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A Note on Meromorphic Functions Associated With Beseel Function Defined by Hilbert Sapce Operator

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Abstract: In this paper,we introduce and study a new subclass of meromorphic functions associated with a certain differential operator on Hilbert space. For this class, we obtain several properties like the coefficient inequality, growth and distortion theorem, radius of close-to-convexity, starlikeness and meromorphically convexity and integral transforms. Further, it is shown that this class is closed under convex linear combinations.

Keywords: Meromorphic functions,Bessel Functions, Coefficient estimates, Hadamard product, Hilbert space operators

MSC: 30C45; 30C50

1 Introduction

Let Σ be denote the class of function $u(z)$ of the form

$$u(z) = \frac{1}{z} + \sum_{\eta=1}^{\infty} a_\eta z^\eta, \eta \in \mathbb{N} = \{1, 2, 3, \cdots\}. \tag{1}$$

Which are analytic in the punctured unit disc$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$. Analytically a function $u \in \Sigma$ given by (1) is said to be meromarphically starlike of order α if if satisfies the following:

$$Re\left\{-\left(\frac{zu'(z)}{u(z)}\right)\right\} > \alpha, \ \ (z \in \mathbb{U})\, forsome\, \alpha\, (0 \le \alpha < 1).$$

We say that u is in the class $\Sigma^*(\alpha)$ of such functions. Similarly, a function $u \in \Sigma$ given by (1) is said to be meromorphically convex of order α if if satisfies the following:

$$Re\left\{-\left(1 + \frac{zu''(z)}{u'(z)}\right)\right\} > \alpha, \ \ (z \in \mathbb{U})\, forsome\, \alpha\, (0 \le \alpha < 1).$$

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We say that u is in the class $\Sigma_k(\alpha)$ of such functions. For a function $u \in \Sigma$ is given by (1) is said to be meromorphically close to convex of order β and α if there exists a function $v \in \Sigma^*(\alpha)$ such that

$$Re\left\{-\left(\frac{zu'(z)}{v(z)}\right)\right\} > \alpha, \ \ (0 \le \alpha < 1, \ 0 \le \beta < 1, \ z \in \mathbb{U}).$$

We say that u is in the class $K(\beta, \alpha)$. The class $\Sigma^*(\alpha)$ and various other subclasses of Σ having been studied rather extensively by Clunie [3],Miller [8],Pommerenke [11],Royster [12], Akgul [1, 2], Venkateswarlu [13], Sakar and Guney [9, 10] et al. In recent years, many authors investigated the subclass of meromorphic functions with positive coefficient Juneja and Reddy [7] class Σ_p functions of the form

$$u(z) = \frac{1}{z} + \sum_{\eta=1}^{\infty} a_\eta z^\eta, (a_\eta \ge 0). \tag{2}$$

Which are regular and univalent in $\mathbb{U}$, the function in this class are said to be meromorphic functions with positive coefficients.
For functions $u \in \Sigma_p$ given by (1) and $v \in \Sigma_p$ given by

$$v(z) = \frac{1}{z} + \sum_{\eta=1}^{\infty} b_\eta z^\eta, (b_\eta \ge 0).$$

We define the Hadamard product(or convolution) of u and v by

$$(u \star v) = \frac{1}{z} + \sum_{\eta=1}^{\infty} a_\eta b_\eta z^\eta.$$

We recall here the generalized Bessel function of the first kind of order γ(see[5]), denote by

$$w(z) = \sum_{\eta=0}^{\infty} \frac{(-c)^\eta}{\eta!\,\Gamma\left(\gamma + \eta + \frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2\eta+\gamma}, \ \ (z \in \mathbb{U}).$$

(Where Γ stands for the Gamma Euler function) which is the particular solution of the second-order linear homogeneous differential equation (see, for details, [14])

$$z^2 w''(z) + bzw'(z) + [cz^2 - \gamma^2 + (1-b)\gamma]w(z) = 0, wherec, \gamma, b \in \mathbb{C}.$$

We introdue the function ψ defined, in terms of the generalized Bessel function w by

$$\psi(z) = 2^\gamma\, \Gamma\left(\gamma + \frac{b+1}{2}\right) z^{-(1+\frac{\gamma}{2})}\, w(\sqrt{z}).$$

By using the well-known Pochhammer symbol $(x)_\mu$ defined for $x \in \mathbb{C}$ and in terms of the Euler gamma function by

$$(x)_\mu = \frac{\Gamma(x+\mu)}{\Gamma(x)} = \begin{cases} 1, & \mu = 0 \\ x(x+1)(x+2)\cdots(x+\eta-1), & \mu = \eta \in \mathbb{N}. \end{cases}$$

We obtain the following series representation for the function $\psi(z)$

$$\psi(z) = \frac{1}{z} + \sum_{\eta=0}^{\infty} \frac{(-c)^{\eta+1}}{4^{\eta+1}(\eta+1)!(\omega)_{\eta+1}} z^\eta,$$
$$here\left(\omega = \gamma + \frac{b+1}{2} \notin \mathbb{Z}_0^- = \{0, -1, -2, \cdots\}\right).$$

Corresponding to the function ψ defined the Bessel operator S_λ^c by the following Hadamard product

$$S_\lambda^c u(z) = (\psi * u)(z) = \frac{1}{z} + \sum_{\eta=0}^{\infty} \frac{\left(\frac{-c}{4}\right)^{\eta+1} a_\eta}{(\eta+1)!(\lambda)_{\eta+1}} z^\eta$$

$$S_\lambda^c u(z) = (\psi * u)(z) = \frac{1}{z} + \sum_{\eta=0}^{\infty} \phi(\eta,\lambda,c) a_\eta z^\eta \tag{3}$$

$$where\ \phi(\eta,\lambda,c) = \frac{\left(\frac{-c}{4}\right)^{\eta}}{(\eta)!(\lambda)_\eta}.$$

Let H be Hilbert space on the complex field and $L(H)$ denote the algebra of all bounded linear operators on H,for a complex valued function u analytic in a domain E of the complex plane containing the spectrum $\sigma(T)$ of the bounded linear operator T. Let $u(T)$ denote the operator on H defined by the Riesz-Dunford integral[4]

$$u(T) = \frac{1}{2\pi i}\int (zI - T)^{-1} u(z) dz.$$

Where I is the identity operator on H and $\mathbb{C}$ is positively oriented simple closed Rectifiable closed contour containing the spectrum $\sigma(T)$ in the interior domain [4]. The operator $u(T)$ can also be defined by the following series

$$u(T) = \sum_{\eta=0}^{\infty} \frac{u^\eta(0)}{\eta!} T^\eta.$$

This converges in the norm topology. The class of all functions $u \in \Sigma$, $a_\eta \geq 0$ is defined by Σ_p. The object of the present paper is to investigate the following subclass Σ_p associated with the differential operator $S_\lambda^c u(z)$.

Definition 1.1. For $0 \leq \beta < 1$ and $0 \leq \alpha < 1$, a function $u \in \Sigma_p$ given by (1) is in the class $\sigma_p(\alpha,\beta,\lambda,c,T)$ if

$$\begin{aligned}&\|T(S_\lambda^c u(T))' - \{(\beta-1)S_\lambda^c u(T) + \beta T(S_\lambda^c u(T))'\}\|\\ &<\|T(S_\lambda^c u(T))' + (1-2\alpha)\{(\beta-1)S_\lambda^c u(T) + \beta T(S_\lambda^c u(T))'\}\|.\end{aligned}$$

The main object of the paper is to study usual properties of the geometric function theory such as coefficients bounds, growth and distortion properties, a radius of convexity, convex linear combination, convolution properties, integral operators andδ-neighborhoods for the class $\sigma_p(\alpha,\beta,\lambda,c,T)$.

2 Coefficient Bounds

We first give a characterization of the class $\sigma_p(\alpha,\beta,\lambda,c,T)$ by finding necessary and sufficient condition for a functions in the class. This characterization implies coefficient estimates.

Theorem 2.1. *A function $u \in \Sigma_p$ given by (2) is in the class $\sigma_p(\alpha,\beta,\lambda,c,T)$ for all contraction T with $T \neq \theta$ if and only if*

$$\sum_{\eta=1}^{\infty} \left(\eta + \alpha - \alpha\beta(\eta+1)\right)\phi(\eta,\lambda,c) a_\eta \leq (1-\alpha). \tag{4}$$

The result is sharp for the function

$$u(z) = \frac{1}{z} + \frac{1-\alpha}{\left(\eta + \alpha - \alpha\beta(\eta+1)\right)\phi(\eta,\lambda,c)} z^\eta, \ (\eta \geq 1). \tag{5}$$

Proof. Suppose that (4) is true for $0 \leq \beta < 1$ and $0 \leq \alpha < 1$. Then

$$\begin{aligned}
&\left\|T(S_\lambda^c u(T))' - \{(\beta-1)S_\lambda^c u(T) + \beta T(S_\lambda^c u(T))'\}\right\| \\
&- \left\|T(S_\lambda^c u(T))' + (1-2\alpha)\{(\beta-1)S_\lambda^c u(T) + \beta T(S_\lambda^c u(T))'\}\right\| \\
&= \left\|\sum_{\eta=1}^{\infty}(\eta+1)(1-\beta)\phi(\eta,\lambda,c)a_{eta}T^{\eta}\right\| \left\|2(1-\alpha)T^{-1} - \sum_{\eta=1}^{\infty}[\eta+(1-2\alpha)(\beta-1+\beta\eta)]\phi(\eta,\lambda,c)a_\eta T^{\eta}\right\| \\
&\leq \sum_{\eta=1}^{\infty}(\eta+1)(1-\beta)\phi(\eta,\lambda,c)a_\eta\|T\|^{\eta} - 2(1-\alpha)\|T^{-1}\| + \sum_{\eta=1}^{\infty}[\eta+(1-2\alpha)(\beta-1+\beta\eta)]\phi(\eta,\lambda,c)a_\eta\|T^{\eta}\| \\
&= 2\sum_{\eta=1}^{\infty}[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)a_\eta\|T^{\eta}\| - 2(1-\alpha)\|T^{-1}\| \\
&\leq 2(1-\alpha) - 2(1-\alpha) = 0, \text{ by using(4)}
\end{aligned}$$

and so $u \in \Sigma_p$ is in the class $\sigma_p(\alpha,\beta,\lambda,c,T)$. Conversely suppose that $u \in \sigma_p(\alpha,\beta,\lambda,c,T)$ satisfies the coefficients inequality (4). Since $u \in \sigma_p(\alpha,\beta,\lambda,c,T)$ then

$$\|T\ (S_\lambda^c u(T))' - \{(\beta-1)S_\lambda^c u(T) + \beta T(S_\lambda^c u(T))'\}\| < \|T\quad (S_\lambda^c u(T))' + (1-2\alpha)\{(\beta-1)S_\lambda^c u(T) + \beta T(S_\lambda^c u(T))'\}\|$$

From this inequality, it is obtained that

$$\begin{aligned}
&\left\|\sum_{\eta=1}^{\infty}(\eta+1)(1-\beta)\phi(\eta,\lambda,c)a_\eta T^{\eta-1}\right\| \\
&< \left\|2(1-\alpha) - \sum_{\eta=1}^{\infty}[\eta+(1-2\alpha)(\beta-1+\beta\eta)]\phi(\eta,\lambda,c)a_\eta T^{\eta+1}\right\|.
\end{aligned}$$

By choosing $T = rI(0 < r < 1)$ in above inequality, we get

$$\frac{\sum_{\eta=1}^{\infty}(\eta+1)(1-\beta)\phi(\eta,\lambda,c)a_{eta}r^{\eta+1}}{2(1-\alpha) - \sum_{\eta=1}^{\infty}[\eta+(1-2\alpha)(\beta-1+\beta\eta)]\phi(\eta,\lambda,c)a_\eta r^{\eta+1}} < 1.$$

Letting $r \to 1$ in the above inequality, we obtain the assertion (4). This completes the proof of the theorem. □

From Theorem 4, we have the following result.

Corollary 2.2. *If a function $u(z) \in \Sigma_p$ given by (2) is in the class $\sigma_p(\alpha,\beta,\lambda,c,T)$ then*

$$a_\eta \leq \frac{(1-\alpha)}{\sum_{\eta=1}^{\infty}[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}, \quad (\eta \geq 1). \tag{6}$$

The result is sharp for the function u of the form (5).

3 Distortion Bounds

In this section we obtain growth and the distortion bounds for the class $\sigma_p(\alpha,\beta,\lambda,c,T)$.

Theorem 3.1. *If $u \in \sigma_p(\alpha,\beta,T)$ then* $0 < |z| = r < 1$,

$$\begin{aligned}
\|u(T)\| &\geq \frac{1}{\|T\|} - \frac{(1-\alpha)}{[1+\alpha-2\alpha\beta]\phi(\eta,\lambda,c)}\|T\|, \\
\|u(T)\| &\leq \frac{1}{\|T\|} + \frac{(1-\alpha)}{[1+\alpha-2\alpha\beta]\phi(1,\lambda,c)}\|T\|.
\end{aligned} \tag{7}$$

The result is sharp for

$$u(z) = \frac{1}{z} + \frac{(1-\alpha)}{\phi(1,\lambda,c)[1+\alpha-2\alpha\beta]}\, z. \tag{8}$$

Proof. Suppose $u(z)$ is in $\sigma_p(\alpha,\beta,\lambda,c,T)$. By Theorem 4, we have

$$\phi(1,\lambda,c)[1+\alpha-2\alpha\beta]\sum_{\eta=1}^{\infty} a_\eta \le \sum_{\eta=1}^{\infty}\phi(\eta,\lambda,c)[\eta+\alpha-\alpha\beta(\eta+1)]a_\eta \le (1-\alpha).$$

Therefore

$$\sum_{\eta=1}^{\infty} a_\eta \le \frac{1-\alpha}{\phi(1,\lambda,c)[1+\alpha-2\alpha\beta]}.$$

Also, if $u(T) = T^{-1} + \sum_{\eta=1}^{\infty} a_\eta T^\eta$, then

$$\frac{1}{\|T\|} - \sum_{\eta=1}^{\infty} a_\eta \|T\|^\eta \le \|u(T)\| \le \frac{1}{\|T\|} + \sum_{\eta=1}^{\infty} a_\eta \|T\|^\eta. \tag{9}$$

Since $\|T\| < 1$, the above inequality becomes

$$\frac{1}{\|T\|} - \|T\|\sum_{\eta=1}^{\infty} a_\eta \le \|u(T)\| \le \frac{1}{\|T\|} + \|T\|\sum_{\eta=1}^{\infty} a_\eta. \tag{10}$$

Using (9), we get the result. □

Theorem 3.2. *If $u(z) \in \sigma_p(\alpha,\beta,\lambda,c,T)$ then*

$$\begin{aligned} \|u'(T)\| &\ge \frac{1}{\|T\|^2} - \frac{(1-\alpha)}{\phi(1,\lambda,c)[1+\alpha-2\alpha\beta]}, \\ \|u'(T)\| &\le \frac{1}{\|T\|^2} + \frac{(1-\alpha)}{\phi(1,\lambda,c)[1+\alpha-2\alpha\beta]}. \end{aligned} \tag{11}$$

The result is sharp for

$$u(z) = \frac{1}{z} + \frac{(1-\alpha)}{\phi(1,\lambda,c)[1+\alpha-2\alpha\beta]}\, z. \tag{12}$$

4 Extreme points

In this section we obtain extreme bounds for the class $\sigma_p(\alpha,\beta,\lambda,c,T)$.

Theorem 4.1. *Let $u_0(z) = \frac{1}{z}$ and*

$$u_\eta(z) = \frac{1}{z} + \frac{(1-\alpha)}{[\eta+\alpha-\alpha\beta(\eta+1)](\eta+2)^\eta}\, z^\eta, \quad (\eta = 1, 2, \cdots). \tag{13}$$

Then $u(z) \in \sigma_p(\alpha,\beta,\lambda,c,T)$ if and only if can be expressed in the form

$$u(z) = \sum_{\eta=0}^{\infty} \zeta_\eta u_\eta(z), \quad \text{here } \zeta_\eta \ge 0,\ \sum_{\eta=0}^{\infty}\zeta_\eta = 1.$$

Proof. Assume that $u(z)=\sum\limits_{\eta=1}^{\infty}\zeta_\eta u_\eta(z)$, here $\zeta_\eta \geq 0$, $\sum\limits_{\eta=0}^{\infty}\zeta_\eta=1$. Then we have

$$\begin{aligned} u(z) &= \sum_{\eta=0}^{\infty}\zeta_\eta u_\eta(z) \\ &= \zeta_0 u_0(z)+\sum_{\eta=1}^{\infty}\zeta_\eta u_\eta(z) \\ &= \frac{1}{z}+\sum_{\eta=1}^{\infty}\frac{1-\alpha}{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}z^\eta. \end{aligned}$$

Therefore

$$\sum_{\eta=1}^{\infty}[\eta+\alpha-\alpha(\eta+1)]\phi(\eta,\lambda,c)\zeta_\eta\,\frac{1-\alpha}{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}$$

$$=(1-\alpha)\sum_{\eta=1}^{\infty}\zeta_\eta=(1-\alpha)(1-\zeta_0)\leq(1-\alpha).$$

Hence by Theorem 4, $u(z)\in\sigma_p(\alpha,\beta,\lambda,c,T)$.

Conversely, suppose that $u\in\sigma_p(\alpha,\beta,\lambda,c,T)$. Since, by Corollary 4,

$$a_\eta \leq \frac{1-\alpha}{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)},\quad (\eta\geq 1),$$

$$\text{setting}\quad \zeta_\eta=\frac{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{1-\alpha}\,a_\eta,$$

$$\text{here}\quad \eta\geq 1,\ \zeta_0=1-\sum_{\eta=1}^{\infty}\zeta_\eta.$$

We obtain $u(z)=\zeta_0u_0(z)+\sum\limits_{\eta=1}^{\infty}\zeta_\eta u_\eta(z)$.

This completes the proof of the theorem. □

Theorem 4.2. *The class $u\in\sigma_p(\alpha,\beta,\lambda,c,T)$ is closed under convex combination.*

Proof. Let the functions $u(z)=\frac{1}{z}+\sum\limits_{\eta=1}^{\infty}a_\eta z^\eta$ and $v(z)=\frac{1}{z}+\sum\limits_{\eta=1}^{\infty}b_\eta z^\eta$ be in the class $\sigma_p(\alpha,\beta,\lambda,c,T)$. Then by Theorem 4, we have

$$\sum_{\eta=1}^{\infty}[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)a_\eta\leq(1-\alpha),$$

$$\sum_{\eta=1}^{\infty}[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)b_\eta\leq(1-\alpha).$$

To $0\leq\zeta\leq 1$, define the function $h(z)$ as $h(z)=\zeta u(z)+(1-\zeta)v(z)$.

Then we get $h(z)=\frac{1}{z}+\sum\limits_{\eta=1}^{\infty}[\zeta a_\eta+(1-\zeta)b_\eta]z^\eta$, now we obtain

$$\begin{aligned}
&\sum_{\eta=1}^{\infty}[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)[\zeta a_\eta+(1-\zeta)b_\eta] \\
&=\zeta\sum_{\eta=1}^{\infty}[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)a_\eta \\
&\quad+(1-\zeta)\sum_{\eta=1}^{\infty}[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)b_\eta \\
&\leq\zeta(1-\alpha)+(1-\zeta)(1-\alpha) \\
&=(1-\alpha).
\end{aligned}$$

Therefore $h\in\sigma_p(\alpha,\beta,\lambda,c,T)$.
Hence completes the proof □

5 Radii of close-to-convexity, starlikeness and convexity

Theorem 5.1. *Let $u\in\sigma_p(\alpha,\beta,\lambda,c,T)$. Then u is meromorphically close-to-convex of order γ $(0\leq\gamma<1)$ in the disc $|z|<r_1$, where*

$$r_1=\inf_{\eta\in N}\left[\frac{(1-\gamma)[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{\eta(1-\alpha)}\right]^{\frac{1}{\eta+1}}. \tag{14}$$

The result is sharp for the extremal function given by (5).

Proof. It sufficient to show that

$$\|u'(T)T^2+1\|<(1-\gamma). \tag{15}$$

By Theorem 4, we have

$$\sum_{\eta=1}^{\infty}\frac{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{1-\alpha}a_\eta\leq 1.$$

So the inequality

$$\left\|u'(T)T^2+1\right\|=\left\|\sum_{\eta=1}^{\infty}\eta a_\eta T^{\eta+1}\right\|\leq\sum_{\eta=1}^{\infty}\eta a_\eta\|T\|^{\eta+1}<(1-\gamma)$$

holds if

$$\frac{\eta\|T\|^{\eta+1}}{1-\gamma}\leq\frac{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{1-\alpha}.$$

Then (13) holds if

$$\|T\|^{\eta+1}\leq\frac{(1-\gamma)[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{\eta(1-\alpha)},\quad(\eta\geq 1).$$

This yields the close-to-convexity of the function and completes the proof. □

Theorem 5.2. *Let $u\in\sigma_p(\alpha,\beta,\lambda,c,T)$. Then u is meromorphically starlike of order γ $(0\leq\gamma<1)$ in the disc $|z|<r_2$, where*

$$r_2=\inf_{\eta\in N}\left[\frac{(1-\gamma)[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{(\eta+2-\gamma)(1-\alpha)}\right]^{\frac{1}{\eta+1}}. \tag{16}$$

The result is sharp for the extremal function given by (5).

Proof. Let $u(T)=T^{-1}+\sum_{\eta=1}^{\infty}a_\eta z^\eta$. Since $u\in\sigma_p(\alpha,\beta,\lambda,c,T)$ is meromorphically starlike of order γ,

$$\left\|\frac{Tu'(T)}{u(T)}+1\right\|\leq(1-\delta). \tag{17}$$

Substituting for u, the above inequality becomes,

$$\sum_{\eta=1}^{\infty}\left(\frac{\eta+2-\gamma}{1-\gamma}\right)\|T\|^{\eta+1}a_{\eta}\leq 1. \tag{18}$$

By Theorem 4,

$$\sum_{\eta=1}^{\infty}\frac{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{1-\alpha}a_{\eta}\leq 1. \tag{19}$$

Then, (18) will be true if

$$\left(\frac{\eta+2-\gamma}{1-\gamma}\right)\|T\|^{\eta+1}\leq\frac{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{1-\alpha}.$$

That is

$$\|T\|\leq\left[\frac{(1-\gamma)[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{(\eta+2-\gamma)(1-\alpha)}\right]^{\frac{1}{\eta+1}}.$$

□

Theorem 5.3. *Let $u\in\sigma_p(\alpha,\beta,\lambda,c,T)$. Then u is meromorphically convex of order γ $(0\leq\gamma<1)$ in the disc $|z|<r_3$, where*

$$r_3=\inf_{\eta\in N}\left[\frac{(1-\gamma)[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{\eta(\eta+2-\gamma)(1-\alpha)}\right]^{\frac{1}{\eta+1}}.$$

The result is sharp for the extremal function given by (5).

Proof. By using the technique employed in the proof of the Theorem 5.1, we can show that

$$\left\|\frac{Tu''(T)}{u'(T)}+2\right\|<1-\gamma$$

for $|z|<r_3$ and prove that the assertion of the theorem is true. □

6 Hadamard product

Theorem 6.1. *For function $u,v\in\Sigma_p$ defined by (1) and (3) respectively, let $u,v\in\sigma_p(\alpha,\beta,\lambda,c,T)$. Then the Hadamard product $u*v\in\sigma_p(\rho,\beta,T)$, where*

$$\rho\leq 1-\frac{(1-\alpha)^2(\eta+1)(1-\beta)}{(1-\alpha)^2[1-\beta(\eta+1)]+[\eta+\alpha-\alpha\beta(\eta+1)]^2\phi(\eta,\lambda,c)}.$$

Proof. We need to find the largest ρ such that

$$\sum_{\eta=1}^{\infty}\frac{[\eta+\rho-\rho\beta(\eta+1)]\phi(\eta,\lambda,c)}{1-\rho}a_{\eta}b_{\eta}\leq 1.$$

Since $u,v\in\sigma_p(\alpha,\beta,\lambda,c,T)$, by Theorem 4, we have

$$\sum_{\eta=1}^{\infty}\frac{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{1-\alpha}a_{\eta}\leq 1 \tag{20}$$

$$\text{and}\ \sum_{\eta=1}^{\infty}\frac{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{1-\alpha}b_{\eta}\leq 1. \tag{21}$$

From (20) and (21), we find , utilizing the Cauchy-Schwartz inequality,

$$\sum_{\eta=1}^{\infty} \frac{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{1-\alpha}\sqrt{a_\eta b_\eta} \le 1. \tag{22}$$

We want only to show that

$$\begin{aligned} &\frac{[\eta+\rho-\rho\beta(\eta+1)]\phi(\eta,\lambda,c)}{1-\rho}\, a_\eta b_\eta \\ &\le \frac{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{1-\alpha}\sqrt{a_\eta b_\eta} \\ &\Rightarrow \sqrt{a_\eta b_\eta} \le \frac{(1-\rho)[\eta+\alpha-\alpha\beta(\eta+1)]}{(1-\alpha)[\eta+\rho-\rho\beta(\eta+1)]}. \end{aligned} \tag{23}$$

On the other hand, from (22), we have

$$\sqrt{a_\eta b_\eta} \le \frac{(1-\alpha)}{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}. \tag{24}$$

Therefore in view of (23) and (24), it is enough to find the largest ρ that

$$\frac{(1-\alpha)}{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)} \le \frac{(1-\rho)[\eta+\alpha-\alpha\beta(\eta+1)]}{(1-\alpha)[\eta+\rho-\rho\beta(\eta+1)]}$$

which yields

$$\begin{aligned} \rho &\le \frac{[\eta+\alpha-\alpha\beta(\eta+1)]^2\phi(\eta,\lambda,c)-n(1-\alpha)^2}{[\eta+\alpha-\alpha\beta(\eta+1)]^2\phi(\eta,\lambda,c)+(1-\alpha)^2[1-\beta(\eta+1)]} \\ \Rightarrow \rho &\le 1-\frac{(1-\alpha)^2(\eta+1)(1-\beta)}{(1-\alpha)^2[1-\beta(\eta+1)]+[\eta+\alpha-\alpha\beta(\eta+1)]^2\phi(\eta,\lambda,c)}. \end{aligned}$$

□

Theorem 6.2. *For function $u, v \in \Sigma_p$ defined by (1) and (3) respectively, let $u, v \in \sigma_p(\alpha,\beta,\lambda,c,T)$. Then the function $f(z) = \frac{1}{z} + \sum_{\eta=1}^{\infty}(a_\eta^2 + b_\eta^2)z^\eta$ is in the class $\sigma_p(\alpha,\beta,\lambda,c,T)$, where $\rho \le 1 - \frac{2(1-\alpha)^2\phi(\eta,\lambda,c)[1-\beta(\eta+1)+\eta]}{\{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)\}^2+2(1-\alpha)^2\phi(\eta,\lambda,c)[1-\beta(\eta+1)]}$.*

Proof. Since $u, v \in \sigma_p(\alpha,\beta,\lambda,c,T)$, we have

$$\sum_{\eta=1}^{\infty}\left[\frac{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{1-\alpha}a_\eta\right]^2 \le 1 \tag{25}$$

$$\text{and} \sum_{\eta=1}^{\infty}\left[\frac{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{1-\alpha}b_\eta\right]^2 \le 1. \tag{26}$$

Combining the last two inequalities, we get

$$\sum_{\eta=1}^{\infty}\left[\frac{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{1-\alpha}\right]^2(a_\eta^2+b_\eta^2) \le 1. \tag{27}$$

But we need to find the largest ρ such that

$$\sum_{\eta=1}^{\infty}\left[\frac{[\eta+\rho-\rho\beta(\eta+1)]\phi(\eta,\lambda,c)(a_\eta^2+b_\eta^2)}{1-\alpha}\right] \le 1. \tag{28}$$

The inequity (28) would hold if

$$\frac{[\eta+\rho-\rho\beta(\eta+1)]\phi(\eta,\lambda,c)}{1-\rho} \le \frac{1}{2}\left[\frac{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{1-\alpha}\right]^2.$$

Then we have

$$\rho \le \frac{\left([\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)\right)^2 - 2\eta(1-\alpha)^2\phi(\eta,\lambda,c)}{\left([\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)\right)^2 + 2(1-\alpha)^2\phi(\eta,\lambda,c)[1-\beta(\eta+1)]}$$
$$= 1 - \frac{2(1-\alpha)^2\phi(\eta,\lambda,c)[1-\beta(\eta+1)+\eta]}{\left\{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)\right\}^2 + 2(1-\alpha)^2\phi(\eta,\lambda,c)[1-\beta(\eta+1)]}$$

□

7 Integral operators

In this section, we consider integral transforms of functions in the class $\sigma_p(\alpha,\beta,\lambda,c,T)$ of the type considered by Goel and Sohi [11].

Theorem 7.1. *Let the function $u \in \Sigma_p$ given by (1) be in the class $\sigma_p(\alpha,\beta,\lambda,c,T)$. Then the integral operator*

$$U(z) = c\int_0^1 q^c u(qz)dq, \quad 0 < q \le 1, \; 0 < c < \infty \tag{29}$$

is in the class $\sigma_p(\alpha,\beta,\lambda,c,T)$, where

$$\rho = 1 - \frac{(1-\alpha)(1+2\beta)+c}{(1+\alpha-2\alpha\beta)(c+2)+(1-\alpha)(1-2\beta)c}.$$

The result is sharp for the function

$$u(z) = \frac{1}{z} + \frac{(1-\alpha)}{\phi(1,\lambda,c)[1+\alpha-2\alpha\beta]}\, z.$$

Proof. Let $u \in \Sigma_p$ given by (1) be in the class $\sigma_p(\alpha,\beta,\lambda,c,T)$. Then

$$U(z) = c\int_0^1 q^c u(qz)dq = \frac{1}{z} + \frac{c}{c+\eta+1}\, a_\eta z^\eta. \tag{30}$$

We have to show that

$$\sum_{\eta=1}^{\infty}\left[\frac{c[\eta+\rho-\rho\beta(\eta+1)]\phi(\eta,\lambda,c)}{(1-\rho)(c+\eta+1)}\right] a_\eta \le 1. \tag{31}$$

Since $u \in \sigma_p(\alpha,\beta,\lambda,c,T)$, we have

$$\sum_{\eta=1}^{\infty}\frac{[\eta+\alpha-\alpha\beta(\eta+1)]\phi(\eta,\lambda,c)}{1-\alpha}\, a_\eta \le 1.$$

The inequality (31) satisfies if

$$\frac{c\,[\eta+\rho-\rho\beta(\eta+1)]}{(1-\rho)(c+\eta+1)} \le \frac{[\eta+\alpha-\alpha\beta(\eta+1)]}{1-\alpha}.$$

Then we get

$$\rho \le \frac{[\eta+\alpha-\alpha\beta(\eta+1)](\eta+c+1)-(1-\alpha)\,c\eta}{[\eta+\alpha-\alpha\beta(\eta+1)](\eta+c+1)+c(1-\alpha)[1-\beta(\eta+1)]}$$
$$= 1 - \frac{(1-\alpha)(1+\beta(\eta+1)+\;c\,\eta)}{[\eta+\alpha-\alpha\beta(\eta+1)](\eta+c+1)+c(1-\alpha)[1-\beta(\eta+1)]}.$$

since

$$\phi(\eta) = 1 - \frac{(1-\alpha)(1+\beta(\eta+1)+\ c\,\eta)}{[\eta+\alpha-\alpha\beta(\eta+1)](\eta+c+1)+c(1-\alpha)[1-\beta(\eta+1)]}$$

is an increasing function of $\eta,\ \eta \geq 1$. We obtained the desired result. □

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