DE GRUYTER Concr. Oper. 2021; 8:135–149

Research Article Open Access

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Cesàro and Abel ergodic theorems for integrated semigroups

https://doi.org/10.1515/conop-2020-0119
Received February 15, 2021; accepted September 7, 2021

Abstract: Let $\{S(t)\}_{t\geq 0}$ be an integrated semigroup of bounded linear operators on the Banach space $\mathcal{X}$ into itself and let A be their generator. In this paper, we consider some necessary and sufficient conditions for the Cesàro mean and the Abel average of $S(t)$ converge uniformly on $\mathcal{B}(\mathcal{X})$. More precisely, we show that the Abel average of $S(t)$ converges uniformly if and only if $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$, if and only if $\mathcal{R}(A^k)$ is closed for some integer k and $\|\lambda^2 R(\lambda, A)\| \longrightarrow 0$ as $\lambda \to 0^+$, where $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $R(\lambda, A)$, be the range, the kernel, the resolvent function of A, respectively. Furthermore, we prove that if $S(t)/t^2 \longrightarrow 0$ as $t \to \infty$, then the Cesàro mean of $S(t)$ converges uniformly if and only if the Abel average of $S(t)$ is also converges uniformly.

Keywords: Cesàro means, Abel averages, Integrated semigroups, Uniform Abel ergodic, Uniform Cesàro ergodic.

MSC: 47D62

1 Introduction

Throughout this paper $\mathcal{B}(\mathcal{X})$ denotes the Banach algebra of all bounded linear operators on Banach space $\mathcal{X}$ into itself. Let A be a closed linear operator on $\mathcal{X}$ with domain $D(A) \subset \mathcal{X}$, we denote by $\mathcal{N}(A)$, $\mathcal{R}(A)$, $\sigma(A)$, $\rho(A)$ and $R(., A)$, the kernel, the range, the spectrum, the resolvent set and the resolvent operator of A, respectively.

The family $\{T(t)\}_{t\geq 0}$ of bounded linear operator on $\mathcal{X}$ is called a strongly continuous semigroup (C_0-semigroup in short [2]) if it has the following properties:

1. $T(0) = I$,
2. $T(t)T(s) = T(t + s)$,
3. The map $t \to T(t)x$ from $[0, +\infty[$ into $\mathcal{X}$ is continuous for all $x \in \mathcal{X}$.

Their infinitesimal generator A is defined by:

$$Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t} \quad \text{for all } x \in D(A),$$

where $$D(A) = \{x \in \mathcal{X} : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \; exists\}.$$
The Laplace transformation $R(\lambda)$ of a C_0-semigroup $T(t)$ on $\mathcal{B}(\mathcal{X})$, defined as

$$R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x dt,$$

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which is exactly the resolvent function of A. Moreover, the infinitesimal generator of a C_0-semigroup is a linear closed densely defined operator on a Banach space $\mathcal{X}$, see for instance [12] and [3].

Integrated semigroups and n-time Integrated semigroups, $n \in \mathbb{N}$, of operators in Banach space were introduced by Arendt [1] and studied by Arendt, Kellermann, Hieber [7], Thieme [17] and many others.
A relevant example is obtained if we assume that $\{T(t)\}_{t\geqslant 0}$ is a C_0-semigroup of bounded linear operator on $\mathcal{X}$, then $S(t) = \int_0^t T(r)dr$ defines an integrated semigroup $\{S(t)\}_{t\geqslant 0}$ having the following three properties:

1. $S(0) = 0,$
2. $S(s)S(t) = \int_0^s S(r+t) - S(r)dr$ for $t, s \geq 0,$
3. The map $t \to S(t)$ from $[0, +\infty[$ into $\mathcal{X}$ is strongly continuous.

Arendt [1] showed that certain natural classes of operators, such as adjoint semigroups of C_0-semigroups on non-reflexive Banach spaces, give rise to integrated semigroups which are not integrals of C_0-semigroups. In contrast to C_0-semigroups, integrated semigroups may be not exponentially bounded, may be locally defined, and their generators may be not densely defined.

Ergodic theorems have a long tradition and are usually formulated in semigroup theory via the existence of limits of the Cesàro means, defined by:

$$\mathcal{C}(t) = t^{-1} \int_0^t T(s)ds, \quad \text{for } t \geq 0, \tag{1.1}$$

where $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(\mathcal{X})$ is a C_0-semigroup. In this case, $\{T(t)\}_{t\geq 0}$ is said to be uniformly Cesàro ergodic if the Cesàro means of $T(t)$ converges uniformly in $\mathcal{B}(\mathcal{X})$, as $t \to \infty$. This notion is completely connected to study the limit of the Abel averages of a C_0-semigroup $\{T(t)\}_{t\geq 0}$, defined as:

$$\lambda R(\lambda, A) = \lambda \int_0^\infty e^{-\lambda t} T(t)x dt. \tag{1.2}$$

where A is the generator of $\{T(t)\}_{t\geq 0}$ and $R(\lambda, A)$ is the resolvent function of A.
Recall that, a C_0-semigroup $T(t)$ is called uniformly Abel ergodic if the limit, as $\lambda \to 0^+$, if their Abel averages exists in the norm operator topology.
We will denote the growth bound of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ by

$$\omega_0 = \inf\left\{w \in \mathbb{R} : \text{ there exists } M \text{ such that } \|T(t)\| \leq Me^{wt},\ t \geq 0\right\}.$$

Usually one assumes $\omega_0 \leq 0$ or the even stronger condition $\|T(t)\|/t \longrightarrow 0$ as $t \to \infty$, to study the convergence of the Cesàro means and the Abel averages of $\{T(t)\}_{t\geq 0}$. Generally, great attention has been focused on the study the relationship between Cesàro ergodicity and Abel ergodicity for different classes of semigroups in $\mathcal{B}(\mathcal{X})$. The result of Hille and Phillips in [6, Theorem 18.8.4] deals with the uniform Abel ergodicity of semigroups of class (A), a class slightly larger than C_0-semigroups, under the assumption $\omega_0 \leq 0$. More precisely, they have shown that $T(t)$ is uniformly Abel ergodic if and only if $\lambda^2 R(\lambda, A)x \longrightarrow 0$ as $\lambda \to 0^+$ for every $x \in \mathcal{X}$ and $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$. Furthermore, if $T(t)$ is uniformly Abel ergodic, then $\mathcal{R}(A^m) = \mathcal{R}(A)$, for all $m \in \mathbb{N}^\star$. A relevant result obtained by S.Y. Shaw in [13] for a locally integrated semigroup, under an assumption weaker than $\omega_0 \leq 0$, that means $T(t)$ is uniformly Cesàro ergodic if and only if it satisfies the following conditions:

(i) The Laplace transformation R_λ exists for every $\lambda > 0$,
(ii) $\|T(t)R_\lambda\|/t \longrightarrow 0$ as $t \to \infty$, for some $\lambda > 0$, and
(iii) $T(t)$ is uniformly Abel ergodic.

The condition (i) holds whenever $\omega_0 \leq 0$, let us also mention that somewhat different necessary and sufficient conditions are obtained in [14]. Clearly, if $T(t)$ is uniformly Cesàro ergodic then is uniformly Abel ergodic, but the reverse is not true, for more information see [9, Chapter 2]. It is useful to mention that the limit of Cesàro averages and of Abel averages of C_0-semigroup $\{T(t)\}_{t\geqslant 0}$ is the same, is the projection P of $\mathcal{X}$ onto $\mathcal{N}(A)$ parallel to $\mathcal{R}(A)$, corresponding to the ergodic decomposition

$$\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A). \tag{1.3}$$

The classical uniform ergodic theorem for C_0-semigroups $\{T(t)\}_{t\geq 0}$ of bounded linear operators on $\mathcal{X}$, goes back to M. Lin in [10], he treats the Cesàro ergodicity of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ under the assumption $\lim\limits_{t\to\infty} \|T(t)\|/t = 0$, it showed that $T(t)$ is uniformly Cesàro ergodic if and only if its infinitesimal generator A has a closed range if and only if $T(t)$ is uniformly Abel ergodic. In this case, and under this latter assumption, we can easy checked that $T(t)$ is uniformly Abel ergodic if and only if $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$. Moreover, this theory also plays an important role in the study of power convergence of linear operators. Recall that, an operator $T \in \mathcal{B}(\mathcal{X})$ is called uniformly power convergent if there exists an operator $P \in \mathcal{B}(\mathcal{X})$ such that $\lim\limits_{n\to\infty} \|T^n - P\| = 0$. Lin, Shoikhet and Suciu in [11], showed that for a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$ satisfying $\lim\limits_{t\to\infty} \|T(t)\|/t = 0$, $\{T(t)\}_{t\geq 0}$ is uniformly ergodic if and only if there exists some $\lambda > 0$ such that the Abel average $\lambda R(\lambda, A)$ is uniformly power convergent. Kozitsky, Shoikhet and Zemànek in [8], obtained necessary and sufficient conditions for which the Abel averages of a C_0-semigroup $T(t)$ can be uniformly power convergent. Further condition have been obtained more recently by several authors [8, 16].

This paper is organized as follows. In section 2, we give some definitions and fundamental properties for an integrated semigroup of bounded linear operators on $\mathcal{X}$. In section 3, we are motivated by application to the ergodic theory for an integrated semigroup $\{S(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$. More precisely, We show that $S(t)$ is uniformly Abel ergodic if and only if $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$, if and only if $\mathcal{R}(A^k)$ is closed for some integer k and $\|\lambda^2 R(\lambda, A)\| \longrightarrow 0$ as $\lambda \to 0^+$. Also, we show that if $S(t)$ satisfying $S(t)/t^2 \longrightarrow 0$ as $t \to \infty$, then we have the following equivalent

(i) $S(t)$ is uniformly Cesàro ergodic,
(ii) $S(t)$ is uniformly Abel ergodic,
(iii) $\mathcal{R}(A^k)$ is closed for some integer $k \geq 1$,
(iv) The descent $des(A)$ of A is finite.

2 Preliminaries

We start this present section by recalling an interesting concept in operator theory that we need in the sequel. Let A be a closed linear operator on a Banach space $\mathcal{X}$, with domain $D(A) \subset \mathcal{X}$. The smallest non-negative integer p such that $\mathcal{N}(A^p) = \mathcal{N}(A^{p+1})$ is called the ascent of A and denoted by $asc(A)$. If such an integer does not exist, we set $asc(A) = \infty$. Similarly, the smallest non-negative integer q such that $\mathcal{R}(A^q) = \mathcal{R}(A^{q+1})$ is called the descent of A and denoted by $des(A)$. If such an integer does not exist, we set $des(A) = \infty$. Let A be a closed linear operator with $D(A) \subsetneq \mathcal{X}$, if $asc(A)$ and $des(A)$ are both finite, then $asc(A) \leq des(A)$, the equality holds when $A \in \mathcal{B}(\mathcal{X})$, see [18, Theorem 6.2]).
For $A \in \mathcal{B}(\mathcal{X})$, we have the following equivalences, see [4, Lemma 1.1]:

$$asc(A) \leq p \iff \mathcal{R}(A^p) \cap \mathcal{N}(A^j) = \{0\};\ j = 1, 2, \ldots.$$

$$des(A) \leq q \iff \mathcal{X} = \mathcal{R}(A^j) + \mathcal{N}(A^q);\ j = 1, 2, \ldots.$$

The family $\{S(t)\}_{t\geqslant 0}$ of bounded linear operator on $\mathcal{B}(\mathcal{X})$ is called integrated semigroup [7, Definition 1.1] if it has the following properties:

1. $S(0) = 0$,

2. $S(s)S(t) = \int_0^s S(r+t) - S(r)dr$ for $t, s \geq 0$,
3. The map $t \to S(t)$ from $[0, +\infty[$ into $\mathcal{X}$ is strongly continuous.

The differentiation spaces C^n, $n \geq 0$, are defined by $C^0 = \mathcal{X}$ and

$$C^n = \left\{x \in \mathcal{X} : S(.)x \in C^n\left(\mathbb{R}^+; \mathcal{X}\right)\right\}.$$

Using this notion (2) can equivalently be formulated by $S(t)x \in C^1$ for all $x \in \mathcal{X}$, and

$$S'(r)S(t) = S(r+t) - S(r).$$

The set $N = \{x \in \mathcal{X}; S(t)x = 0, \forall t \geq 0\}$ is called the degeneration space of the integrated semigroup $\{S(t)\}_{t\geqslant 0}$. In this case, $\{S(t)\}_{t\geqslant 0}$ is called non-degenerate if $N = \{0\}$ and degenerate otherwise.
The generator $A : D(A) \subset \mathcal{X} \to \mathcal{X}$ of a non-degenerate integrated semigroup $\{S(t)\}_{t\geqslant 0}$ is defined as follows: $x \in D(A)$ and $Ax = y$ if and only if $x \in C^1$ and $S'(t)x - x = S(t)y$ for $t \geq 0$. Usually, a non-degenerate integrated semigroup is uniquely determined by its generator. Motivated from the Laplace transform theory we can define the generator of an integrated semigroup as

$$Ax = (\lambda - R_\lambda^{-1}) \text{ for all } x \in D(A),$$

where $R_\lambda = \lambda \int_0^{+\infty} e^{-\lambda t} S(t)dt$ is the The Laplace Transformation of $\{S(t)\}_{t\geqslant 0}$, this integral does not exist in general, for more details see [7, Exemple 1.2]. To ensure the existence of this integral, we recall the following definition. An integrated semigroup $\{S(t)\}_{t\geqslant 0}$ is called exponentially bounded, if there exist constants $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\left\|S(t)\right\| \leqslant Me^{\omega t} \text{ for all } t \geqslant 0.$$

In this case, the Laplace Transformation R_λ exists for all λ with $Re\lambda > \omega$. In General, an operator A is called generator of an integrated semigroup, if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$, and there exists a strongly continuous exponentially bounded family $\{S(t)\}_{t\geqslant 0}$ of bounded linear operators on $\mathcal{X}$ such that

$$\begin{aligned} S(0) &= 0, \text{ and} \\ (\lambda - A)^{-1} &= \lambda \int_0^\infty e^{-\lambda t} S(t)dt, \text{ for all } \lambda \text{ with } Re\lambda > \omega. \end{aligned}$$

In this case, the Laplace Transformation R_λ of $\{S(t)\}_{t\geqslant 0}$ is exactly the resolvent function of A and satisfies the pseudo-resolvent:

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu).$$

For more details, we refer the interested reader to [1, 5, 7].

Example 1. *1. Let $\mathcal{X} = C[0,1]$ with $Af = f'$ on $D(A) = \{f \in C^1([0,1]) : f(1) = 0\}$. The integrated semigroup generated by A is given by*

$$S(t)f(x) = \int_x^m f(s)ds, \text{ where } m = \min\{1, x+t\}.$$

2. We consider $\mathcal{X} = \ell^2$ and the family $\{S(t)\}_{t\geqslant 0}$ of bounded linear operators on $\mathcal{X}$ defined by:

$$S(t)(x_n)_{n\in\mathbb{N}^*} = \left(\int_0^t e^{a_n s} ds x_n\right)_{n\in\mathbb{N}^*},$$

where $a_n = n + 2^{n^2}\pi i$. *Then* $\{S(t)\}_{t\geqslant 0}$ *is an integrated semigroup on* $\mathcal{X}$.

3. *Consider the Schrödinger operator* $A = i.\Delta$ *on* $L^p(\mathbb{R})$ *for* $p \geq 1$. *Then A generates an integrated semigroup* $(\{S(t)\}_{t\geq 0}$ *given by*

$$f \mapsto \mathcal{F}^{-1}(u_t \mathcal{F} f) \text{ with } u_t(\xi) = \int_0^t e^{-is|\xi|^2} ds.$$

Definition 2.1. *Let* $\{S(t)\}_{t\geqslant 0}$ *be an integrated semigroup on* $\mathcal{B}(\mathcal{X})$.

(i) *We say that* $\{S(t)\}_{t\geqslant 0}$ *is uniformly Cesàro ergodic if the Cesàro averages of* $\{S(t)\}_{t\geq 0}$ *defined by*

$$\mathcal{C}(t) = \frac{1}{t^2}\int_0^t S(r)dr; \quad \text{for } t \geq 0,$$

converges in the norm operator topology as t tend to infinity. Moreover, $\{S(t)\}_{t\geqslant 0}$ *is said to be uniformly Cesàro bounded if there exists* $M > 0$ *such that*

$$\sup_{t\geq 0} \|\mathcal{C}(t)\| \leq M.$$

(ii) *We say that* $\{S(t)\}_{t\geqslant 0}$ *is uniformly Abel ergodic if the Abel averages of* $\{S(t)\}_{t\geqslant 0}$ *defined by*

$$\mathcal{A}(\lambda) = \lambda^2 \int_0^\infty e^{-\lambda t} S(t)dt; \quad \text{for } t \geq 0,$$

converges in the norm operator topology as λ *tend to zero.*

Now, we will need the following relations between integrated semigroups and their generators. The first result was investigated by W. Arendt [1] in the case of n-times integrated semigroup on $\mathcal{B}(\mathcal{X})$, where $n \in \mathbb{N}$.

Proposition 2.1. *[1, Proposition 3.3] Let A be the generator of an integrated semigroup* $\{S(t)\}_{t\geqslant 0}$ *on* $\mathcal{B}(\mathcal{X})$. *Then*

1. *For all* $x \in D(A)$ *and all* $t \geq 0$, *we have:* $S(t)x \in D(A)$ *and* $AS(t)x = S(t)Ax$.
   Moreover, $S(t)x = \int_0^t S(s)Axds + tx$.
2. *For all* $x \in \mathcal{X}$, $\int_0^t S(s)xds \in D(A)$, *and*

$$A\int_0^t S(s)xds = S(t)x - tx.$$

Lemma 2.1. *[15, Lemma 2.3] Let A be the generator of an integrated semigroup* $\{S(t)\}_{t\geqslant 0}$ *on* $\mathcal{B}(X)$. *Then we have the following assertions:*

1. $\mathcal{R}(A) = \big(\lambda R(\lambda, A) - I\big)\mathcal{X}$.
2. $\mathcal{N}(A) = \mathcal{N}\big(\lambda R(\lambda, A) - I\big) = \{x \in \mathcal{X} \,:\, S(t)x = tx; \text{ for all } t \geq 0\}$.

We recall the following result which was recently proved in [16] for an α-times integrated semigroup $\{S(t)\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{X})$, where $\alpha \geq 0$.

Theorem 2.1. *[16, Theorem 2.2] Let A be the generator of an integrated semigroup* $\{S(t)\}_{t\geqslant 0}$ *on* $\mathcal{B}(\mathcal{X})$ *such that* $\lim_{t\to\infty}\left\|\frac{S(t)}{t}\right\| = 0$. *If* $\mathcal{R}(A)$ *is closed, then* $\{S(t)\}_{t\geq 0}$ *is uniformly Cesàro ergodic.*

3 Main results

We began this section by the following two lemmas which will be widely used in the sequel.

Lemma 3.1. *Let A be the generator of an integrated semigroup $\{S(t)\}_{t\geqslant 0}$ on $\mathcal{B}(\mathcal{X})$. If $\|\lambda^2 R(\lambda, A)\| \longrightarrow 0$ as $\lambda \to 0^+$, then $\overline{\mathcal{R}(A)} \cap \mathcal{N}(A) = \{0\}$, which yields $asc(A) \leq 1$.*

Proof. Let $\{S(t)\}_{t\geqslant 0}$ be an integrated semigroup on $\mathcal{B}(\mathcal{X})$ with A be their generator and $R(\lambda, A)$ the resolvent function of A. We assume that $\|\lambda^2 R(\lambda, A)\| \to 0$ as $\lambda \to 0^+$ and let $y \in \overline{\mathcal{R}(A)} \cap \mathcal{N}(A)$, then by the second assertion of Lemma 2.1, we get

$$\lambda R(\lambda, A)y = y\,, \text{ for all } \lambda \in \rho(A).$$

Since $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}\big(\lambda R(\lambda, A) - I\big)}$, then there exists $x \in \mathcal{X}$ and $M > 0$, such that

$$y = \big(\lambda R(\lambda, A) - I\big)x \text{ and } \|x\| \leq M\|y\|.$$

Next, from the resolvent equation:

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A), \text{ for all } \lambda \neq \mu \in \rho(A).$$

We get the following inequality, for all $\lambda \neq \mu$,

$$\begin{aligned} \|\lambda R(\lambda, A)y\| &\leq |\mu - \lambda|^{-1}\big[\,\|\lambda^2 R(\lambda, A)\| + |\lambda|\|\mu R(\mu, A)\|\,\big]\|x\| \\ &\leq M\,|\mu - \lambda|^{-1}\big[\,\|\lambda^2 R(\lambda, A)\| + |\lambda|\|\mu R(\mu, A)\|\,\big]\|y\|, \end{aligned}$$

It follows that $\lambda R(\lambda, A)y \longrightarrow 0$ as $\lambda \to 0^+$, which yields that $y = 0$. Therefore $\overline{\mathcal{R}(A)} \cap \mathcal{N}(A) = \{0\}$. □

Lemma 3.2. *Let A be a closed linear operator with domain $D(A) \subset \mathcal{X}$, such that $asc(A) = d < \infty$. If either of the following hold:*

(i) $\mathcal{R}(A^n)$ is closed for some $n > d$, or
(ii) $\mathcal{R}(A^j) + \mathcal{N}(A^k)$ is closed for some positive integers with $j + k = n \geq d$,
then $\mathcal{R}(A^n)$ is closed for all $n \geq d$, and $\mathcal{R}(A^j) + \mathcal{N}(A^k)$ is closed for all $j + k \geq d$.

Proof. Let A and B are closed linear operators on a Banach space $\mathcal{X}$, if $\mathcal{R}(B) \subset D(A)$ then we have the following identity:

$$A^{-1}\mathcal{R}(AB) = \mathcal{R}(B) + \mathcal{N}(A).$$

Then, for a linear operator A on a Banach space $\mathcal{X}$ and for some integers j and k, we get

$$A^{-k}\mathcal{R}(A^j A^k) = \mathcal{R}(A^j) + \mathcal{N}(A^k).$$

Then, we obtain the following results:
(i) If $\mathcal{R}(A^n)$ is closed, so is $\mathcal{R}(A^j) + \mathcal{N}(A^k)$ whenever $j + k = n$.
(ii) For $n \geq d$, $\mathcal{R}(A^n)$ is closed whenever $\mathcal{R}(A^n) + \mathcal{N}(A^m)$ is closed for some $m \geq 1$.
Assume that A be a closed linear operator with domain $D(A) \subset \mathcal{X}$ such that $asc(A) = d < \infty$, then

$$\mathcal{R}(A^d) \cap \mathcal{N}(A^m) = \{0\}, \text{ for all } m = 1, 2, \ldots.$$

Now, we separate the hypothesis. Let $\mathcal{R}(A^n)$ is closed for some $n > d$, hence it follows from (i) that $\mathcal{R}(A^j) + \mathcal{N}(A^k)$ is closed whenever $j + k = n$. So take $j = n - 1$ and $k = 1$, then by (ii), $\mathcal{R}(A^{n-1})$ is closed. Therefore, by induction $\mathcal{R}(A^j)$ is closed for all $d \leq j \leq n$. Since the ascent $asc(A)$ is finite which means that $\mathcal{R}(A^{n-1}) \cap \mathcal{N}(A) = \{0\}$, then the restriction of A to the closed invariant subspace $\mathcal{R}(A^{n-1})$ is one to one. Thus the restriction is a Banach space isomorphism from the closed subspace $\mathcal{R}(A^{n-1})$ onto the closed subspace $\mathcal{R}(A^n)$. It carries the subspace $\mathcal{R}(A^n)$ onto $\mathcal{R}(A^{n+1})$, which must be also closed. Consequently $\mathcal{R}(A^n)$ is closed for all $n \geq d$.

Next, we assume that $\mathcal{R}(A^n) + \mathcal{N}(A^m)$ is closed for some $n > d$ and $m \geq 1$. It follows from (ii) that $\mathcal{R}(A^n)$ is closed for some $n > d$. Again, we applied the first argument, we deduce $\mathcal{R}(A^n)$ is closed for all $n \geq d$, so is $\mathcal{R}(A^n) + \mathcal{N}(A^m)$ for all $n \geq d$ and $m \geq 1$. □

The first main result of this paper can be described as follows.

Theorem 3.1. *Let A be the generator of an integrated semigroup $\{S(t)\}_{t\geqslant 0}$ on $\mathcal{B}(\mathcal{X})$. Then the following assertions are equivalent:*

1. *$S(t)$ is uniform Abel ergodic,*
2. *$\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$,*
3. *$\|\lambda^2 R(\lambda, A)\| \longrightarrow 0$ as $\lambda \to 0^+$ and $\mathcal{R}(A)$ is closed,*
4. *$\|\lambda^2 R(\lambda, A)\| \longrightarrow 0$ as $\lambda \to 0^+$ and the descent $des(A)$ is finite.*

Proof. Let A be the generator of an integrated semigroup $\{S(t)\}_{t\geqslant 0}$ on $\mathcal{B}(\mathcal{X})$. then there exist constants $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\| \leqslant Me^{\omega t} \text{ for all } t \geqslant 0.$$

Therefore, the Laplace Transformation R_λ of $\{S(t)\}_{t\geqslant 0}$ is exactly the resolvent of A. It follows that the Abel averages $\mathcal{A}(\lambda)$ of $\{S(t)\}_{t\geqslant 0}$ satisfying the following equality

$$\begin{aligned}\lim_{\lambda\to 0^+} \mathcal{A}(\lambda) &= \lim_{\lambda\to 0^+} \lambda^2 \int_0^\infty e^{-\lambda t} S(t)dt; \quad \text{for } t \geq 0,\\ &= \lim_{\lambda\to 0^+} \lambda R(\lambda, A).\end{aligned}$$

(1) $\Longrightarrow$ (2) It is known from the mean ergodic theorem [19, p. 217] that if there exists an operator $P \in \mathcal{B}(\mathcal{X})$ such that $\|\lambda R(\lambda, A) - P\| \longrightarrow 0$ as $\lambda \to 0^+$, then P is the projection of $\mathcal{X}$ onto $\mathcal{N}\big(\lambda R(\lambda, A) - I\big)$ parallel to $\big(\lambda R(\lambda, A) - I\big)\mathcal{X}$, and by Lemma 2.1, we get

$$\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A).$$

(2) $\Longrightarrow$ (3) Assume that $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$ where $\mathcal{R}(A)$ is closed. It follows from the second assertion of Lemma 2.1 that for all $x \in \mathcal{N}(A)$ we have $\lambda R(\lambda, A)x = x$. Then

$$\|\lambda^2 R(\lambda, A)|_{\mathcal{N}(A)}\| \longrightarrow 0 \text{ as } \lambda \to 0^+.$$

So, to complete the proof we show that $\|\lambda^2 R(\lambda, A)|_{\mathcal{R}(A)}\| \longrightarrow 0$ as $\lambda \to 0^+$. Indeed, let's denote $Y = \mathcal{R}(A)$ and A_1 be the generator of the restriction of $T(t)$ to Y, which is equal to the restriction of A to $Y \cap D(A)$. From the first assertion of Lemma 2.1 we have $Y = \mathcal{R}\big(\lambda R(\lambda, A) - I\big)$. Since $\mathcal{R}\big(\lambda R(\lambda, A) - I\big) \cap \mathcal{N}\big(\lambda R(\lambda, A) - I\big) = \{0\}$, then the operator $\big(\lambda R(\lambda, A) - I\big)$ is invertible on Y.
Now, let $y \in Y \cap D(A)$ such that $A_1 y = 0$, hence

$$\begin{aligned} y &= R(\lambda, A)(\lambda - A)y\\ &= \lambda R(\lambda, A)y - R(\lambda, A)Ay\\ &= \lambda R(\lambda, A)y - R(\lambda, A)A_1 y\\ &= \lambda R(\lambda, A)y.\end{aligned}$$

Then $y \in \mathcal{N}\big(\lambda R(\lambda, A) - I\big)$, which implies that $y = 0$. Thus A_1 is one to one.
Clearly, we have $R(\lambda, A)Y \subset Y$, hence we obtain that $\big(\lambda R(\lambda, A) - I\big)Y \subset \mathcal{R}(A_1)$.
Then, we get the following

$$Y \supset \mathcal{R}(A_1) \supset \big(\lambda R(\lambda, A) - I\big)Y = \big(\lambda R(\lambda, A) - I\big)\mathcal{X} = \mathcal{R}(A) = Y.$$

Hence $Y = \mathcal{R}(A_1)$, so A_1^{-1} is defined on all Y, since A_1 is closed, therefore A_1^{-1} is closed, and by the closed graph theorem A_1^{-1} is continuous.
Let $0 < \lambda < \delta < \dfrac{1}{\|A_1^{-1}\|}$ and $y \in Y$, we get

$$\|\lambda^2 R(\lambda, A)y\| = \|\lambda^2 R(\lambda, A)A_1 A_1^{-1} y\|$$

$$\leq \|\lambda^2 \left(\lambda R(\lambda, A) - I\right)\| \, \|A_1^{-1}\| \, \|y\|.$$

Hence

$$\|\lambda^2 R(\lambda, A)y\| \leq \lambda^2 \left(\|\lambda R(\lambda, A)\| + 1\right) \|A_1^{-1}\| \, \|y\|.$$

Also, we have

$$\|\lambda R(\lambda, A)\| \leq \delta\left(\|\lambda R(\lambda, A)\| + 1\right) \|A_1^{-1}\|.$$

Then, we get

$$\|\lambda R(\lambda, A)\| \leq \frac{\delta\|A_1^{-1}\|}{1 - \delta\|A_1^{-1}\|} = M.$$

Therefore

$$\begin{aligned} \|\lambda^2 R(\lambda, A)y\| &\leq \|\lambda^2 \left(\lambda R(\lambda, A) - I\right)\| \, \|A_1^{-1}\| \, \|y\| \\ &\leq \lambda^2 \left(\|\lambda R(\lambda, A)\| + 1\right) \|A_1^{-1}\| \, \|y\| \\ &\leq \lambda^2 (M + 1) \|A_1^{-1}\| \, \|y\|, \end{aligned}$$

which implies that $\|\lambda^2 R(\lambda, A)|_Y\| \longrightarrow 0$ as $\lambda \to 0^+$. Hence the assertion (3) holds.

(3) $\Longrightarrow$ (4) suppose that $\|\lambda^2 R(\lambda, A)\| \longrightarrow 0$ as $\lambda \to 0^+$, and $\mathcal{R}(A)$ is closed.
By Lemma 2.1, we have $\mathcal{R}(A) = \left(\lambda R(\lambda, A) - I\right)\mathcal{X}$, which means that for all $\lambda > 0$, the operator $\lambda R(\lambda, A) - I$ has a closed range. Fix $\mu > 0$ such that for each $y \in \left(\mu R(\mu, A) - I\right)\mathcal{X}$, there exists a $M > 0$ and $x \in \mathcal{X}$ such that $y = \left(\mu R(\mu, A) - I\right)x$ and $\|x\| \leq M\|y\|$. So, we have

$$\lambda R(\lambda, A)\left(\mu R(\mu, A) - I\right) = \lambda\mu R(\lambda, A)R(\mu, A) - \lambda R(\lambda, A).$$

By the resolvent equation:

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A),$$

we obtain

$$\begin{aligned} \lambda R(\lambda, A)\left(\mu R(\mu, A) - I\right) &= \lambda\mu R(\lambda, A)R(\mu, A) - \lambda(\mu - \lambda)R(\lambda, A)R(\mu, A) - \lambda R(\mu, A) \\ &= \lambda^2 R(\lambda, A)R(\mu, A) - \lambda R(\mu, A) \\ &= \lambda^2 (\mu - \lambda)^{-1} \left[R(\lambda, A) - R(\mu, A)\right] - \lambda R(\mu, A) \\ &= (\mu - \lambda)^{-1} \left[\lambda^2 R(\lambda, A) - \lambda\mu R(\mu, A)\right]. \end{aligned}$$

This gives

$$\begin{aligned} \|\lambda R(\lambda, A)y\| &= \|\lambda R(\lambda, A)\left(\mu R(\mu, A) - I\right)x\| \\ &= \left\|(\mu - \lambda)^{-1} \left[\lambda^2 R(\lambda, A) - \lambda\mu R(\mu, A)\right]x\right\| \\ &\leq |\mu - \lambda|^{-1} \left[\|\lambda^2 R(\lambda, A)\| + |\lambda| \, \|\mu R(\mu, A)\|\right] M\|y\|. \end{aligned}$$

It follows that

$$\|\lambda R(\lambda, A)|_{(\lambda R(\lambda, A) - I)\mathcal{X}}\| \longrightarrow 0 \text{ as } \lambda \to 0^+. \tag{3.1}$$

Then for a small $\lambda > 0$, the operator $\lambda R(\lambda, A) - I$ is invertible on $\left(\lambda R(\lambda, A) - I\right)\mathcal{X}$. Therefore

$$\left(\lambda R(\lambda, A) - I\right)^2 \mathcal{X} = \left(\lambda R(\lambda, A) - I\right)\mathcal{X},$$

which yields $\mathcal{X} = \left(\lambda R(\lambda, A) - I\right)\mathcal{X} + \mathcal{N}\left(\lambda R(\lambda, A) - I\right)$. It follows from Lemma 2.1 that $\mathcal{X} = \mathcal{R}(A) + \mathcal{N}(A)$, therefore the $des(A)$ is finite. Hence the assertion (4) holds.

(4) $\Longrightarrow$ (1) Since the $des(A)$ is finite then $\mathcal{X} = \mathcal{R}(A) + \mathcal{N}(A^d)$ for some integer $d > 0$, and by Lemma 3.1, we have $\overline{\mathcal{R}(A)} \cap \mathcal{N}(A) = \{0\}$, then $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$. Hence, it follows from Lemma 2.1

$$\mathcal{X} = \left(\lambda R(\lambda, A) - I\right)\mathcal{X} \oplus \mathcal{N}\left(\lambda R(\lambda, A) - I\right). \tag{3.2}$$

Moreover, as shown above that $\|\lambda R(\lambda, A)|_{(\lambda R(\lambda, A) - I)\mathcal{X}}\| \longrightarrow 0$ as $\lambda \to 0^+$. Since for all $x \in \mathcal{N}(\lambda R(\lambda, A) - I)$, we have $\lambda R(\lambda, A)x = x$, then $\lambda R(\lambda, A)|_{\mathcal{N}(\lambda R(\lambda, A) - I)}$ converge to the identity operator I as $\lambda \to 0^+$. It follows from the decomposition (3.2) that $\lambda R(\lambda, A)$ converges uniformly on $\mathcal{X}$. Hence the assertion (1) holds. □

The following corollary is an immediate consequence of the Theorem 3.2 and of the Lemma 3.2.

Corollary 3.1. *Let A be the generator of an integrated semigroup $\{S(t)\}_{t\geqslant 0}$ on $\mathcal{B}(\mathcal{X})$. Then the following assertions are equivalent:*

1. *$S(t)$ is uniform Abel ergodic,*
2. *$\|\lambda^2 R(\lambda, A)\| \longrightarrow 0$ as $\lambda \to 0^+$ and $\mathcal{R}(A^k)$ is closed for some integer k,*
3. *$\|\lambda^2 R(\lambda, A)\| \longrightarrow 0$ as $\lambda \to 0^+$ and $\mathcal{R}(A^k) + \mathcal{N}(A^j)$ is closed for some $k, j > 0$.*

Corollary 3.2. *Let A be the generator of an integrated semigroup $\{S(t)\}_{t\geqslant 0}$ on $\mathcal{B}(\mathcal{X})$. Then the following assertions are equivalent:*

1. *$S(t)$ is uniformly Abel ergodic,*
2. *$\mathcal{R}(A)$ is closed and $\|\lambda R(\lambda, A)|_{\mathcal{R}(A)}\| \longrightarrow 0$ as $\lambda \to 0^+$.*

Proof. Let A be the generator of an integrated semigroup $\{S(t)\}_{t\geqslant 0}$ on $\mathcal{B}(\mathcal{X})$ and let $R(\lambda, A)|_{\mathcal{R}(A)}$ be the restriction of $R(\lambda, A)$ onto $\mathcal{R}(A)$. As shown above in the proof of Theorem 3.2 that if $S(t)$ is uniformly Abel ergodic, then $\mathcal{R}(A)$ is closed and $\|\lambda R(\lambda, A)|_{\mathcal{R}(A)}\| \longrightarrow 0$ as $\lambda \to 0^+$ by (3.1).

Conversely, we assume that $\mathcal{R}(A)$ is closed and $\|\lambda R(\lambda, A)|_{\mathcal{R}(A)}\| \longrightarrow 0$ as $\lambda \to 0^+$.
Since $\mathcal{R}(A) = (\lambda R(\lambda, A) - I)\mathcal{X}$, then $\| \lambda R(\lambda, A)|_{(\lambda R(\lambda,A)-I)\mathcal{X}}\| \longrightarrow 0$ as $\lambda \to 0^+$, then for a small λ, the operator $(\lambda R(\lambda, A) - I)|_{(\lambda R(\lambda,A)-I)\mathcal{X}}$ is invertible and we have

$$\mathcal{R}(\lambda R(\lambda, A) - I) = \mathcal{R}((\lambda R(\lambda, A) - I)|_{\mathcal{R}(A)}) = \mathcal{R}\left[(\lambda R(\lambda, A) - I)^2\right].$$

Therefore

$$\mathcal{X} = (\lambda R(\lambda, A) - I)\mathcal{X} + \mathcal{N}(\lambda R(\lambda, A) - I). \tag{3.3}$$

Now, let $y \in \mathcal{R}(\lambda R(\lambda, A)-I) \cap \mathcal{N}(\lambda R(\lambda, A)-I)$, so $\lambda R(\lambda, A)y = y$ for all $\lambda > 0$, and by assumption $\lambda R(\lambda, A)y \longrightarrow 0$ as $\lambda \to 0^+$, hence $y = 0$ which yields that

$$(\lambda R(\lambda, A) - I)\mathcal{X} \cap \mathcal{N}(\lambda R(\lambda, A) - I) = \{0\}.$$

Then, the summation in (3.3) is direct and from Lemma 2.1 $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$. Finally, Theorem 3.2 implies that $S(t)$ is uniformly Abel ergodic. □

Now, we present our second main result as follows.

Theorem 3.2. *Let A be the generator of an integrated semigroup $\{S(t)\}_{t\geqslant 0}$ on $\mathcal{B}(\mathcal{X})$. Assume that $\lim\limits_{t\to\infty} \|S(t)\|/t^2 = 0$, then $S(t)$ is uniformly Abel ergodic if and only if $S(t)$ is uniformly Cesàro ergodic.*

Proof. Let A be the generator of an integrated semigroup $\{S(t)\}_{t\geqslant 0}$ on $\mathcal{B}(\mathcal{X})$ such that $\lim\limits_{t\to\infty} \|S(t)\|/t^2 = 0$. Let $S(t)$ be uniformly Abel ergodic, then by Theorem 3.1, we obtain the decomposition $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$, with $\mathcal{R}(A)$ is closed.
From the second assertion of Lemma 2.1, we can easy check that for all $x \in \mathcal{N}(A)$.

$$\lim_{t\to\infty} \left\| \frac{1}{t^2} \int_0^t S(r)x dr - \frac{Ix}{2} \right\| = 0.$$

So, to complete the proof we show that $\frac{1}{t^2} \int_0^t S(r)y dr \longrightarrow 0$ as $t \to \infty$, for all $y \in \mathcal{R}(A)$. Since $\mathcal{R}(A)$ is closed, let's denote A_1 be the generator of the restriction of $S(t)$ to $\mathcal{R}(A)$, which is equal to the restriction of A to $\mathcal{R}(A) \cap D(A)$. As shown in the proof of Theorem 3.1 that A_1^{-1} is defined on all $\mathcal{R}(A)$ and continuous, then for all

$y \in \mathcal{R}(A)$, there exists $x \in D(A)$ such that $y = A_1 x$ and $\|x\| \leq \|A_1^{-1}\|\|y\|$. the second assertion of Proposition 2.1 implies that for all $x \in D(A)$, we have

$$\int_0^t S(s)Axds = S(t)x - tx; \text{ for all } t \geq 0.$$

It follows that, we get

$$\begin{aligned} \left\|\frac{1}{t^2}\int_0^t S(r)ydr\right\| &= \left\|\frac{1}{t^2}\big(S(t)x - tx\big)\right\| \\ &\leq \|A_1^{-1}\|\left(\left\|\frac{S(t)}{t^2}\right\| + \left|\frac{1}{t}\right|\right)\|y\|. \end{aligned}$$

Since $\lim_{t\to\infty} \|S(t)\|/t^2 = 0$, then $\left\|\frac{1}{t^2}\int_0^t S(r)ydr\right\| \longrightarrow 0$ as $t \to \infty$ for all $y \in \mathcal{R}(A)$. Therefore $S(t)$ is uniformly Cesàro ergodic.

Conversely, let $S(t)$ be uniformly Cesàro ergodic, hence there exists an operator $P \in \mathcal{B}(\mathcal{X})$ such that $\|\mathcal{C}(t) - P\| \longrightarrow 0$ as $t \to \infty$, where $\mathcal{C}(t) = \frac{1}{t^2}\int_0^t S(r)dr$ for $t \geq 0$. So, there exists $\varepsilon > 0$ and $a > 0$ such that $\|\mathcal{C}(t) - P\| \leq \varepsilon$ for all $t > a$.

Now, let's denote $\mathcal{W}(t) = \int_0^t S(r)dr$ for all $t \geq 0$. Then, using the integration by parts, we get the following identity:

$$R(\lambda, A) = \lambda^2 \int_0^\infty e^{-\lambda t}\mathcal{W}(t)dt, \text{ for all } t \geq 0. \tag{3.4}$$

Moreover, we have the following identity for any bounded linear operator $P \in \mathcal{B}(\mathcal{X})$

$$\lambda^{\alpha+1}\int_0^\infty e^{-\lambda t}t^\alpha P\,dt = (\alpha!)P, \text{ for all } \lambda \in \mathbb{C} \text{ and } \alpha, t \geq 0. \tag{3.5}$$

It follows from (3.4) and (3.5), we have for every $x \in \mathcal{X}$

$$\begin{aligned} \left\|\lambda R(\lambda, A)x - 2Px\right\| &= \left\|\lambda R(\lambda, A)x - \lambda^3\int_0^\infty e^{-\lambda t}t^2 Px\,dt\right\| \\ &= \left\|\lambda^3\int_0^\infty e^{-\lambda t}\mathcal{W}(t)xdt - \lambda^3\int_0^\infty e^{-\lambda t}t^2Px\,dt\right\| \\ &\leq \left\|\lambda^3\int_0^\infty e^{-\lambda t}\big(\mathcal{W}(t) - (t^2)P\big)dt\right\|\,\|x\| \\ &\leq \Big[|\lambda^3|\int_0^a e^{-\lambda t}\Big(\|\mathcal{W}(t)\| + t^2\|P\|\Big)dt \\ &\quad + \left\|\lambda^3\int_a^\infty e^{-\lambda t}\mathcal{W}(t) - t^2Pdt\right\|\Big]\|x\| \\ &\leq \Big[|\lambda^3|\int_0^a e^{-\lambda t}\Big(\|\mathcal{W}(t)\| + t^2\|P\|\Big)dt \end{aligned}$$

$$+\left\|\lambda^3\int_a^\infty e^{-\lambda t}t^2\big(\mathcal{C}(t)-P\big)dt\right\|\Big]\|x\|$$

$$\le \Big[|\lambda^3|\int_0^a e^{-\lambda t}\Big(\|\mathcal{W}(t)\|+t^2\|P\|\Big)dt+2\|\mathcal{C}(t)-P\|\Big]\|x\|$$

$$\le \Big[|\lambda^3|a\big(\sup_{t\le a}\|\mathcal{W}(t)\|+a^2\|P\|\big)+2\varepsilon\Big]\|x\|.$$

Therefore the above estimate implies that $\|\lambda R(\lambda,A)-2P\|\longrightarrow 0$ as $\lambda\to 0^+$, which yields that $S(t)$ is uniformly Abel ergodic and the proof is finished. □

Corollary 3.3. *Let A be the generator of an integrated semigroup $\{S(t)\}_{t\geqslant 0}$ on $\mathcal{B}(\mathcal{X})$. Assume that $\lim_{t\to\infty}\|S(t)\|/t^2=0$, then the following assertions are equivalent:*

1. *$S(t)$ is uniformly Cesàro ergodic,*
2. *$\mathcal{R}(A^k)$ is closed for some integer $k\ge 1$,*
3. *$\mathcal{R}(A^k)+\mathcal{N}(A^j)$ is closed for some integers $k,j\ge 1$,*
4. *The descent $des(A)$ of A is finite.*

Proof. Let A be the generator of an integrated semigroup $\{S(t)\}_{t\geqslant 0}$ on $\mathcal{B}(\mathcal{X})$. First we must show that $\lim_{t\to\infty}\|S(t)\|/t^2=0$ implies that $\|\lambda^2R(\lambda,A)\|\longrightarrow 0$ as $\lambda\to 0^+$. Indeed, if $\lim_{t\to\infty}\|S(t)\|/t^2=0$, then there exists $\varepsilon>0$ and $a>0$ such that

$$\|S(t)\|\le\varepsilon t^2;\ \text{ for all } t>a$$

Using the resolvent equation, we obtain for all $x\in\mathcal{X}$ and $\mu\neq\lambda$;

$$\begin{aligned}
\|\lambda^2R(\lambda,A)x\| &= \|\lambda^2\left[R(\mu,A)+(\mu-\lambda)R(\lambda,A)R(\mu,A)\right]x\|\\
&\le \|\lambda^2R(\mu,A)\|\|x\| + |\mu-\lambda|\,\lambda^2\|R(\lambda,A)R(\mu,A)x\|\\
&\le \|\lambda^2R(\mu,A)\|\|x\| + |\mu-\lambda|\,\lambda^3\int_0^\infty e^{-\lambda t}\|S(t)R(\mu,A)x\|dt\\
&\le \|\lambda^2R(\mu,A)\|\|x\| + |\mu-\lambda|\,\Big[\lambda^3\int_0^a e^{-\lambda t}\|S(t)R(\mu,A)x\|dt\\
&\qquad + \varepsilon\lambda^3\int_a^\infty e^{-\lambda t}t^2\|R(\mu,A)x\|dt\Big].
\end{aligned}$$

From the identity (3.5), we obtain

$$\begin{aligned}
\|\lambda^2R(\lambda,A)x\| \le\ & \|\lambda^2R(\mu,A)\|\|x\|+|\mu-\lambda|\,\Big[\lambda^3a\big(\sup_{t\le a}\|S(t)\|\|R(\mu,A)\|\big)\\
&+2\varepsilon\|R(\mu,A)\|\Big]\|x\|.
\end{aligned}$$

It follows from the above estimate that $\|\lambda^2R(\lambda,A)\|\longrightarrow 0$ as $\lambda\to 0^+$. So, according to Corollary 3.1, Theorem 3.1 and Theorem 3.2, the equivalents hold. □

Proposition 3.1. *Let A be the generator of an integrated semigroup $\{S(t)\}_{t\geqslant 0}$ on $\mathcal{B}(\mathcal{X})$. If $\{S(t)\}_{t\geqslant 0}$ is uniformly Cesàro ergodic, then the Laplace Transformation R_λ of $\{S(t)\}_{t\geqslant 0}$ exists for all λ with the real part $Re(\lambda)>0$.*

Proof. Assume that $\{S(t)\}_{t\geqslant 0}\subset\mathcal{B}(\mathcal{X})$ is uniformly Cesàro ergodic, then there exists $M>0$ such that

$$\sup_{t\ge 0}\left\|\frac{1}{t^2}\int_0^t S(s)ds\right\|\le M.$$

Let's denote $\mathcal{W}(t) = \int_0^t S(s)ds$ for all $t \geq 0$, and let $\lambda \in \mathbb{C}$ such that $Re(\lambda) > 0$. Then, from the integration by parts, we have for all $0 < u < v$ and $x \in \mathcal{X}$

$$\begin{aligned}\|\lambda \int_u^v e^{-\lambda t} S(t)xdt\| &= \left\|\lambda\left[e^{-\lambda t}\mathcal{W}(t)x\right]_u^v + \lambda^2 \int_u^v e^{-\lambda t}\mathcal{W}(t)xdt\right\| \\ &\leq M\left\|\lambda\left[e^{-\lambda t}t^2\right]_u^v + \lambda^2 \int_u^v e^{-\lambda t}t^2 dt\right\|\|x\| \\ &\leq M\left[|\lambda|\left(e^{-\lambda v}v^2 + e^{-\lambda u}u^2\right) + |\lambda^2|\int_u^v e^{-Re\lambda\, t}t^2 dt\right]\|x\| .\end{aligned}$$

It follows that $\|\lambda \int_u^v e^{-\lambda t} S(t)dt\| \longrightarrow 0$ as $u \to \infty$. Finally, we deduce that the Laplace Transformation R_λ of $\{S(t)\}_{t\geqslant 0}$ exists for all λ with $Re\lambda > 0$. □

Proposition 3.2. *Let $\{S(t)\}_{t\geqslant 0}$ be an integrated semigroup on $\mathcal{B}(\mathcal{X})$. Then, $S(t)$ is uniformly Cesàro ergodic if and only if the restriction of $S(t)$ to $\overline{\mathcal{R}(A)}$ is uniformly Cesàro ergodic.*

Proof. Let A be the generator of an integrated semigroup $\{S(t)\}_{t\geqslant 0}$ on $\mathcal{B}(\mathcal{X})$. Let's denote $\{\mathcal{Q}(t)\}_{t\geqslant 0}$ be the restriction of $\{S(t)\}_{t\geqslant 0}$ to $\overline{\mathcal{R}(A)}$, and A_1 be their generator, which is exactly the restriction of A to $\overline{\mathcal{R}(A)} \cap D(A)$. As shown above that if $S(t)$ is uniformly Cesàro ergodic, then the Cesàro averages of $S(t)$ converges uniformly to the projection P of $\mathcal{X}$ onto $\mathcal{N}(A)$ along $\mathcal{R}(A)$, corresponding to the ergodic decomposition $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$ where $\mathcal{R}(P) = \mathcal{N}(A)$ and $\mathcal{N}(P) = \mathcal{R}(A)$. Since $\mathcal{R}(A)$ is closed and $S(t)$-invariant for all $t \geq 0$, then we easily check that $\mathcal{Q}(t)$ is uniformly Cesàro ergodic to 0, which means that the Cesàro averages of $\mathcal{Q}(t)$ converge to 0 as $t \to \infty$.

Conversely, we assume that $\mathcal{Q}(t)$ is uniformly Cesàro ergodic, so it follows from Theorem 3.2 that $\mathcal{Q}(t)$ is also uniformly Abel ergodic and $\overline{\mathcal{R}(A)} = \mathcal{R}(A_1) \oplus \mathcal{N}(A_1)$ where $\mathcal{R}(A_1)$ is closed. Moreover, we have

$$\left\|\frac{1}{t^2}\int_0^t S(r)xdr\right\| \to 0 \text{ as } t \to \infty, \text{ for all } x \in \overline{\mathcal{R}(A)}.$$

Let $x \in \overline{\mathcal{R}(A)}$, hence from the above decomposition, we have $x = y + z$ where $y \in \mathcal{R}(A_1)$ and $z \in \mathcal{N}(A_1)$. Since $\mathcal{N}(A_1) \subset \mathcal{N}(A)$, then

$$z = \mathcal{Q}(t)z = \frac{1}{t^2}\int_0^t \mathcal{Q}(r)zdr, \text{ for all } t \geq 0.$$

Therefore, we obtain

$$\begin{aligned}\left\|\frac{1}{t^2}\int_0^t \mathcal{Q}(r)zdr\right\| &= \left\|\frac{1}{t^2}\int_0^t \mathcal{Q}(r)(x-y)dr\right\| \\ &\leq \left\|\frac{1}{t^2}\int_0^t \mathcal{Q}(r)xdu\right\| + \left\|\frac{1}{t^2}\int_0^t \mathcal{Q}(r)ydu\right\|.\end{aligned}$$

It follows that $\left\|\frac{1}{t^2}\int_0^t \mathcal{Q}(r)zdr\right\| \longrightarrow 0$ as $t \to \infty$, hence $z = 0$ which gives that $x = y$. Therefore $\overline{\mathcal{R}(A)} = \mathcal{R}(A_1)$, since $\mathcal{R}(A_1) \subset \mathcal{R}(A)$, then $\mathcal{R}(A)$ is closed.

Furthermore, since $\mathcal{Q}(t)$ is uniformly Abel ergodic and their Cesàro averages converge to 0, then

$$\left\|\lambda R(\lambda, A)|_{\mathcal{R}(A)}\right\| \longrightarrow 0 \text{ as } \lambda \to 0^{+}.$$

Consequently, $T(t)$ is uniformly Abel ergodic by Corollary 3.2, and we obtain

$$\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A).$$

The projection P of $\mathcal{X}$ onto $\mathcal{N}(A)$ along $\mathcal{R}(A)$ corresponding to this decomposition is bounded, so we have $\mathcal{N}(I-P) = \mathcal{R}(P) = \mathcal{N}(A)$, and $\mathcal{R}(I-P) = \mathcal{N}(P) = \mathcal{R}(A)$. Then it follows from Lemma 2.1 that for all $x \in \mathcal{X}$

$$\frac{1}{t^2}\int_0^t S(r)Pxdr = \frac{1}{2}Px.$$

So, we obtain

$$\begin{aligned}
\left\|\mathcal{C}(t)x - \frac{1}{2}Px\right\| &= \left\|\frac{1}{t^2}\int_0^t S(r)xdr - \frac{1}{t^2}\int_0^t S(r)Pxdr\right\| \\
&= \left\|\frac{1}{t^2}\int_0^t S(r)(x-Px)dr\right\| \\
&= \left\|\frac{1}{t^2}\int_0^t \mathcal{Q}(r)(x-Px)dr\right\| \\
&\leq \left\|\frac{1}{t^2}\int_0^t \mathcal{Q}(r)dr\right\| \|x-Px\| \\
&\leq \left\|\frac{1}{t^2}\int_0^t \mathcal{Q}(r)dr\right\| (1+\|P\|)\|x\|.
\end{aligned}$$

Then $\left\|\mathcal{C}(t)x - \frac{1}{2}Px\right\| \longrightarrow 0$ as $t \to \infty$. Finally, $S(t)$ is uniformly Cesàro ergodic. □

Remark 3.1. *Let A be the generator of an integrated semigroup $\{S(t)\}_{t\geqslant 0}$ on $\mathcal{B}(\mathcal{X})$. If $\lim_{t\to\infty} \|S(t)\|/t = 0$, then A is one to one by [15, Corollary 2.4]. In this case, if the Cesàro averages and the Abel averages converge, they will converge to zero and we obtain $\mathcal{X} = \mathcal{R}(A)$. Moreover, the strong limit of $\frac{1}{t}\int_0^t S(r)dr$ may be divergent when $t \to \infty$, as the following example shows.*

Example: Let $X = C([0,\infty])$, we consider the derivation operator $Af = -f'$ for all $f \in D(A)$, with $D(A) = \{f \in C^1([0,1]) : f(0) = 0\}$. Since the domain $D(A)$ is not dense in X, then A cannot be an infinitesimal generator of a C_0-semigroup. Furthermore, the semigroup $S(t)$ generated by A, is given by:

$$\big(S(t)f\big)(x) = \begin{cases} -\int_x^{x-t} f(s)ds, & \text{si } x > t, \\ \int_x^0 f(s)ds, & \text{si } 0 \leq x \leq t, \end{cases}.$$

Note that $S(t)$ is an integrated semigroup of type $\omega_0 = 0$, where

$$\omega_0 = \inf\{w \in \mathbb{R} : \text{ there exists } M \text{ such that } \|S(t)\| \leq Me^{wt},\ t \geq 0\}.$$

It follows that $\|S(t)\|/t \longrightarrow 0$ as $t \to \infty$.
On the other hand, since the generator A has an empty spectrum, then $X = R(A)$, and by Theorem 3.1 and Corollary 3.3, we deduce that $S(t)$ is both uniformly Cesàro ergodic and uniformly Abel ergodic. More precisely, we have

$$\lim_{t\to\infty}\left\|\frac{1}{t^2}\int_0^t S(r)dr\right\| = \lim_{\lambda\to 0^+}\left\|\lambda^2\int_0^\infty e^{-\lambda t}S(t)dt\right\| = 0.$$

Now, we suppose that then there exists an operator P such that

$$\lim_{t\to\infty}\left\|\frac{1}{t}\int_0^t S(r)dr - P\right\| = 0.$$

Hence P is a projection operator of X onto $\mathcal{N}(A)$ parallel to $\mathcal{R}(A)$, corresponding to ergodic decomposition $X = \mathcal{R}(A) \oplus \mathcal{N}(P)$. As shown above that $X = \mathcal{R}(A)$, then we deduce that

$$\lim_{t\to\infty}\left\|\frac{1}{t}\int_0^t S(r)dr\right\| = 0.$$

Next, let f be a non-zero function defined on X, hence there exists $g \in D(A)$ such that $f = Ag = -g'$. Therefore, we obtain for $0 < x \leq s$

$$\begin{aligned}\int_0^t \big(S(s)f\big)(x)ds &= \int_0^t \big(S(s)Ag\big)(x)ds \\ &= -\int_0^t \big(S(s)g'\big)(x)ds \\ &= -\int_0^t\int_x^0 g'(r)dr \\ &= -\int_0^t \big(g(0)-g(x)\big)ds \\ &= g(x)t.\end{aligned}$$

By assumption $\lim_{t\to\infty}\left\|\frac{1}{t}\int_0^t S(r)dr\right\| = 0$, hence $g(x) = 0$ for all $0 < x \leq s$, absurd.

Finally, $\frac{1}{t}\int_0^t S(r)dr$ is divergent as t tends to infinity.

Acknowledgement: The author wishes to express their indebtedness to the referee, for his suggestions and valuable comments on this paper.

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