} \binom{m+j}{j} v_j = v_j,
\end{aligned}$$

where $\sum_{k=0}^{j-i} (-1)^k \binom{n+j-k-i-1}{j-k-i} \binom{n}{k} = 0$ for $j \neq i$. Hence, we conclude that $u \in C_p^{n/m}$ and $T(u) = v$ which implies T is onto. Also, since $\|u\|_{C_p^{n/m}} = \|C^{n/m}u\|_{\ell_p}$ holds we obtain that T preserves norms. This completes the proof. □

Remark 2.6. *If we choose $n = m$, then we obtain that the Cesàro sequence spaces C_p^n and C_∞^n are linearly isomorphic to ℓ_p and ℓ_∞, respectively.*

Remark 2.7. *The space $C_2^{n/m}$ is an inner product space with the inner product defined as $\langle u,\tilde{u}\rangle_{C_2^{n/m}} = \langle C^{n/m}u, C^{n/m}\tilde{u}\rangle_2$, where $\langle .,.\rangle_2$ is the inner product on ℓ_2.*

Theorem 2.8. *The space $C_p^{n/m}$ is not an inner product space except for $p = 2$. Thus, the space $C_p^{n/m}$ is not a Hilbert space except for $p = 2$.*

Proof. Let us define the sequences $u = (u_j)$ and $\tilde{u} = (\tilde{u}_j)$ as

$$u_j = \begin{cases} 1 & , \quad j = 0 \\ \sum_{k=0}^{1}(-1)^{j-k}\binom{n}{j-k}\binom{m+k}{k} & , \quad j \in \mathbb{N} \end{cases}$$

and

$$\tilde{u}_j = \begin{cases} 1 & , \quad j = 0 \\ (-1)^j\sum_{k=0}^{1}\binom{n}{j-k}\binom{m+k}{k} & , \quad j \in \mathbb{N}. \end{cases}$$

Then, we have $C^{n/m}u = (1,1,0,\dots,0,\dots) \in \ell_p$ and $C^{n/m}\tilde{u} = (1,-1,0,\dots,0,\dots) \in \ell_p$. Hence, we observe that

$$\left(\|u+\tilde{u}\|_{C_p^{n/m}}\right)^2 + \left(\|u-\tilde{u}\|_{C_p^{n/m}}\right)^2 = 8 \neq 4.2^{2/p} = 2\left[\|u\|^2_{C_p^{n/m}} + \|\tilde{u}\|^2_{C_p^{n/m}}\right]$$

for $p \neq 2$. Since the parallelogram equality does not hold, we conclude that $C_p^{n/m}$ is not an inner product space for $p \neq 2$ and so it is not a Hilbert space . □

Theorem 2.9. *The inclusion $C_p^{n/m} \subset C_q^{n/m}$ strictly holds, where $1 \le p < q < \infty$.*

Proof. Let $u \in C_p^{n/m}$. Then, we have $C^{n/m}u \in \ell_p$. Since the inclusion $\ell_p \subset \ell_q$ holds for $1 \le p < q < \infty$, we have $C^{n/m}u \in \ell_q$. This implies that $u \in C_q^{n/m}$. Hence, we conclude that the inclusion $C_p^{n/m} \subset C_q^{n/m}$ holds.

Now, we show that the inclusion is strict. Since the inclusion $\ell_p \subset \ell_q$ is strict, we can choose $v = (v_j) \in \ell_q \backslash \ell_p$. Define a sequence $u = (u_j)$ as

$$u_j = \sum_{k=0}^{j}(-1)^{j-k}\binom{m+k}{k}\binom{n}{j-k}v_k \quad (j \in \mathbb{N}_0).$$

Then, we have

$$(C^{n/m}u)_j = v_j$$

for every $j \in \mathbb{N}_0$ which means $C^{n/m}u = v$ and so $C^{n/m}u \in \ell_q \backslash \ell_p$. Hence, we conclude that $u \in C_q^{n/m} \backslash C_p^{n/m}$ and so the inclusion $C_p^{n/m} \subset C_q^{n/m}$ is strict. □

Theorem 2.10. *The inclusion $C_p^{n/m} \subset C_\infty^{n/m}$ strictly holds, where $1 \le p < \infty$.*

Proof. Let $u \in C_p^{n/m}$. Then, we have $C^{n/m}u \in \ell_p$. Since the inclusion $\ell_p \subset \ell_\infty$ holds for $1 \le p < \infty$, we have $C^{n/m}u \in \ell_\infty$. This implies that $u \in C_\infty^{n/m}$. Hence, we conclude that the inclusion $C_p^{n/m} \subset C_\infty^{n/m}$ holds.

Now, we show that the inclusion is strict. Consider the sequence $u = (u_j)$ defined as

$$u_j = (-1)^j\sum_{k=0}^{j}\binom{m+k}{k}\binom{n}{j-k} \quad (j \in \mathbb{N}_0).$$

We deduce that

$$\begin{aligned}(C^{n/m}u)_j &= \frac{1}{\binom{m+j}{j}}\sum_{k=0}^{j}\binom{n+j-k-1}{j-k}u_k \\ &= \frac{1}{\binom{m+j}{j}}\sum_{k=0}^{j}\binom{n+j-k-1}{j-k}(-1)^k\sum_{i=0}^{k}\binom{m+i}{i}\binom{n}{k-i} \\ &= \frac{1}{\binom{m+j}{j}}\sum_{i=0}^{j}\left(\sum_{k=0}^{j-i}(-1)^{k+i}\binom{n+j-k-i-1}{j-k-i}\binom{n}{k}\right)\binom{m+i}{i} \\ &= \frac{1}{\binom{m+j}{j}}(-1)^j\binom{n-1}{0}\binom{m+j}{j} = (-1)^j,\end{aligned}$$

where $\sum_{k=0}^{j-i}(-1)^k\binom{n+j-k-i-1}{j-k-i}\binom{n}{k} = 0$ for $j \neq i$. Since we have $C^{n/m}u = ((-1)^j) \in \ell_\infty \backslash \ell_p$, we obtain that $u \in C_\infty^{n/m}\backslash C_p^{n/m}$. Consequently, the inclusion $C_p^{n/m} \subset C_\infty^{n/m}$ is strict. □

Theorem 2.11. *Define the sequence $(e^{(k)}) = (e_j^{(k)})$ for each $k \in \mathbb{N}$ by*

$$(e^{(k)})_j = \begin{cases} (-1)^{j-k}\binom{m+k}{k}\binom{n}{j-k} & , \quad j \geq k \\ 0 & , \quad j < k. \end{cases} \quad (j \in \mathbb{N}_0)$$

Then, the sequence $(e^{(k)})$ is a basis for the space $C_p^{n/m}$, and each $u \in C_p^{n/m}$ has a unique representation of the form $u = \sum_k (C^{n/m}u)_k e^{(k)}$.

Proof. Let A be a triangle. By Theorem 2.3 of Jarrah and Malkowsky [17], the matrix domain U_A has a basis if and only if the normed sequence space U has a basis. Hence the proof follows immediately. □

We use the following lemma to compute the dual spaces. By $\mathcal{N}$, we denote the family of all finite subsets of $\mathbb{N}$.

Lemma 2.12. *[29] The following statements hold:*

(i) $A = (a_{j,k}) \in (\ell_1, \ell_1)$ if and only if

$$\sup_k \sum_{j=0}^{\infty} |a_{j,k}| < \infty.$$

(ii) $A = (a_{j,k}) \in (\ell_p, \ell_1)$ if and only if

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty}|a_{j,k}|\right)^{p^*} < \infty,$$

where $1 < p < \infty$.

(iii) $A = (a_{j,k}) \in (\ell_\infty, \ell_1)$ if and only if

$$\sup_{K\in\mathcal{N}} \sum_{j=0}^{\infty}\left|\sum_{k\in K} a_{j,k}\right| < \infty.$$

(iv) $A = (a_{j,k}) \in (\ell_1, c)$ if and only if

$$\lim_{j\to\infty} a_{j,k} \text{ exists for each } k \in \mathbb{N}$$

and

$$\sup_{j,k}|a_{j,k}| < \infty. \tag{2.1}$$

(v) $A = (a_{j,k}) \in (\ell_p, c)$ *if and only if* (2.12) *holds and*

$$\sup_j \sum_{k=0}^{\infty} |a_{j,k}|^{p^\star} < \infty, \tag{2.2}$$

where $1 < p < \infty$.

(vi) $A = (a_{j,k}) \in (\ell_\infty, c)$ *if and only if* (2.12) *holds and*

$$\lim_{j\to\infty} \sum_{k=0}^{\infty} |a_{j,k}| = \sum_{k=0}^{\infty} \left| \lim_{j\to\infty} a_{j,k} \right|.$$

(vii) $A = (a_{j,k}) \in (\ell_1, \ell_\infty)$ *if and only if* (2.1) *holds.*

(viii) $A = (a_{j,k}) \in (\ell_p, \ell_\infty)$ *if and only if* (2.2) *holds, where* $1 < p < \infty$.

(ix) $A = (a_{j,k}) \in (\ell_\infty, \ell_\infty)$ *if and only if*

$$\sup_j \sum_{k=0}^{\infty} |a_{j,k}| < \infty.$$

The α-dual of a sequence space U consists of all sequences $a = (a_k) \in \omega$ such that $au = (a_k u_k) \in \ell_1$ for all $u = (u_k) \in U$.

Theorem 2.13. *The α-duals of the spaces $C_1^{n/m}$, $C_p^{n/m}$ $(1 < p < \infty)$ and $C_\infty^{n/m}$ are as follows:*

$$(C_1^{n/m})^\alpha = \left\{ a = (a_j) \in \omega : \sup_k \sum_{j=0}^{\infty} \left| (-1)^{j-k} \binom{m+k}{k} \binom{n}{j-k} a_j \right| < \infty \right\},$$

$$(C_p^{n/m})^\alpha = \left\{ a = (a_j) \in \omega : \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} \left| (-1)^{j-k} \binom{m+k}{k} \binom{n}{j-k} a_j \right| \right)^{p^\star} < \infty \right\}$$

and

$$(C_\infty^{n/m})^\alpha = \left\{ a = (a_j) \in \omega : \sup_{K\in\mathbb{N}} \sum_{j=0}^{\infty} \left| \sum_{k\in K} (-1)^{j-k} \binom{m+k}{k} \binom{n}{j-k} a_j \right| < \infty \right\}.$$

Proof. Let $a = (a_j) \in \omega$ and define the matrix $D = (d_{j,k})$ as

$$d_{j,k} = \begin{cases} (-1)^{j-k} \binom{n}{j-k} \binom{m+k}{k} a_j &, \quad 0 \le k \le j \\ 0 &, \quad \text{otherwise.} \end{cases}$$

For any $u = (u_j) \in C_p^{n/m}$ $(1 < p < \infty)$, we have $a_j u_j = (Dv)_j$ for all $j \in \mathbb{N}$. Thus $au \in \ell_1$ with $u \in C_p^{n/m}$ if and only if $Dv \in \ell_1$ with $v \in \ell_p$. Hence, we conclude that $a \in (C_p^{n/m})^\alpha$ if and only if $D \in (\ell_p, \ell_1)$. This completes the proof by part *(ii)* of Lemma 2.12. The other cases can be proved similarly. □

The β-dual of a sequence space U consists of all sequences $a = (a_k) \in \omega$ such that $\left(\sum_{k=1}^{n} a_k u_k\right) \in c$ for all $u = (u_k) \in U$.

Theorem 2.14. *Let define the following sets.*

$$P_1 = \left\{ a = (a_k) \in \omega : \lim_{j\to\infty} \sum_{i=k}^{j} (-1)^{i-k} \binom{m+k}{k} \binom{n}{i-k} a_i \text{ exists for each } k \in \mathbb{N} \right\},$$

$$P_2 = \left\{ a = (a_k) \in \omega : \sup_{j,k} \left| \sum_{i=k}^{j} (-1)^{i-k} \binom{m+k}{k} \binom{n}{i-k} a_i \right| < \infty \right\},$$

$$P_3 = \left\{ a = (a_k) \in \omega : \sup_j \sum_{k=0}^{\infty} \left| \sum_{i=k}^{j} (-1)^{i-k} \binom{m+k}{k} \binom{n}{i-k} a_i \right|^{p^\star} < \infty \right\},$$

and

$$P_4 = \left\{ a = (a_k) \in \omega : \lim_{j\to\infty} \sum_{k=0}^{\infty} \left| \sum_{i=k}^{j} (-1)^{i-k} \binom{m+k}{k} \binom{n}{i-k} a_i \right| = \sum_{k=0}^{\infty} \left| \sum_{i=k}^{\infty} (-1)^{i-k} \binom{m+k}{k} \binom{n}{i-k} a_i \right| \right\}.$$

Then, $(C_1^{n/m})^\beta = P_1 \cap P_2$, $(C_p^{n/m})^\beta = P_1 \cap P_3$ $(1 < p < \infty)$ *and* $(C_\infty^{n/m})^\beta = P_1 \cap P_4$ *hold.*

Proof. $a = (a_k) \in (C_1^{n/m})^\beta$ if and only if the series $\sum_{k=0}^{\infty} a_k u_k$ is convergent for all $u = (u_k) \in C_1^{n/m}$. From the equality

$$\begin{aligned}\sum_{k=0}^{j} a_k u_k &= \sum_{k=0}^{j} a_k \left(\sum_{i=0}^{k} (-1)^{k-i} \binom{m+i}{i} \binom{n}{k-i} v_i \right) \\ &= \sum_{k=0}^{j} \left(\sum_{i=k}^{j} (-1)^{i-k} \binom{m+k}{k} \binom{n}{i-k} a_i \right) v_k,\end{aligned}$$

it follows that $a = (a_k) \in (C_1^{n/m})^\beta$ if and only if the matrix $P = (p_{j,k})$ is in (ℓ_1, c), where

$$p_{j,k} = \begin{cases} \sum_{i=k}^{j} (-1)^{i-k} \binom{m+k}{k} \binom{n}{i-k} a_i &, \quad 0 \le k \le j \\ 0 &, \quad \text{otherwise.} \end{cases}$$

Hence, we conclude by part (iv) of Lemma 2.12 that

$$\lim_{j\to\infty} \sum_{i=k}^{n} (-1)^{i-k} \binom{m+k}{k} \binom{n}{i-k} a_i \text{ exists for each } k \in \mathbb{N}$$

and

$$\sup_{j,k} \left| \sum_{i=k}^{j} (-1)^{i-k} \binom{m+k}{k} \binom{n}{i-k} a_i \right| < \infty$$

which means $a = (a_k) \in P_1 \cap P_2$ and so we have $(C_1^{n/m})^\beta = P_1 \cap P_2$. The other cases can be proved similarly. □

The γ-dual of a sequence space U consists of all sequences $a = (a_k) \in \omega$ such that $\left(\sum_{k=1}^{n} a_k u_k\right) \in \ell_\infty$ for all $u = (u_k) \in U$.

Theorem 2.15. *The γ-duals of the spaces $C_1^{n/m}$, $C_p^{n/m}$ $(1 < p < \infty)$ and $C_\infty^{n/m}$ are as follows:*

$$(C_1^{n/m})^\gamma = \left\{ a = (a_k) \in \omega : \sup_{j,k} \left| \sum_{i=k}^{j} (-1)^{i-k} \binom{m+k}{k} \binom{n}{i-k} a_i \right| < \infty \right\},$$

$$(C_p^{n/m})^\gamma = \left\{ a = (a_k) \in \omega : \sup_j \sum_{k=0}^{\infty} \left| \sum_{i=k}^{j} (-1)^{i-k} \binom{m+k}{k} \binom{n}{i-k} a_i \right|^{p^\star} < \infty \right\}$$

and

$$(C_\infty^{n/m})^\gamma = \left\{ a = (a_k) \in \omega : \sup_j \sum_{k=0}^{\infty} \left| \sum_{i=k}^{j} (-1)^{i-k} \binom{m+k}{k} \binom{n}{i-k} a_i \right| < \infty \right\}.$$

Proof. By using the same technique in the proof of Theorem 2.14, we compute the gamma duals. □

3 Factorization of the Hilbert matrix based on Cesàro matrices

In this section, we introduce several factorizations for the Hilbert matrix based on fractional Cesàro matrix. Throughout this section we suppose that m and n are two non-negative integers that $n \geq m$.

Let us define the matrix $B^{n/m} = (b^{n/m}_{j,k})$ by

$$b^{n/m}_{j,k} = \binom{k+m}{k}\beta(j+k+1, n+1),$$

where the β function is

$$\beta(s,t) = \int_0^1 z^{s-1}(1-z)^{t-1}dz \qquad (s,t = 1,2,\ldots).$$

Note that, in case $n = m$ we have the matrix B^m that has the entries

$$b^m_{j,k} = \binom{k+m}{k}\beta(j+k+1, m+1),$$

and according to [24] has the ℓ_p-norm $\|B^n\|_{\ell_p} = \frac{\Gamma(n+1/p^*)\Gamma(1/p)}{\Gamma(n+1)}$.

Lemma 3.1. *For $|z| < 1$, we have*

$$\sum_{j=0}^{\infty}\binom{n+j-1}{j}z^j = (1-z)^{-n}.$$

Proof. By differentiating $n-1$ times the identity $(1-z)^{-1} = \sum_{j=0}^{\infty} z^j$, we get the result. □

Theorem 3.2. *The Hilbert matrix H, has a factorization of the form $H = B^{n/m}C^{n/m}$, where $C^{n/m}$ is the fractional Cesàro matrix and the matrix $B^{n/m}$ is a bounded operator on ℓ_p with*

$$\|B^{n/m}\|_{\ell_p} \leq \frac{\Gamma(m+1/p^\star)\Gamma(1/p)}{\Gamma(m+1)}.$$

In particular, for $n = m$, H has Roopaei's factorization $H = B^nC^n$, where $\|B^n\|_{\ell_p} = \frac{\Gamma(n+1/p^\star)\Gamma(1/p)}{\Gamma(n+1)}$.

Proof. By applying Lemma 3.1 we have

$$\begin{aligned}(B^{n/m}C^{n/m})_{i,k} &= \sum_{j=k}^{\infty}\binom{m+j}{j}\beta(i+j+1, n+1)\frac{\binom{n+j-k-1}{j-k}}{\binom{m+j}{j}}\\ &= \sum_{j=0}^{\infty}\binom{n+j-1}{j}\beta(i+j+k+1, n+1)\\ &= \int_0^1\sum_{j=0}^{\infty}\binom{n+j-1}{j}z^jz^{i+k}(1-z)^n dz\\ &= \int_0^1 z^{i+k}dz = \frac{1}{i+k+1} = h_{i,k}.\end{aligned}$$

which results in the claimed factorization. It is obvious that for $n \geq m$,

$$b^{n/m}_{j,k} = \binom{k+m}{k}\beta(j+k+1, n+1) \leq \binom{k+m}{k}\beta(j+k+1, m+1) = b^m_{j,k},$$

hence $\|B^{n/m}\|_{\ell_p} \leq \|B^m\|_{\ell_p}$. Now, by letting $n = m$ we have the factorization $H = B^nC^n$. □

Lemma 3.3. *We have*

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{1}{j+m}=\int_0^1 x^{m-1}(1-x)^n\,dx=\beta(m,n+1).$$

Proof. By multiplying both sides of the identity

$$(1-x)^n=\sum_{j=0}^{n}(-1)^j\binom{n}{j}x^j,$$

in x^{m-1} and also integrating from 0 to 1, we have the desired result. □

Remark 3.4. *By applying Lemma 3.3 one can straightly obtain*

$$\begin{aligned}(HC^{-n/m})_{i,k} &= \sum_{j=k}^{n+k}\frac{1}{i+j+1}(-1)^{j-k}\binom{n}{j-k}\binom{m+k}{k}\\ &= \binom{m+k}{k}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{1}{i+j+k+1}\\ &= \binom{m+k}{k}\beta(i+k+1,n+1)=b_{i,k}^{n/m},\end{aligned}$$

which results the factor $B^{n/m}$ in Theorem 3.2.

We are ready to generalize the Hilbert's inequality.

Corollary 3.5. *Let $p>1$ and $x\in\ell_p$. Then*

$$\|Hx\|_{\ell_p}\le\frac{\Gamma(m+1/p^\star)\Gamma(1/p)}{\Gamma(m+1)}\|C^{n/m}x\|_{\ell_p}.$$

In particular, for $n=m$ we obtain the inequality (1.3) and for $n=m=0$ we obtain the Hilbert's inequality.

Proof. Since $H=B^{n/m}C^{n/m}$, according to Theorem 3.2 we have

$$\|Hx\|_{\ell_p}=\|B^{n/m}C^{n/m}x\|_{\ell_p}\le\frac{\Gamma(m+1/p^\star)\Gamma(1/p)}{\Gamma(m+1)}\|C^{n/m}x\|_{\ell_p}.$$

Consider that for $n=m$ we have $C^{n/m}=C^n$ and for $n=m=0$, $C^0=I$, hence we have the (1.3) inequality and Hilbert's. □

Corollary 3.6. *The Hilbert matrix H, has a symmetric factorization of the form $H=(C^{n/m})^tU^{n/m}C^{n/m}$, where the matrix $U^{n/m}=(u_{j,k}^{n/m})$ has the entries*

$$u_{j,k}^{n/m} = \binom{m+j}{j}\binom{m+k}{k}\beta(j+k+1,2n+1),$$

is a bounded operator on ℓ_p and

$$\|U^{n/m}\|\le\frac{\Gamma(m+1/p)\Gamma(m+1/p^\star)}{(\Gamma(m+1))^2}.$$

Proof. According to Theorem 3.2 it is sufficient to prove $U^{n/m} = (C^{-n/m})^t B^{n/m}$. By applying the identity $\sum_{j=0}^{n}(-1)^j \binom{n}{j} z^j = (1-z)^n$ we have

$$\begin{aligned} u_{i,k}^{n/m} &= \sum_{j=i}^{n+i}(-1)^{j-i}\binom{n}{j-i}\binom{m+i}{i}\binom{m+k}{k}\beta(j+k+1,n+1) \\ &= \binom{m+i}{i}\binom{m+k}{k}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\beta(j+i+k+1,n+1) \\ &= \binom{m+i}{i}\binom{m+k}{k}\int_0^1\sum_{j=0}^{n}(-1)^j\binom{n}{j}z^j z^{i+k}(1-z)^n dz \\ &= \binom{m+i}{i}\binom{m+k}{k}\beta(i+k+1,2n+1). \end{aligned}$$

Similar to the proof of Theorem 3.2, for $n \geq m$, $u_{j,k}^{n/m} \leq u_{j,k}^{m}$ which results $\|U^{n/m}\|_{\ell_p} \leq \|U^m\|_{\ell_p}$. But Roopaei in [24] Corollary 2.6 has proved that the matrix U^n is a bounded operator on ℓ_p and $\|U^n\|_{\ell_p} = \frac{\Gamma(n+1/p)\Gamma(n+1/p^*)}{(\Gamma(n+1))^2}$. □

Remark 3.7. *By letting $n = m$ in Corollary 3.6 and using the notation $U^{n,n} = U^n$ we obtain the Corollary 2.6 of [24] which states that the Hilbert matrix H, has a symmetric factorization of the form $H = (C^n)^t U^n C^n$, where the matrix U^n is a bounded operator on ℓ_p with*

$$\begin{aligned} u_{j,k}^{n} &= \binom{n+j}{j}\binom{n+k}{k}\beta(j+k+1,2n+1) \\ &= \binom{2n}{n}\binom{j+k}{k}\beta(j+n+1,k+n+1) \\ &= \binom{2n}{n}\frac{(j+1)\cdots(j+n)(k+1)\cdots(k+n)}{(j+k+1)\cdots(j+k+2n+1)}, \end{aligned}$$

and

$$\|U^n\|_{\ell_p} = \frac{\Gamma(n+1/p)\Gamma(n+1/p^*)}{(\Gamma(n+1))^2}.$$

3.1 Factorization of the Hilbert operator based on Gamma matrices

In this part of our study, we state some factorizations for the Hilbert matrix based on fractional Gamma matrix. Suppose that n, m are two non-negative integers that $n \geq m$. We define the fractional Gamma matrix $\Gamma^{n/m} = (\gamma_{j,k}^{n/m})$ by

$$\gamma_{j,k}^{n/m} = \begin{cases} \frac{\binom{n+k-1}{k}}{\binom{m+j}{j}} & 0 \leq k \leq j \\ 0 & otherwise. \end{cases}$$

Obviously, for $n = m$, $\Gamma^{n/m} = \Gamma^n$.

The fractional Gamma matrix, $\Gamma^{n/m}$, is invertible and it's inverse $\Gamma^{-n/m} = (\gamma_{j,k}^{-n/m})$ is defined by

$$\gamma_{j,k}^{-n/m} = \begin{cases} \frac{\binom{m+k}{k}}{\binom{n+k-1}{k}} & j = k, \\ -\frac{\binom{m+k}{k}}{\binom{n+k}{k+1}} & j = k+1, \\ 0 & otherwise. \end{cases}$$

Lemma 3.8. *For the fractional Cesàro and Gamma matrices the following identity holds:*

$$C^{n/m} = \Gamma^{n/m} C^{n-1}.$$

In particular, for $n = m$, we have the identity $C^n = \Gamma^n C^{n-1} = C^{n-1}\Gamma^n$.

Proof. By using the identity $\sum_{k=0}^{j} \binom{n+k-1}{k} = \binom{n+j}{j}$, we have

$$\begin{aligned}(\Gamma^{n/m} C^{n-1})_{i,k} &= \sum_{j=k}^{i} \gamma_{i,j}^{n/m} c_{j,k}^{n-1} = \sum_{j=k}^{i} \frac{\binom{n+j-1}{j}}{\binom{m+i}{i}} \frac{\binom{n+j-k-2}{j-k}}{\binom{n-1+j}{j}} \\ &= \frac{1}{\binom{m+i}{i}} \sum_{j=0}^{i-k} \binom{n-1+j-1}{j} = \frac{\binom{n+i-k-1}{i-k}}{\binom{m+i}{i}} = C_{i,k}^{n/m}.\end{aligned}$$

Since every two Hausdorff matrices commute ([13], Theorem 197), we have the claimed identity for $n = m$. □

Theorem 3.9. *The Hilbert matrix has a factorization of the form $H = S^{n/m}\Gamma^{n/m}$, where the matrix $S^{n/m} = (s_{j,k}^{n/m})$ has the entries*

$$s_{j,k}^{n/m} = \frac{\binom{m+k}{k}}{\binom{n+k}{k}} \frac{(1-1/n)(j+1)+(k+1)}{(j+k+1)(j+k+2)} \qquad (j,k = 0,1,\dots),$$

is a bounded operator on ℓ_p and

$$\|S^{n/m}\|_{\ell_p} \le \pi\left(1 - \frac{1}{np}\right)\csc(\pi/p).$$

In particular, for $n = m$, $H = S^n\Gamma^n$, where Γ^n is the Gamma matrix of order n, $S^n = (s_{j,k}^n)$ has the entries

$$s_{j,k}^n = \frac{(1-1/n)(j+1)+(k+1)}{(j+k+1)(j+k+2)},$$

and is a bounded operator with ℓ_p-norm $\|S^n\|_{\ell_p} = \pi(1-\frac{1}{np})\csc(\pi/p)$.

Proof. Let $\lambda = (k+1)(k+2)\dots(k+n-1)$, we have

$$\begin{aligned}(S^{n/m}\Gamma^{n/m})_{j,k} &= \sum_{i=k}^{\infty} s_{j,i}^{n/m}\gamma_{i,k}^{n/m} = \sum_{i=k}^{\infty} \frac{\binom{m+i}{i}}{\binom{n+i}{i}} \frac{[(1-1/n)(j+1)+(i+1)]}{(j+i+1)(j+i+2)} \frac{\binom{n+k-1}{k}}{\binom{m+i}{i}} \\ &= \binom{n+k-1}{k} \sum_{i=k}^{\infty} \frac{[(n-1)(j+1)+n(i+1)]}{(j+i+1)(j+i+2)} \frac{i!(n-1)!}{(n+i)!} \\ &= \lambda \sum_{i=k}^{\infty} \frac{(n-1)(j+1)+n(i+1)}{(i+1)\cdots(i+n)(j+i+1)(j+i+2)} \\ &= \lambda \sum_{i=k}^{\infty} \frac{(i+n)(j+i+2)-(i+1)(j+i+1)}{(i+1)\cdots(i+n)(j+i+1)(j+i+2)} \\ &= \lambda \sum_{i=k}^{\infty} \left\{ \frac{1}{(i+1)\cdots(i+n-1)(j+i+1)} - \frac{1}{(i+2)\cdots(i+n)(j+i+2)} \right\} \\ &= \frac{1}{j+k+1} = h_{j,k},\end{aligned}$$

Since $n \ge m$, hence $s_{j,k}^{n/m} \le s_{j,k}^n$, where

$$s_{j,k}^n = \frac{(1-1/n)(j+1)+(k+1)}{(j+k+1)(j+k+2)}.$$

But Roopaei in [24] Theorem 2.8 has proved that S^n is a bounded operator and $\|S^n\|_{\ell_p} = \pi(1-\frac{1}{np})\csc(\pi/p)$, which completes the proof. □

Remark 3.10. *One can straightly obtain the factor $S^{n/m}$ by*

$$
\begin{aligned}
s_{i,k}^{n/m} &= \sum_{j=k,k+1} h_{i,j}\gamma_{j,k}^{-n/m} \\
&= \frac{1}{i+j+1}\frac{\binom{m+k}{k}}{\binom{n+k-1}{k}} - \frac{1}{i+j+2}\frac{\binom{m+k}{k}}{\binom{n+k}{k+1}} \\
&= \frac{\binom{m+k}{k}}{\binom{n+k}{k}}\frac{(1-1/n)(i+1)+(k+1)}{(i+k+1)(i+k+2)}.
\end{aligned}
$$

As an immediate consequence of the above theorem, we generalize inequality (1.4) again.

Corollary 3.11. *Let $p > 1$and $x \in l_p$. Then*

$$\|Hx\|_{\ell_p} \le \pi(1-1/np)\csc(\pi/p)\|\Gamma^{n/m}x\|_{\ell_p}.$$

In particular, for $n = m$ and $n = m = 1$ inequalities (1.3) and (1.4) occur.

Proof. According to the above theorem, we have

$$\|Hx\|_{\ell_p} = \|S^{n/m}\Gamma^{n/m}x\|_{\ell_p} \le \pi(1-1/np)\csc(\pi/p)\|\Gamma^{n/m}x\|_{\ell_p}.$$

□

4 Some applications

At the end of this research, as an application of the Hilbert's factorizations based on the Cesàro and Gamma matrices, we compute the norm of this operator on the matrix domains $\Gamma_p^{n/m}$ and $C_p^{n/m}$. In so doing, the following lemma seems essential.

Lemma 4.1. *Let T be a matrix such that $T = BA$ and $A_p \cong \ell_p$. If B is a bounded operator on ℓ_p, then T is a bounded operator from A_p into ℓ_p and*

$$\|T\|_{A_p,\ell_p} = \|B\|_{\ell_p}.$$

Proof. Since A_p and ℓ_p are isomorphic spaces, hence

$$
\begin{aligned}
\|T\|_{A_p,\ell_p} &= \sup_{x\in A_p}\frac{\|Tx\|_{\ell_p}}{\|x\|_{A_p}} = \sup_{x\in A_p}\frac{\|BAx\|_{\ell_p}}{\|Ax\|_{\ell_p}} \\
&= \sup_{y\in \ell_p}\frac{\|By\|_{\ell_p}}{\|y\|_{\ell_p}} = \|B\|_{\ell_p}.
\end{aligned}
$$

□

The matrix domain associated with matrix $\Gamma^{n/m}$, is the set

$$\Gamma_p^{n/m} = \left\{ x=(x_k)\in\omega : \sum_{j=0}^{\infty}\left|\frac{1}{\binom{m+j}{j}}\sum_{k=0}^{j}\binom{n+k-1}{k}x_k\right|^p < \infty \right\},$$

which is called the Gamma space of order n and has the following norm

$$\|x\|_{\Gamma_p^{n/m}} = \left(\sum_{j=0}^{\infty}\left|\frac{1}{\binom{m+j}{j}}\sum_{k=0}^{j}\binom{n+k-1}{k}x_k\right|^p\right)^{\frac{1}{p}}.$$

Theorem 4.2. *The Hilbert operator H, is a bounded operator from $\Gamma_p^{n/m}$ into ℓ_p and*

$$\|H\|_{\Gamma_p^{n/m},\ell_p} \leq \pi(1-1/np)\csc(\pi/p).$$

In particular, the Hilbert operator is a bounded operator from Γ_p^n into ℓ_p and $\|H\|_{\Gamma_p^n,\ell_p} = \pi(1-1/np)\csc(\pi/p)$.

Proof. According to Lemma 4.1 and Theorem 3.9 we have

$$\|H\|_{\Gamma_p^{n/m},\ell_p} = \|S^{n/m}\|_{\ell_p} \leq \pi(1-1/np)\csc(\pi/p).$$

□

Theorem 4.3. *The Hilbert operator H, is a bounded operator from $C_p^{n/m}$ into ℓ_p and*

$$\|H\|_{C_p^{n/m},\ell_p} \leq \frac{\Gamma(m+1/p^*)\Gamma(1/p)}{\Gamma(m+1)}.$$

In particular, the Hilbert operator H is a bounded operator from C_p^n into ℓ_p and $\|H\|_{C_p^n,\ell_p} = \frac{\Gamma(n+1/p^*)\Gamma(1/p)}{\Gamma(n+1)}$.

Proof. According to Lemma 4.1 and Theorem 3.2 we have

$$\|H\|_{C_p^n,\ell_p} = \|B^{n/m}\|_{\ell_p} \leq \frac{\Gamma(m+1/p^*)\Gamma(1/p)}{\Gamma(m+1)}.$$

□

Let H_p be the sequence space associated with H which is

$$H_p = \left\{ x=(x_k)\in\omega : \sum_{j=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{x_k}{j+k+1}\right|^p < \infty \right\},$$

and has the norm

$$\|x\|_{H_p} = \left(\sum_{j=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{x_k}{j+k+1}\right|^p\right)^{\frac{1}{p}}.$$

Corollary 4.4. *Let $n \geq m \geq 0$, H_p be the Hilbert matrix domain and $\Gamma_p^{n/m}$ and $C_p^{n/m}$ be the fractional Gamma and Cesàro matrix domains, respectively. Then*

$$\Gamma_p^{n/m} \subset H_p \quad and \quad C_p^{n/m} \subset H_p.$$

Proof. Let $x \in \Gamma_p^{n/m}$. The inequality

$$\|Hx\|_{\ell_p} = \|S^{n/m}\Gamma^{n/m}x\|_{\ell_p} \leq \pi(1-\frac{1}{np})\csc(\pi/p)\|\Gamma^{n/m}x\|_{\ell_p},$$

indicates that $x \in H_p$. Now let $x \in C_p^{n/m}$. According to the Corollary 3.5 we have

$$\|Hx\|_{\ell_p} = \|B^{n/m}C^{n/m}x\|_{\ell_p} \leq \frac{\Gamma(m+1/p^*)\Gamma(1/p)}{\Gamma(m+1)}\|C^{n/m}x\|_{\ell_p}.$$

which results in $x \in H_p$. □

Some challenging problems

1- What is the ℓ_p-norm of fractional Cesàro and Gamma matrices?

2- What is the ℓ_p-norm of matrices $B^{n/m}$ and $S^{n/m}$ who are the factors in the factorization of Hilbert matrix based on the fractional Cesàro and Gamma matrices, respectively?

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