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Ashish Pathak* and Shrish Pandey

Besov-type spaces for the κ-Hankel wavelet transform on the real line

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Abstract: In this paper, we shall introduce functions spaces as subspaces of $L^p_\kappa(\mathbb{R})$ that we call Besov-κ-Hankel spaces and extend the concept of κ-Hankel wavelet transform in $L^p_\kappa(\mathbb{R})$ space. Subsequently we will characterize the Besov-κ-Hankel space by using κ-Hankel wavelet coefficients.

Keywords: Besov κ-Hankel space,Continuous κ-Hankel wavelet transform, κ-Hankel transform, κ-Hankel convolution

MSC: 33A40, 44A05, 42C40

Dedicated to the memory of Prof. R. S. Pathak

1 Introduction

Besov spaces $B^{p,q}_\alpha(\mathbb{R})$ are subspaces of $L^p(\mathbb{R})$, having functions of smoothness α and q gives a finer graduation to the smoothness. It is extension of classical Sobolev and Hölder spaces. It is also expressed as interpolation space lies in between two Sobolev spaces H^x_p and H^y_p $(1 \le p, q \le \infty)$ with $\alpha = (1-\beta)x + \beta y \quad ; \alpha, x, y \in \mathbb{R}, \beta \in (0, 1)$.
The Besov spaces $B^{p,q}_\alpha$ $(\alpha \in \mathbb{R}\ and\ 1 \le p, q \le \infty)$ were recognized in about $sixty^{th}$ decade of $ninteen^{th}$ century [3, 4]. They were generalized in mid of seventies by various authors in different directions with different ideas. The classical definition of Besov spaces depends on the modulus of smoothness [11, 13]. The Littlewood-Paley theory interlink Besov spaces with the fourier transform. Michael Frazier and Björn Jawerth [6] characterise Besov spaces with the help of Calderon's formula while Dang Vu Giang and Ferenc Moricz [7] characterise Besov spaces in terms of it's Riesz mean and Dirichlet integral. In 1996, Valerie Perrier and Claude Basdevant [10] characterise Besov spaces by the behavior of the continuous wavelet coefficients for $\alpha \in \mathbb{R}^+ \ , \ \alpha \notin \mathbb{Z}^+$.
Jorge J. Betancor and L. Rodríguez-Mesa [5] pave the way for exploration of Besov-Hankel spaces and characterized by mean of the Bochner-Riesz mean and the partial Hankel integrals. Recently Salem Ben Saïd, Mohamed Amine Boubatra, Mohamed Sifi [2] come out with deformed Besov-Hankel spaces and characterised it in terms of the deformed Bochner-Riesz means and the deformed partial Hankel integral.
Hatem Mejjaoli, khalifa Trimèche [8] presented κ-Hankel wavelet transform in the year 2020. Using the approach of Betancor et al. [5] and Ben Saïd et al. [2], we define Besov κ-Hankel space and by exploiting the technique of Perrier et al. [10] we characterise Besov κ-Hankel space with the help of continuous κ-Hankel wavelet transform.
Present paper is organized in following manner: section 1 is introductory, in which we define the development from Besov space to Besov κ-Hankel space with the help of κ-Hankel wavelet and it's characterisation

***Corresponding Author: Ashish Pathak:** Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi-221005, India, E-mail: ashishpathak@bhu.ac.in
Shrish Pandey: Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi-221005, India, E-mail: shrish20594@gmail.com

with time period. section 2 is preliminary, in which we recall some properties of κ-Hankel transform , Besov κ-Hankel space and continuous κ-Hankel wavelet transform. Section 3 is related the continuous κ-Hankel wavelet transform in $L^p_\kappa(\mathbb{R})$. In the section 4, we characterize Besov κ-Hankel norms in terms of continuous κ-Hankel wavelet transform.

2 Preliminary

In this paper, we denotes the weighted $L^p_\kappa(\mathbb{R})$ space as

$$\|f\|_{L^p_\kappa} = \|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p d\sigma_\kappa(x)\right)^{\frac{1}{p}}, (1 \le p < \infty),$$

where $d\sigma_\kappa(x) = |x|^{2\kappa-1}dx$, $\kappa \ge \frac{1}{2}$.

$$\|f\|_{L^\infty_\kappa} = \|f\|_\infty = esssup|f(x)|.$$

The κ-Hankel transformation of the function $f \in L^1_\kappa(\mathbb{R})$ for order $\kappa \ge \frac{1}{2}$ is defined as see [8]

$$(\Im_\kappa f)(\mu) = \hat{f}(\mu) := \frac{1}{\rho_\kappa}\int_{\mathbb{R}} \mathcal{B}_\kappa(\mu, x)f(x)d\sigma_\kappa(x), \quad x \in \mathbb{R},$$

where

$$\rho_\kappa = \int_{\mathbb{R}} e^{-|x|}d\sigma_\kappa(x) = 2\Gamma(2\kappa)$$

and $\mathcal{B}_\kappa(\mu, x)$ is the κ-Hankel kernel given as

$$\mathcal{B}_\kappa(\mu, x) = j_{2\kappa-1}\left(2\sqrt{|\mu x|}\right) - \frac{\mu x}{2\kappa(2\kappa+1)}j_{2\kappa+1}\left(2\sqrt{|\mu x|}\right).$$

Here

$$j_\lambda(\omega) = \Gamma(\lambda+1)\left(\frac{\omega}{2}\right)^{-\lambda}J_\lambda(\omega) = \Gamma(\lambda+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\lambda+n+1)}\left(\frac{\omega}{2}\right)^{2n}$$

denote the normalized Bessel function of index λ.
If $\hat{f} \in L^1_\kappa(\mathbb{R})$, then the inverse of κ-Hankel transformations is given by

$$f(x) := \frac{1}{\rho_\kappa}\int_{\mathbb{R}} \mathcal{B}_\kappa(\mu, x)\hat{f}(\mu)d\sigma_\kappa(\mu), \quad x \in \mathbb{R}. \tag{1}$$

Also, Parseval's formula of the κ-Hankel transformation for $f, g \in L^1_\kappa(\mathbb{R}) \bigcap L^2_\kappa(\mathbb{R})$ is given by

$$\int_{\mathbb{R}} \hat{f}(\mu)\overline{\hat{g}(\mu)}d\sigma_\kappa(\mu) = \int_{\mathbb{R}} f(x)\overline{g(x)}d\sigma_\kappa(x).$$

By denseness and continuity the Parseval's formula can be extended to all $f, g \in L^2_\kappa$. Hence $\Im_\kappa$ is isometry on $L^2_\kappa(\mathbb{R})$.
If $f, g \in L^2_\kappa(\mathbb{R})$, then the convolution associated with the κ-Hankel transform is defined as see [1]

$$(f\#_\kappa g)(x) = \frac{1}{\rho_\kappa}\int_{\mathbb{R}} f(y)\tau^\kappa_x g(y)d\sigma_\kappa(y), \tag{2}$$

where the operatore τ_x^κ is κ-Hankel translation is given by

$$f^\kappa(x,y)=\tau_x^\kappa f(y)=\int_{\mathbb{R}} f(z)\mathcal{K}_\kappa(x,y,z)d\sigma_\kappa(z)$$

and

$$\int_{\mathbb{R}} \mathcal{B}_\kappa(\mu,z)\mathcal{K}_\kappa(x,y,z)d\sigma_\kappa(z)=\mathcal{B}_\kappa(\mu,y)\mathcal{B}_\kappa(\mu,x). \tag{3}$$

From (1) and (3), we have

$$\mathcal{K}_\kappa(x,y,z)=\frac{1}{{\rho_\kappa}^2}\int_{\mathbb{R}} \mathcal{B}_\kappa(\mu,x)\mathcal{B}_\kappa(\mu,y)\mathcal{B}_\kappa(\mu,z)d\sigma_\kappa(\mu)$$

and moreover,

$$\int_{\mathbb{R}} \mathcal{K}_\kappa(x,y,z)d\sigma_\kappa(z)=1$$

$$\int_{\mathbb{R}} |\mathcal{K}_\kappa(x,y,z)|d\sigma_\kappa(z)\le M_\kappa$$

where M_κ is independent of x and y such that $M_\kappa \overset{<}{\to} 2$ as $\kappa\to\infty$ whenever $xy<0$ and $M_\kappa \overset{<}{\to} 3$ as $k\to\infty$ elsewhere.

$$(f\widehat{\#_\kappa} g)(x)=\hat{f}(x)\hat{g}(x).$$

Now, we recall some properties of κ-Hankel convolution [1] which are useful throught the paper.

Lemma 2.1. *Let $f\in L_\kappa^p(\mathbb{R})$, $1\le p\le\infty$. Then we have*

$$||\tau_x^\kappa f(y)||_p\le M_\kappa||f||_p.$$

Lemma 2.2. *Let $f\in L_\kappa^p(\mathbb{R})$ and $g\in L_\kappa^q(\mathbb{R})$, $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$. Then we have*

$$||f\#_\kappa g||_r\le M_\kappa||f||_p||g||_q.$$

Definition 2.3. **(Besov κ-Hankel Space):** Let measurable function ϕ defined on $\mathbb{R}$ belongs to $\mathcal{BH}_{\alpha,\kappa}^{p,q}$ if $\phi\in L_\kappa^p(\mathbb{R})$ and

$$\int_0^\infty \left(h^{-\alpha}w_{p,\kappa}(\phi)(h)\right)^q\frac{dh}{h}<\infty \quad \text{for },\ 1\le p,q<\infty,$$

$$esssup_{h>0}\left(h^{-\alpha}w_{p,\kappa}(\phi)(h)\right)<\infty \quad \text{for} \quad q=\infty,$$

where $w_{p,\kappa}(\phi)(h)=:\|\tau_h^\kappa\phi-\phi\|_{L_\kappa^p}$, $h\in\mathbb{R}^+$ and $0<\alpha<1$.

2.1 κ-Hankel wavelet

Using the properites of κ-Hankel transform see [8] define the κ-Hankel wavelet for $\varphi\in L_\kappa^p(\mathbb{R})$, $1\le p<\infty$, $b\in\mathbb{R}$ and $a>0$ as

$$\begin{aligned}\varphi_{b,a}^\kappa(x): &= D_a\tau_b^\kappa\varphi(x)\\ &= D_a\varphi^\kappa(x,b)\\ &= a^{-2\kappa}\varphi^\kappa(\frac{x}{a},\frac{b}{a})\\ &= a^{-2\kappa}\int_{\mathbb{R}}\varphi(z)\mathcal{K}_\kappa(\frac{b}{a},\frac{x}{a},z)d\sigma_\kappa(z)\end{aligned}$$

where D_a denote the dilation operatore such that

$$D_a\varphi(x,y) = a^{-2\kappa}\varphi(\frac{x}{a},\frac{y}{a}).$$

The continuous κ-Hankel wavelet transform of $f \in L^2_\kappa(\mathbb{R})$ with respect to a wavelet $\varphi \in L^2_\kappa(\mathbb{R})$ is defined as [8]

$$\begin{aligned}(\mathcal{H}^\kappa_\varphi f)(b,a) &:= \frac{1}{\rho_\kappa}\int_{\mathbb{R}} f(x)\overline{\varphi}^\kappa_{b,a}(x)d\sigma_\kappa(x)\\ &= \frac{a^{-2\kappa}}{\rho_\kappa}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)\overline{\varphi}(z)\mathcal{K}_\kappa\left(\frac{b}{a},\frac{x}{a},z\right)d\sigma_\kappa(z)\ d\sigma_\kappa(x).\end{aligned}$$

Moreover , using (2), we have

$$(\mathcal{H}^\kappa_\varphi f)(b,a) = (f\#_\kappa\varphi^\kappa_a)(b)$$

where $\varphi^\kappa_a(t) = a^{-2\kappa}\overline{\varphi}(t/a)$
for more about κ-Hankel wavelet see [8].

3 The Continuous κ-Hankel Wavelet Transform in $L^p_\kappa(\mathbb{R})$

In this section we extend the concept of κ-Hankel wavelet transform on $L^p_\kappa(\mathbb{R})$.

Theorem 3.1. *Suppose that a function $\psi \in L^2_\kappa(\mathbb{R})$ satisfies the admissibility condition*

$$C_{\kappa,\phi} = \int_0^\infty \omega^{-1}|\hat{\psi}(\omega)|^2 d\omega < \infty, \tag{4}$$

where $\hat{\psi}$ denote the κ-Hankel transform of ψ then continuous κ-Hankel wavelet transform is a bounded linear operator

$$L^p_\kappa(\mathbb{R}) \to L^2_\kappa(\mathbb{R}^+,\frac{d\sigma_\kappa(a)}{a^{2\kappa}})\times L^p_\kappa(\mathbb{R}),$$

moreover, for any $f \in L^p_\kappa(\mathbb{R})$, $1 < p < \infty$

$$\|f\|_{L^p_\kappa(\mathbb{R})} \simeq \left(\int_{-\infty}^{\infty}\left(\int_0^\infty |(\mathcal{H}^\kappa_\psi f)(b,a)|^2\frac{d\sigma_\kappa(a)}{a^{2\kappa}}\right)^{\frac{p}{2}} d\sigma_\kappa(b)\right)^{\frac{1}{p}}. \tag{5}$$

Proof. Let S_p denote the space $L^2_\kappa(\mathbb{R}^+,\frac{d\sigma_\kappa(a)}{a^{2\kappa}})\times L^p_\kappa(\mathbb{R})$ associated to the norm

$$\|f\|_{S_p} = \left\{\int_{-\infty}^{\infty}\left(\int_0^\infty |f(b,a)|^2\frac{d\sigma_\kappa(a)}{a^{2\kappa}}\right)^{\frac{p}{2}} d\sigma_\kappa(b)\right\}^{\frac{1}{p}}.$$

If we take $p = 2$, then from Plancherel's theorem [8]:

$$\begin{aligned}\|(\mathcal{H}^\kappa_\psi f)\|_{S_2} &= \left\{\int_{-\infty}^{\infty}\left(\int_0^\infty |(\mathcal{H}^\kappa_\psi f)(b,a)|^2\frac{d\sigma_\kappa(a)}{a^{2\kappa}}\right) d\sigma_\kappa(b)\right\}^{\frac{1}{2}}\\ &= \sqrt{C_{\kappa,\psi}}\|f\|_{L^2_\kappa(\mathbb{R})},\end{aligned}$$

where $C_{\kappa,\psi} = \int_0^\infty \omega^{-1}|\hat{\psi}(\omega)|^2 d\omega < \infty$, if ψ is real. From singular integral theorem, the operators on $L^2_\kappa(\mathbb{R}^+, \frac{d\sigma_\kappa(a)}{a^{2\kappa}})$ holds inequality:

$$\|\mathcal{H}^\kappa_\psi f\|_{S_p} \le C_p \|f\|_{L^p_\kappa(\mathbb{R})} \text{ for } 1 < p \le 2,$$

where the constant C_p depends only on p and ψ(see [12]). Due to duality the inequality is also valid for $1 < p < \infty$. It follows that

$$\left\{ \int_{-\infty}^{\infty} \left(\int_0^\infty |(\mathcal{H}^\kappa_\psi f)(b,a)|^2 \frac{d\sigma_\kappa(a)}{a^{2\kappa}} \right)^{\frac{p}{2}} d\sigma_\kappa(b) \right\}^{\frac{1}{p}} \le C_p \|f\|_{L^p_\kappa(\mathbb{R})}. \tag{6}$$

Conversely suppose that $f \in L^2_\kappa(\mathbb{R}) \cap L^p_\kappa(\mathbb{R})$. Since continuous κ-Hankel wavelet transform is isomerty for every $g \in L^2_\kappa(\mathbb{R}) \cap L^q_\kappa(\mathbb{R})$, we can write

$$\int_{-\infty}^{\infty}\int_0^\infty (\mathcal{H}^\kappa_\psi f)(b,a)\overline{(\mathcal{H}^\kappa_\psi g)(b,a)} a^{-2\kappa} d\sigma_\kappa(a) d\sigma_\kappa(b) = C_{\kappa,\psi}\langle f, g\rangle$$

$$\frac{1}{C_{\kappa,\psi}}\int_{-\infty}^{\infty}\int_0^\infty (\mathcal{H}^\kappa_\psi f)(b,a)\overline{(\mathcal{H}^\kappa_\psi g)(b,a)} a^{-2\kappa} d\sigma_\kappa(a) d\sigma_\kappa(b) = \int_{-\infty}^{\infty} f(x)\overline{g(x)} d\sigma_\kappa(x).$$

Now,

$$\begin{aligned} \left|\int_{-\infty}^{\infty} f(x)\overline{g(x)} d\sigma_\kappa(x)\right| &= \frac{1}{C_{\kappa,\psi}}\left|\int_{-\infty}^{\infty}\int_0^\infty (\mathcal{H}^\kappa_\psi f)(b,a)\overline{(\mathcal{H}^\kappa_\psi g)(b,a)} a^{-2\kappa} d\sigma_\kappa(a) d\sigma_\kappa(b)\right| \\ &\le \frac{1}{C_{\kappa,\psi}}\int_{-\infty}^{\infty}\int_0^\infty |(\mathcal{H}^\kappa_\psi f)(b,a)\overline{(\mathcal{H}^\kappa_\psi g)(b,a)}| a^{-2\kappa} d\sigma_\kappa(a) d\sigma_\kappa(b) \end{aligned}$$

using Schwarz inequality and then Holder's inequality, we have

$$\begin{aligned} &\le \frac{1}{C_{\kappa,\psi}} \left(\int_{-\infty}^{\infty} \left(\int_0^\infty |(\mathcal{H}^\kappa_\psi f)(b,a)|^2 a^{-2\kappa} d\sigma_\kappa(a) \right)^{\frac{p}{2}} d\sigma_\kappa(b) \right)^{\frac{1}{p}} \\ &\times \left(\int_{-\infty}^{\infty} \left(\int_0^\infty |(\mathcal{H}^\kappa_\psi g)(b,a)|^2 a^{-2\kappa} d\sigma_\kappa(a) \right)^{\frac{q}{2}} d\sigma_\kappa(b) \right)^{\frac{1}{q}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.
From equation (6), we get

$$\le \frac{C_q}{C_{\kappa,\psi}} \left(\int_{-\infty}^{\infty} \left(\int_0^\infty |(\mathcal{H}^\kappa_\psi f)(b,a)|^2 a^{-2\kappa} d\sigma_\kappa(a) \right)^{\frac{p}{2}} d\sigma_\kappa(b) \right)^{\frac{1}{p}} \|g\|_{L^q_\kappa(\mathbb{R})}.$$

By Density theorem

$$\|f\|_{L^p_\kappa(\mathbb{R})} \le A \left(\int_{-\infty}^{\infty} \left(\int_0^\infty |(\mathcal{H}^\kappa_\psi f)(b,a)|^2 a^{-2\kappa} d\sigma_\kappa(a) \right)^{\frac{p}{2}} d\sigma_\kappa(b) \right)^{\frac{1}{p}},$$

where $A = \frac{C_q}{C_{\kappa,\psi}}$. □

Theorem 3.2. ***(Parseval's formula)*** *Let us assume $\phi_1 \in L^p_\kappa(\mathbb{R})$, $\phi_2 \in L^q_\kappa(\mathbb{R})$ with $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If ψ is a real wavelet then*

$$\frac{1}{C_{\kappa,\psi}} \int_{-\infty}^{\infty} \int_{0}^{\infty} (\mathcal{H}^\kappa_\psi \phi_1)(b,a)\overline{(\mathcal{H}^\kappa_\psi \phi_2)(b,a)} a^{-2\kappa} d\sigma_\kappa(a) d\sigma_\kappa(b) = \int_{-\infty}^{\infty} \phi_1(x)\overline{\phi_2(x)} d\sigma_\kappa(x),$$

where $C_{\kappa,\psi} = \int_0^\infty \omega^{-1}|\hat{\psi}(\omega)|^2 d\omega < \infty$ and $\hat{\psi}$ denote the κ-Hankel transform.

Proof. Let us define bilinear transform $T : L^p_\kappa(\mathbb{R}) \times L^q_\kappa(\mathbb{R}) \to \mathbb{R}$ by

$$T(\phi_1, \phi_2) = \langle (\mathcal{H}^\kappa_\psi \phi_1)(b,a), (\mathcal{H}^\kappa_\psi \phi_2)(b,a) \rangle_{\left(\frac{d\sigma_\kappa(a)}{a^{2\kappa}} d\sigma_\kappa(b)\right)}.$$

Now, applying Hölder's inequality two times we obtain

$$\begin{aligned} |T(\phi_1,\phi_2)| &= |\langle (\mathcal{H}^\kappa_\psi \phi_1)(b,a), (\mathcal{H}^\kappa_\psi \phi_2)(b,a) \rangle_{\frac{d\sigma_\kappa(a)}{a^{2\kappa}} d\sigma_\kappa(b)}| \\ &\le \int_{-\infty}^{\infty} \left(\int_0^\infty |(\mathcal{H}^\kappa_\psi \phi_1)(b,a)|^2 \frac{d\sigma_\kappa(a)}{a^{2\kappa}} \right)^{\frac{1}{2}} \left(\int_0^\infty |(\mathcal{H}^\kappa_\psi \phi_2)(b,a)|^2 \frac{d\sigma_\kappa(a)}{a^{2\kappa}} \right)^{\frac{1}{2}} d\sigma_\kappa(b) \\ &\le \left(\int_{-\infty}^{\infty} \left(\int_0^\infty |(\mathcal{H}^\kappa_\psi \phi_1)(b,a)|^2 \frac{d\sigma_\kappa(a)}{a^{2\kappa}} \right)^{\frac{p}{2}} d\sigma_\kappa(b) \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_{-\infty}^{\infty} \left(\int_0^\infty |(\mathcal{H}^\kappa_\psi \phi_2)(b,a)|^2 \frac{d\sigma_\kappa(a)}{a^{2\kappa}} \right)^{\frac{q}{2}} d\sigma_\kappa(b) \right)^{\frac{1}{q}} \end{aligned}$$

using Theorem 3.1. we have

$$|T(\phi_1,\phi_2)| \le A \|\phi_1\|_{L^p_\kappa(\mathbb{R})} \|\phi_2\|_{L^q_\kappa(\mathbb{R})}. \tag{7}$$

Moreover for all $\phi_1 \in L^2_\kappa(\mathbb{R}) \cap L^p_\kappa(\mathbb{R})$ and $\phi_2 \in L^2_\kappa(\mathbb{R}) \cap L^q_\kappa(\mathbb{R})$ we get

$$T(\phi_1,\phi_2) = \langle (\mathcal{H}^\kappa_\psi \phi_1)(b,a), (\mathcal{H}^\kappa_\psi \phi_2)(b,a) \rangle_{\frac{d\sigma_\kappa(a)}{a^{2\kappa}}, d\sigma_\kappa(b)} = C_{\kappa,\psi} \langle \phi_1, \phi_2 \rangle. \tag{8}$$

From equations (7), (8) and density of spaces $L^2_\kappa(\mathbb{R}) \cap L^p_\kappa(\mathbb{R})$ in $L^p_\kappa(\mathbb{R})$ gives the result. □

3.1 An inversion formula

Theorem 3.3. *Let us consider $\phi \in L^p_\kappa(\mathbb{R})$ with $1 < p < \infty$ and ψ is a real wavelet. Then*

$$\phi(x) = \frac{1}{C_{\kappa,\psi}\rho_\kappa} \int_{-\infty}^{\infty} \int_0^\infty (\mathcal{H}^\kappa_\psi \phi)(b,a) \psi^\kappa_{b,a}(x) \frac{d\sigma_\kappa(a)}{a^{2\kappa}} d\sigma_\kappa(b).$$

The equality holds in $L^p_\kappa(\mathbb{R})$ sense and the integral of right hand side have to be taken in the sense of distributions.

Proof. The proof followed from theorem 3.2. □

4 Characterization of Besov κ-Hankel Norms

In present section, By using the above results, we characterize the Besov κ-Hankel norms associated with the κ-Hankel wavelet transform.

Theorem 4.1. *Let $f \in \mathcal{BH}^{p,q}_{\alpha,\kappa}(\mathbb{R})\,(p,q \geq 1, 0<\alpha<1)$ and ψ, $z^\alpha\psi \in L^1_\kappa(\mathbb{R})$, then the wavelet transform of function f holds following conditions:*

$$\text{if } q<\infty, \quad \int_0^\infty \left[a^{-\alpha}\|(\mathcal{H}^\kappa_\psi f)(.,a)\|_{L^p_\kappa}\right]^q \frac{da}{a} < \infty$$
$$\text{if } q=\infty, \quad a \to a^{-\alpha}\|(\mathcal{H}^\kappa_\psi f)(.,a)\|_{L^p_\kappa} \in L^\infty_\kappa(\mathbb{R}^+).$$

Moreover the function $a \to a^{-\alpha}\|(\mathcal{H}^\kappa_\psi f)(.,a)\|_{L^p_\kappa} \in L^q_\kappa(\mathbb{R}^+, \frac{da}{a})$.

Proof. By the definition of continuous κ-Hankel wavelet transform and equation (4), we have

$$\begin{aligned}
(\mathcal{H}^\kappa_\psi f)(b,a) &= \frac{1}{\rho_\kappa}\int_{-\infty}^{\infty} f(x)\overline{\psi^\kappa_{b,a}(x)}d\sigma_\kappa(x) \\
&= \frac{1}{\rho_\kappa}\int_{-\infty}^{\infty} f(x)\left(\int_{-\infty}^{\infty} a^{-2\kappa}\mathcal{K}_\kappa\left(\frac{b}{a},\frac{x}{a},z\right)\overline{\psi(z)}d\sigma_\kappa(z)\right)d\sigma_\kappa(x) \\
&= \frac{1}{\rho_\kappa}\int_{-\infty}^{\infty} \overline{\psi(z)}\left(\int_{-\infty}^{\infty} a^{-2\kappa}\mathcal{K}_\kappa\left(\frac{b}{a},\frac{x}{a},z\right)f(x)d\sigma_\kappa(x)\right)d\sigma_\kappa(z) \\
&= \frac{1}{\rho_\kappa}\int_{-\infty}^{\infty} \overline{\psi(z)}\left(\int_{-\infty}^{\infty} \mathcal{K}_\kappa(b,x,az)f(x)d\sigma_\kappa(x)\right)d\sigma_\kappa(z) \\
&= \frac{1}{\rho_\kappa}\left\{\int_{-\infty}^{\infty}(\tau^\kappa_{az}f)(b)\overline{\psi(z)}d\sigma_\kappa(z) - \int_{-\infty}^{\infty} f(b)\overline{\psi(z)}d\sigma_\kappa(z)\right\} \\
&= \frac{1}{\rho_\kappa}\int_{-\infty}^{\infty}\overline{\psi(z)}\left((\tau^\kappa_{az}f)(b)-f(b)\right)d\sigma_\kappa(z).
\end{aligned}$$

Taking L^p_κ– norm of the wavelet coefficient

$$\|(\mathcal{H}^\kappa_\psi f)(b,a)\|_{L^p_\kappa} = \frac{1}{\rho_\kappa}\left\{\int_{-\infty}^{\infty}\left\{|\int_{-\infty}^{\infty}\overline{\psi(z)}\left((\tau^\kappa_{az}f)(b)-f(b)\right)d\sigma_\kappa(z)|^p\right\}d\sigma_\kappa(b)\right\}^{\frac{1}{p}}.$$

Using Minkowski inequality of integrability for $p \neq \infty$

$$\|(\mathcal{H}^\kappa_\psi f)(b,a)\|_{L^p_\kappa} \leq \frac{1}{\rho_\kappa}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}|(\tau^\kappa_{az}f(b)-f(b)|^p d\sigma_\kappa(b)\right\}^{\frac{1}{p}}|\psi(z)|d\sigma_\kappa(z).$$

Suppose that $q<\infty$ and integrating w.r.t. a, we get

$$\int_0^\infty\left[a^{-\alpha}\|\mathcal{H}^\kappa_\psi f(b,a)\|_{L^p_\kappa}\right]^q\frac{da}{a} \leq \frac{1}{\rho_\kappa}\int_0^\infty\left[a^{-\alpha}\int_{-\infty}^{\infty}|\psi(z)|\omega_{p,\kappa}(f,az)d\sigma_\kappa(z)\right]^q\frac{da}{a}.$$

Again using Minkowski integrabilty inequality

$$\int_0^\infty\left[a^{-\alpha}\|\mathcal{H}^\kappa_\psi f(b,a)\|_{L^p_\kappa}\right]^q\frac{da}{a} \leq \frac{1}{\rho_\kappa}\left[\int_{-\infty}^{\infty}|\psi(z)|d\sigma_\kappa(z)\left\{\int_0^\infty\left(a^{-\alpha}\omega_{p,\kappa}(f,az)\right)^q\frac{da}{a}\right\}^{\frac{1}{q}}\right]^q.$$

Applying change of variable $h = az$

$$\begin{aligned}
&= \frac{1}{\rho_\kappa}\left[\int_{-\infty}^{\infty} z^\alpha|\psi(z)|d\sigma_\kappa(z)\left\{\int_0^\infty \left(h^{-\alpha}\omega_{p,\kappa}(f,h)\right)^q \frac{dh}{h}\right\}^{\frac{1}{q}}\right]^q \\
&\leq \frac{1}{\rho_\kappa}\left\{\int_{-\infty}^{\infty} z^\alpha|\psi(z)|d\sigma_\kappa(z)\right\}^q \times \left\{\int_0^\infty \left(h^{-\alpha}\omega_{p,\kappa}(f,h)\right)^q \frac{dh}{h}\right\} \\
&= \frac{1}{\rho_\kappa}\left\{\int_{-\infty}^{\infty} |z^\alpha\psi(z)|d\sigma_\kappa(z)\right\}^q \times \left\{\int_0^\infty \left(h^{-\alpha}\omega_{p,\kappa}(f,h)\right)^q \frac{dh}{h}\right\} \\
&< \infty.
\end{aligned}$$

If $q = \infty$ the hypothesis on f says that $h^{-\alpha}\omega_{p,\kappa}(f,h) \in L^\infty_\kappa(\mathbb{R}^+)$, so

$$\begin{aligned}
\|(\mathcal{H}^\kappa_\psi f)(b,a)\|_{L^p_\kappa} &\leq \frac{a^\alpha}{\rho_\kappa}\|h^{-\alpha}\omega_{p,\kappa}(f,h)\|_{L^\infty_\kappa(\mathbb{R}^+)} \times \int_{-\infty}^{\infty} |z^\alpha\psi(z)|d\sigma_\kappa(z) \\
&\leq \frac{a^\alpha}{\rho_\kappa}\|h^{-\alpha}\omega_{p,\kappa}(f,h)\|_{L^\infty_\kappa(\mathbb{R}^+)} \times \|z^\alpha\psi\|_{L^1_\kappa(\mathbb{R})}.
\end{aligned}$$

□

Next theorem is the converse of the above theorem. The κ-Hankel wavelet coefficients is sufficient to characterize Besov κ-Hankel spaces.

Theorem 4.2. *Suppose $0 < \alpha < 1$, and a function ψ is a real C^1- regular wavelet with first derivative rapidly decreasing. If $f \in L^p_\kappa(\mathbb{R})$ $(1 < p < \infty)$, and if $a^{-\alpha}\|(\mathcal{H}^\kappa_\psi f)(a,\cdot)\|_{L^p_\kappa} \in L^q_\kappa(\mathbb{R}^+, \frac{da}{a})$, $(1 \leq q \leq \infty)$, then $f \in \mathcal{BH}^{p,q}_{\alpha,\kappa}$ and we have*

$$\begin{aligned}
\|h^{-\alpha}w_{p,\kappa}(f,h)\|_{L^q_\kappa(\frac{dh}{h})} &\leq \frac{M_\kappa}{C_{\kappa,\psi}\rho_\kappa}\left(\frac{M_\kappa+1}{\alpha}\|\psi\|_{L^1_\kappa} + \frac{M_\kappa}{(1-\alpha)}\|\psi'\|_{L^1_\kappa}\right) \\
&\quad \times\|a^{-\alpha}\|(\mathcal{H}^\kappa_\psi f)(a,\cdot)\|_{L^p_\kappa}\|_{L^q_\kappa(\frac{da}{a})}
\end{aligned}$$

Proof. Let $f \in L^p_\kappa(\mathbb{R})$. By inversion formula of κ-Hankel wavelet transform

$$f(x) = \frac{1}{C_{\kappa,\psi}\rho_\kappa}\int_0^\infty \frac{d\sigma_\kappa(a)}{a^{2\kappa}}\int_{-\infty}^{\infty}(\mathcal{H}^\kappa_\psi f)(a,b)\psi^\kappa_{a,b}(x)d\sigma_\kappa(b)$$

and

$$\tau^\kappa_h f(x) = \frac{1}{C_{\kappa,\psi}\rho_\kappa}\int_0^\infty \frac{d\sigma_\kappa(a)}{a^{2\kappa}}\int_{-\infty}^{\infty}(\mathcal{H}^\kappa_\psi f)(a,b)\tau^\kappa_h\psi^\kappa_{a,b}(x)d\sigma_\kappa(b).$$

Then

$$\begin{aligned}
\tau_h^\kappa f(x) - f(x) &= \frac{1}{C_{\kappa,\psi}\rho_\kappa}\int_0^\infty \frac{d\sigma_\kappa(a)}{a^{2\kappa}}\int_{-\infty}^\infty (\mathcal{H}_\psi^\kappa f)(a,b)\left\{\tau_h^\kappa \psi_{a,b}^\kappa(x) - \psi_{a,b}^\kappa(x)\right\} d\sigma_\kappa(b)\\
&= \frac{1}{C_{\kappa,\psi}\rho_\kappa}\int_0^\infty \frac{d\sigma_\kappa(a)}{a^{2\kappa}}\int_{-\infty}^\infty (\mathcal{H}_\psi^\kappa f)(a,b)a^{-2\kappa}\left\{\tau_{\frac{h}{a}}^\kappa \tau_{\frac{b}{a}}^\kappa \psi(\frac{x}{a}) - \tau_{\frac{b}{a}}^\kappa \psi(\frac{x}{a})\right\} d\sigma_\kappa(b)\\
&= \frac{1}{C_{\kappa,\psi}\rho_\kappa}\int_0^\infty \frac{d\sigma_\kappa(a)}{a^{2\kappa}}\int_{-\infty}^\infty (\mathcal{H}_\psi^\kappa f)(a,b)a^{-2\kappa}\mathcal{K}_\kappa(\frac{b}{a},\frac{x}{a},y)d\sigma_\kappa(b)\\
&\quad\times\int_{-\infty}^\infty \left\{\tau_{\frac{h}{a}}^\kappa \psi(y) - \psi(y)\right\} d\sigma_\kappa(y)\\
&= \frac{1}{C_{\kappa,\psi}\rho_\kappa}\int_0^\infty \frac{d\sigma_\kappa(a)}{a^{2\kappa}}\tau_{ay}^\kappa(\mathcal{H}_\psi^\kappa f)(a,x)\int_{-\infty}^\infty \left\{\tau_{\frac{h}{a}}^\kappa \psi(y) - \psi(y)\right\} d\sigma_\kappa(y).
\end{aligned}$$

Taking L_κ^p- norm on both side and applying Minkowski's inequality , we have

$$\begin{aligned}
w_{p,\kappa}(f,h) &= \frac{1}{C_{\kappa,\psi}\rho_\kappa}\left\{\int_{-\infty}^\infty \left|\int_0^\infty \frac{d\sigma_\kappa(a)}{a^{2\kappa}}\tau_{ay}^\kappa(\mathcal{H}_\psi^\kappa f)(a,x)\int_{-\infty}^\infty \left\{\tau_{\frac{h}{a}}^\kappa \psi(y) - \psi(y)\right\} d\sigma_\kappa(y)\right|^p d\sigma_\kappa(x)\right\}^{\frac{1}{p}}\\
&\le \frac{1}{C_{\kappa,\psi}\rho_\kappa}\int_0^\infty \left|\frac{d\sigma_\kappa(a)}{a^{2\kappa}}\right|\int_{-\infty}^\infty |\tau_{\frac{h}{a}}^\kappa \psi(y) - \psi(y)| d\sigma_\kappa(y)\left\{\int_{-\infty}^\infty |\tau_{ay}^\kappa(\mathcal{H}_\psi^\kappa f)(a,x)|^p d\sigma_\kappa(x)\right\}^{\frac{1}{p}}\\
&\le \frac{M_\kappa}{C_{\kappa,\psi}\rho_\kappa}\int_0^\infty \frac{d\sigma_\kappa(t)}{t^{2\kappa}}\|(\mathcal{H}_\psi^\kappa f)(\frac{h}{t},\cdot)\|_{L_\kappa^p}\int_{-\infty}^\infty |\tau_t^\kappa \psi(y) - \psi(y)| d\sigma_\kappa(y).
\end{aligned}$$

Now

$$\begin{aligned}
\left\{\int_0^\infty \frac{dh}{h}h^{-\alpha q}w_{p,\kappa}(f,h)^q\right\}^{\frac{1}{q}} &\le \frac{M_\kappa}{C_{\kappa,\psi}\rho_\kappa}\int_0^\infty \frac{d\sigma_\kappa(t)}{t^{2\kappa}}\int_{-\infty}^\infty |\tau_t^\kappa \psi(y) - \psi(y)| d\sigma_\kappa(y)\\
&\quad\times\left\{\int_0^\infty \frac{dh}{h}h^{-\alpha q}\|(\mathcal{H}_\psi^\kappa f)(\frac{h}{t},\cdot)\|_{L_\kappa^p}^q\right\}^{\frac{1}{q}}\\
&= \frac{M_\kappa}{C_{\kappa,\psi}\rho_\kappa}\int_0^\infty \frac{d\sigma_\kappa(t)}{t^{2\kappa+\alpha}}\int_{-\infty}^\infty |\tau_t^\kappa \psi(y) - \psi(y)| d\sigma_\kappa(y)\\
&\quad\times\left\{\int_0^\infty \frac{da}{a}a^{-\alpha q}\|(\mathcal{H}_\psi^\kappa f)(a,\cdot)\|_{L_\kappa^p}^q\right\}^{\frac{1}{q}}\\
&= \frac{M_\kappa}{C_{\kappa,\psi}\rho_\kappa}\int_0^\infty \frac{dt}{t^{1+\alpha}}\int_{-\infty}^\infty |\tau_t^\kappa \psi(y) - \psi(y)| d\sigma_\kappa(y)\\
&\quad\times\left\{\int_0^\infty \frac{da}{a}a^{-\alpha q}\|(\mathcal{H}_\psi^\kappa f)(a,\cdot)\|_{L_\kappa^p}^q\right\}^{\frac{1}{q}}
\end{aligned}$$

$$= \frac{M_\kappa}{C_{\kappa,\psi}\rho_\kappa} \int_1^\infty \frac{dt}{t^{1+\alpha}} \int_{-\infty}^\infty |\tau_t^\kappa \psi(y) - \psi(y)| d\sigma_\kappa(y)$$
$$\times \left\{ \int_0^\infty \frac{da}{a} a^{-\alpha q} \|(\mathcal{H}_\psi^\kappa f)(a,\cdot)\|_{L_\kappa^p}^q \right\}^{\frac{1}{q}} + \frac{M_\kappa}{C_{\kappa,\psi}\rho_\kappa} \int_0^1 \frac{dt}{t^{1+\alpha}}$$
$$\times \int_{-\infty}^\infty |\tau_t^\kappa \psi(y) - \psi(y)| d\sigma_\kappa(y) \left\{ \int_0^\infty \frac{da}{a} a^{-\alpha q} \|(\mathcal{H}_\psi^\kappa f)(a,\cdot)\|_{L_\kappa^p}^q \right\}^{\frac{1}{q}}.$$

Using Lemma 2.1, we obtain

$$\left\{ \int_0^\infty \frac{dh}{h} h^{-\alpha q} w_{p,\kappa}(f,h)^q \right\}^{\frac{1}{q}} \le \frac{M_\kappa}{C_{\kappa,\psi}\rho_\kappa} \left((M_\kappa + 1)\|\psi\|_{L_\kappa^1} \int_1^\infty \frac{dt}{t^{1+\alpha}} + M_\kappa \|\psi'\|_{L_\kappa^1} \int_0^1 \frac{dt}{t^\alpha} \right)$$
$$\times \|a^{-\alpha} \|(\mathcal{H}_\psi^\kappa f)(a,\cdot)\|_{L_\kappa^p}\|_{L_\kappa^q(\frac{da}{a})}$$
$$= \frac{M_\kappa}{C_{\kappa,\psi}\rho_\kappa} \left(\frac{M_\kappa + 1}{\alpha} \|\psi\|_{L_\kappa^1} + \frac{M_\kappa}{(1-\alpha)} \|\psi'\|_{L_\kappa^1} \right)$$
$$\times \|a^{-\alpha} \|(\mathcal{H}_\psi^\kappa f)(a,\cdot)\|_{L_\kappa^p}\|_{L_\kappa^q(\frac{da}{a})}$$

□

Corollary 4.3. *Let $f \in \mathcal{BH}_{\alpha,\kappa}^{p,q}(\mathbb{R})$ $(p, q > 1, 0 < \alpha < 1)$, then*

$$||f||_{\mathcal{BH}_{\alpha,\kappa}^{p,q}} = ||f||_{L_\kappa^p(\mathbb{R})} + |f|_{\mathcal{BH}_{\alpha,\kappa}^{p,q}}$$

where $|f|_{\mathcal{BH}_{\alpha,\kappa}^{p,q}}$ is equal to

$$|f|_{\mathcal{BH}_{\alpha,\kappa}^{p,q}}^q = \int_0^\infty \left(h^{-\alpha} w_{p,\kappa}(\phi)(h)\right)^q \frac{dh}{h} \approx \int_0^\infty \left[a^{-\alpha} \|\mathcal{H}_\psi^\kappa(.,a)\|_{L_\kappa^p}\right]^q \frac{da}{a}.$$

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