Research Article

Open Access

B.P. Duggal*

On (m, P) -expansive operators: products, perturbation by nilpotents, Drazin invertibility

<https://doi.org/10.1515/conop-2020-0120>

Received February 14, 2021; accepted October 8, 2021

Abstract: A generalisation of m -expansive Hilbert space operators $T \in B(\mathcal{H})$ [18, 20] to Banach space operators $T \in B(\mathcal{X})$ is obtained by defining that a pair of operators $A, B \in B(\mathcal{X})$ is (m, P) -expansive for some operator $P \in B(\mathcal{X})$ if $\Delta_{A,B}^m(P) = (I - L_A R_B)^m(P) = \sum_{j=0}^m (-1)^j \binom{m}{j} A^j P B^j \leq 0$; $L_A(X) = AX$ and $R_B(X) = XB$.

Unlike m -isometric and m -left invertible operators, commuting products and perturbations by commuting nilpotents of (m, I) -expansive operators do not result in expansive operators: using elementary algebraic properties of the left and right multiplication operators, a sufficient condition is proved. For Drazin invertible A and $B \in B(\mathcal{H})$, with Drazin inverses A_d and B_d , a sufficient condition proving $(A_d, B_d) \wedge (A, B)$ is $(m-1, P)$ -isometric (resp., $(m-1, P)$ -contractive) for m even (resp., m odd) is given, and a Banach space analogue of this result is proved.

Keywords: Hilbert/Banach space, Left/right multiplication operator, (m, P) -expansive operators, m -left invertible/ m -isometric operators, product of operators, perturbation by nilpotents, Drazin invertible operators

MSC: 47A05, 47A55; Secondary 47A11, 47B47

1 Introduction

Let $\mathcal{H}$ (resp., $\mathcal{X}$) denote an infinite dimensional complex Hilbert (resp., Banach) space, and let $B(\mathcal{H})$ (resp., $B(\mathcal{X})$) denote the algebra of operators, i.e. bounded linear transformations, on $\mathcal{H}$ (resp., $\mathcal{X}$) into itself. An operator $T \in B(\mathcal{H})$ is m -expansive for some positive integer m if

$$\Delta_{T^*, T}^m(I) = \sum_{j=0}^m (-1)^j \binom{m}{j} T^{*j} T^j \leq 0$$

[20]; T is m -hyperexpansive (resp., completely hyperexpansive) if $\Delta_{T^*, T}^t(I) \leq 0$ for all integers $1 \leq t \leq m$ (resp., for all positive integers t). Agler [1, Theorem 3.1] characterised subnormality with positivity of $\Delta_{T^*, T}^m(I)$: $\Delta_{T^*, T}^m(I) \geq 0$ if and only if $\|T\| \leq 1$ and T is subnormal. Operators T such that $\Delta_{T^*, T}^m(I) \geq 0$ have been called m -contractive and operators T such that $\Delta_{T^*, T}^m(I) = 0$ are said to be m -isometric [2]. A study of m -isometries was initiated by Agler and Stankus in their series of papers [2–4] and further studies of the structure of this class of operators has since been carried out by a number of authors, amongst them [5–17]. Amongst m -expansive operators, 2-expansive operators stand out: 2-expansive operators are 2-hyperexpansive. A. Athavale studied completely hyperexpansive operators [18], and a study of m -expansive operators, including various of their avatars, has since been widely carried out in [19–23].

*Corresponding Author: B.P. Duggal: University of Niš, Faculty of Sciences and Mathematics, P.O. Box 224, 18000 Niš, Serbia, E-mail: bpduggal@yahoo.co.uk

For $A, B \in B(\mathcal{X})$, let L_A and $R_B \in B(B(\mathcal{X}))$ denote respectively the operators

$$L_A(X) = AX \text{ and } R_B(X) = XB$$

of left multiplication by A and right multiplication by B . If we define the elementary operator $\triangle_{A,B} \in B(B(\mathcal{X}))$ by

$$\triangle_{A,B}(X) = (I - L_A R_B)(X) = X - AXB,$$

then

$$\begin{aligned} (1) \quad \triangle_{A,B}^m(X) &= (I - L_A R_B)^m(X) = \sum_{j=0}^m (-1)^j \binom{m}{j} (L_A R_B)^j(X) \\ &= \sum_{j=0}^m (-1)^j \binom{m}{j} A^j X B^j. \end{aligned}$$

Recall [24] that a pair of operators $A, B \in B(\mathcal{X})$ is (m, X) -left invertible for some operator $X \in B(\mathcal{H})$ if $\triangle_{A,B}^m(X) = 0$. Generalising to m -expansive operators, we say that a pair $(A, B) \in B(\mathcal{X})$ is (m, X) -expansive for an $X \in B(\mathcal{X})$ if $\triangle_{A,B}^m(X) \leq 0$. Trivially, every (m, X) -left invertible pair (A, B) is (m, X) -expansive; (m, X) -expansive pairs, however, fail in general to satisfy properties satisfied by (m, X) -left invertible pairs (even if one limits oneself to Hilbert space pairs (T^*, T)). Thus $(A, B) \in (m, X)$ -left invertible implies $(A, B) \in (n, X)$ -left invertible for all integers $n \geq m$: $(T^*, T) \in (m, X)$ -expansive, $T \in B(\mathcal{H})$, fails to imply $(T^*, T) \in (m+1, X)$ -expansive. Since

$$\triangle_{A^n, B^n}^m(X) = (I - L_A^n R_B^n)^m(X) = \{I + L_A R_B + \cdots + L_A^{n-1} R_B^{n-1}\}^m \triangle_{A,B}^m(X),$$

$(A, B) \in (m, X)$ -left invertible implies $(A^n, B^n) \in (m, X)$ -left invertible for all integers $n \geq m$, and if $A = B^* = T \in B(\mathcal{H})$ then $(A, B) \in (m, X)$ -expansive for an $X \in B(\mathcal{H})$ implies $(A^n, B^n) \in (m, X)$ -expansive for all integers $n \geq m$ [20, 24]. The passage of the (m, X) -expansive property from (A, B) to (A^n, B^n) however fails for general (A, B) , and a sufficient condition here is the “existence of certain partial hyperexpansive sequences” (see Theorem 2.6, *infra*).

Richter [25, Lemma 1] observed that $(2, I)$ -expansive Hilbert space operators are $(2, I)$ -hyperexpansive (see also Athavale [18]; see Gu [26] for the Banach space analogue). This property is inherited by $(2, X)$ -expansive Hilbert space pairs (T^*, T) for $X \geq 0$. We start in this paper by proving that: if $(A, B) \in \{(m_0, P)\text{-cotractive}\} \wedge \{(m_0 + 2)\text{-expansive}\} \wedge \{(m, P)\text{-expansive}\}$, where $A, B, P \in B(\mathcal{H})$, $m_0 \geq 0$ and $m > m_0 + 2$ are integers, and the triplet $(A, B, \triangle_{A,B}^s(P))$ satisfies an “order preserving property” (of type: if $\triangle_{A,B}^s(P) \geq 0$ for $m_0 + 2 \leq s \leq m$, then $(L_A R_B)^t(\triangle_{A,B}^{s_0}(P)) \geq 0$ for all positive integers t – see Theorem 2.2 below), then $(A, B) \in (r, P)$ -expansive for all integers $m_0 + 1 \leq r \leq m$. For invertible (A, B) such that both the triplets $(A, B, \triangle_{A,B}^s(P))$ and $(A^{-1}, B^{-1}, \triangle_{A,B}^s(P))$ preserve order for $s = 0, m-1, m$ and $(A, B) \in (m, P)$ -expansive, we prove that $(A, B) \wedge (A^{-1}, B^{-1}) \in (n, P)$ -isometric for all $n \geq m-1$ if m is even and $(A, B) \wedge (A^{-1}, B^{-1}) \in (m-1, P)$ -contractive if m is odd. An analogous result holds for Drazin invertible pairs (A, B) .

If $A_i, B_i, X \in B(\mathcal{X})$ are such that $(A_i, B_i) \in (m_i, X)$ -left invertible ($i = 1, 2$), A_1 commutes with A_2 and B_1 commutes with B_2 , then the products pair $(A_1 A_2, B_1 B_2)$ is $(m_1 + m_2 - 1, X)$ -left invertible; again, if $N \in B(\mathcal{X})$ is an n -nilpotent which commutes with A_1 and B_1 , then $(A_1, B_1) \in (m, X)$ -left invertible implies $(A_1 + N, B_1 + N) \in (m_1 + 2n - 2)$ -left invertible (see [5, 16, 24, 27]). These properties do not transfer to (m, X) -expansive operators. We prove that a sufficient condition for a successful transfer of these properties to (m, X) -expansive operators is the existence of “partial expansive sequences”. The existence of such sequences for left invertible pairs is a (trivial) consequence of the structure of such pairs.

We prove our results, including any complementary results, in the following two sections. In addition to the notation and terminology already introduced, any further notation/terminology will be explained at the first instance of occurrence of such notation/terminology.

2 (m, P) -hyperexpansive operators on Hilbert and Banach spaces.

An important, and well studied, class of (m, X) -expansive operators in $B(\mathcal{H})$ is that of m -isometric operators, where $S \in B(\mathcal{H})$ is m -isometric, equivalently $(S^*, S) \in (m, I)$ -isometric, if

$$\Delta_{S^*, S}^m(I) = (I - L_{S^*} R_S)^m = \sum_{j=0}^m (-1)^j \binom{m}{j} S^{*j} S^j = 0$$

(see [2,5–8,13,16,28] for further references). It is clear that m -isometric operators are (m, I) -left invertible operators. Since $\Delta_{A,B}^n(X) = \Delta_{A,B}^{n-m}(\Delta_{A,B}^m(X))$, $\Delta_{A,B}^0(X) = X$, for all integers $n \geq m$, it follows that

$$(A, B) \in (m, X) - \text{left invertible} \iff (A, B) \in (n, X) - \text{left invertible}$$

for all integers $n \geq m$ [24, Proposition 2.1]. Let $[A, B] = AB - BA$ denote the commutant of the operators $A, B \in B(\mathcal{X})$. If $(A_i, B_i) \in B(\mathcal{X})$, $i = 1, 2$, are such that $[A_1, A_2] = [B_1, B_2] = 0$ and $(A_i, B_i) \in (m_i, X)$ -left invertible, then

$$\begin{aligned} (2) \quad & \Delta_{A_1 A_2, B_1 B_2}^{m_1+m_2-1}(X) = [L_{A_2} \Delta_{A_1, B_1} R_{B_2} + \Delta_{A_2, B_2}]^{m_1+m_2-1}(X) \\ &= \sum_{j=0}^{m_2-1} \binom{m_1+m_2-1}{j} (L_{A_2} R_{B_2})^{m_1+m_2-1-j} \Delta_{A_2, B_2}^j (\Delta_{A_1, B_1}^{m_1+m_2-1-j}(X)) + \\ &+ \sum_{j=m_2}^{m_1+m_2-1} \binom{m_1+m_2-1}{j} (L_{A_2} R_{B_2})^{m_1+m_2-1-j} \Delta_{A_1, B_1}^{m_1+m_2-1-j} (\Delta_{A_2, B_2}^j(X)) \\ &= 0 \end{aligned}$$

i.e., $(A_1 A_2, B_1 B_2) \in (m_1 + m_2 - 1, X)$ -left invertible [24, Remark 2.3]. Again, if $A, B, N_1 \in B(\mathcal{X})$, N_1 is n_1 -nilpotent and $[B, N_1] = 0$, then

$$\begin{aligned} (3) \quad & \Delta_{A, B+N_1}^{m+n_1-1}(X) = [\Delta_{A, B} + L_A R_{N_1}]^{m+n_1-1}(X) \\ &= \sum_{j=0}^{n_1-1} \binom{m+n_1-1}{j} (L_A R_{N_1})^j \Delta_{A, B}^{m+n_1-1-j}(X) + \\ &+ \sum_{j=n_1}^{m+n_1-1} \binom{m+n_1-1}{j} \Delta_{A, B}^{m+n_1-1-j} (L_A R_{N_1})^j(X) \\ &= 0, \end{aligned}$$

since $\Delta_{A, B}^{m+n_1-1-j}(X) = 0$ for all $j \leq n_1 - 1$ (implies $m + n_1 - 1 - j \geq m$) and $N_1^j = 0$ for all $j \geq n_1$. Again, if N_2 is an n_2 -nilpotent which commutes with A , then

$$\begin{aligned} \Delta_{A+N_2, B+N_1}^{m+n_1+n_2-2}(X) &= [\Delta_{A, B+N_1} + L_{N_2} R_{B+N_1}]^{m+n_1+n_2-2}(X) \\ &= \sum_{j=0}^{n_2-1} \binom{m+n_1+n_2-2}{j} (L_{N_2} R_{B+N_1})^j \Delta_{A, B+N_1}^{m+n_1+n_2-2-j}(X) + \\ &+ \sum_{j=n_2}^{m+n_1+n_2-2} \binom{m+n_1+n_2-2}{j} \Delta_{A, B+N_1}^{m+n_1+n_2-2-j} (L_{N_2} R_{B+N_1})^j(X) \\ &= 0, \end{aligned}$$

since $\Delta_{A, B+N_1}^{m+n_1+n_2-2-j}(X) = 0$ for all $j \leq n_2 - 1$ (implies $m + n_1 + n_2 - 2 - j \geq m + n_1 - 1$) and $N_2^j = 0$ for all $j \geq n_2$ (cf. [24]).

These results fail for (non (m, X) -left invertible) (m, X) -expansive operators.

Example 2.1. If we let $T = \alpha I$, α some complex number with modulus greater than one and I the identity of $B(\mathcal{H})$, then

$$\Delta_{T^*, T}^m(I) = \sum_{j=0}^m (-1)^j \binom{m}{j} |\alpha|^{2j} = (1 - |\alpha|^2)^m I \leq 0$$

for all odd positive integers m , hence $(T^*, T) \in (2n+1, I)$ -expansive for all non-negative integers n . However, $(T^*, T) \notin (2n, I)$ -expansive for any integer $n \geq 1$. Let $S = I$. Then, trivially, S is isometric, hence $(S^*, S) \in (2, I)$ -isometric. Trivially, $[S, T] = 0$ and $(S^* T^*, ST) \notin (2n+2, I)$ -expansive for any non-negative integer n .

Before going on to consider our next example (showing that the result on perturbation by commuting nilpotents does not extend from (m, X) -left invertible operators to (m, X) -expansive operators), we prove a generalisation of Richter's result [25, Lemma 1] that 2-expansive operators are 2-hyperexpansive. Given Hilbert space operators $A_i, P \in B(\mathcal{H})$, $i = 1, 2$, let $X_s = X_s(A_1, A_2, P)$ be the operator $X_s = \Delta_{A_1, A_2}^s(P)$ for some non-negative integer s . (Thus $X_0 = P$.) We say that the triplet (A_1, A_2, X_s) preserves order if $L_{A_1}^t R_{A_2}^t(X_s) \geq 0$ (resp., $L_{A_1}^t R_{A_2}^t(X_s) \leq 0$) for all integers $t \geq 1$ whenever $X_s \geq 0$ (resp., $X_s \leq 0$). (Indeed, what we are saying here is that an application of powers of the elementary operator $E_{A, B} = L_A R_B$ preserves the positivity, or otherwise, of our operator X_s ; it is however convenient to shorten it to "triplet (A, B, X_s) preserves order".) Order preserving triplets occur naturally. Thus, if $A, P \in B(\mathcal{H})$ with $P \geq 0$, then $A^{*t} P A^t \geq 0$ for all positive integers t ; more generally, if $A \in (m, P)$ -expansive (resp., $A \in (m, P)$ -contractive), then $A^{*t} \Delta_{A^*, A}^m(P) A^t \leq 0$ (resp., $A^{*t} \Delta_{A^*, A}^m(P) A^t \geq 0$) for all positive integers t (and the triplet $(A^*, A, \Delta_{A^*, A}^m(P))$ is an order preserving triplet). Again, if A_1, A_2 and $P \in B(\mathcal{H})$ are such that $(A_1, A_2) \in (m, P)$ -expansive for all $m \geq t$ for some positive integer t , then for all integers $s \geq t$,

$$\begin{aligned} 0 &\geq \Delta_{A_1, A_2}^{s+1}(P) = A_1 \Delta_{A_1, A_2}^s(P) A_2 - \Delta_{A_1, A_2}^s(P) \\ \implies L_{A_1}^t R_{A_2}^t \Delta_{A_1, A_2}^s(P) &\leq \Delta_{A_1, A_2}^s(P) \leq 0 \end{aligned}$$

for all positive integers t (so that the triplet $(A_1, A_2, \Delta_{A_1, A_2}^t(P))$ preserves order).

Theorem 2.2. Let $A, B, P \in B(\mathcal{H})$ be such that (A, B) is (m_0, P) -contractive for some non-negative integer m_0 (thus: $P \geq 0$ if $m_0 = 0$),

$$(A, B) \in \{(m_0 + 2, P) - \text{expansive}\} \cap \{(m, P) - \text{expansive}\}$$

for some integer $m > m_0 + 2$ and the triplet $(A, B, \Delta_{A, B}^r(P))$, $m_0 + 2 \leq r \leq m$, preserve order. Then

$$\Delta_{A, B}^r(P) \leq 0 \text{ for all } m_0 + 1 \leq r \leq m.$$

Proof. Define the operator $X \in B(\mathcal{H})$ by $X = \Delta_{A, B}^{m_0}(P)$; then $X \geq 0$. If we now let

$$\tilde{\Delta}_{A, B}(S) = (L_A R_B - I)(S) = -\Delta_{A, B}(S),$$

then

$$\begin{aligned} (A, B) \in (m_0 + 2, P) - \text{expansive} &\iff \Delta_{A, B}^{m_0+2}(P) = \Delta_{A, B}^2(X) \leq 0 \\ &\iff A^2 X B^2 - 2 \tilde{\Delta}_{A, B}(X) - X \leq 0 \\ &\iff A^2 X B^2 \leq 2 \tilde{\Delta}_{A, B}(X) + X, \end{aligned}$$

and this, since the triplet $(A, B, \Delta^2 A, B(X))$ preserves order, implies

$$\begin{aligned} 0 &\geq A^3 X B^3 - 2 A \tilde{\Delta}_{A, B}(X) B - A X B \implies A^3 X B^3 - 3 \tilde{\Delta}_{A, B}(X) - X \leq 0 \\ \implies A^3 X B^3 &\leq 3 \tilde{\Delta}_{A, B}(X) + X. \end{aligned}$$

Using an induction argument, it follows that

$$A^n X B^n \leq n \tilde{\Delta}_{A, B}(X) + X \iff \tilde{\Delta}_{A, B}(X) \geq \frac{1}{n} \{A^n X B^n - X\}$$

for all positive integers n . Since $X \geq 0$ implies $(L_A R_B)^n(X) \geq 0$, letting n go to infinity in the above we have

$$\tilde{\Delta}_{A,B}(X) \geq 0 \iff \Delta_{A,B}(X) = \Delta_{A,B}^{m_0+1}(P) \leq 0.$$

The hypothesis

$$\begin{aligned} \tilde{\Delta}_{A,B}^2(X) = \Delta_{A,B}^2(X) \leq 0 &\iff A\tilde{\Delta}_{A,B}(X)B - \tilde{\Delta}_{A,B}(X) \leq 0 \\ \implies A^2\tilde{\Delta}_{A,B}(X)B^2 - A\tilde{\Delta}_{A,B}(X)B &\leq 0 \\ \dots \\ \implies A^n\tilde{\Delta}_{A,B}(X)B^n - A^{n-1}\tilde{\Delta}_{A,B}(X)B^{n-1} &\leq 0 \end{aligned}$$

for all integers $n \geq 1$. Hence

$$A^n\tilde{\Delta}_{A,B}(X)B^n \leq A^{n-1}\tilde{\Delta}_{A,B}(X)B^{n-1} \leq \dots \leq A\tilde{\Delta}_{A,B}(X)B \leq \tilde{\Delta}_{A,B}(X) (\geq 0),$$

and $\{A^n\tilde{\Delta}_{A,B}(X)B^n\}_{n \geq 0}$ is a bounded below decreasing sequence of non-negative operators. Consequently, the sequence converges to a non-negative operator, say Q .

The hypothesis $(A, B) \in (m, P)$ -expansive also implies that

$$\Delta_{A,B}^m(P) = \Delta_{A,B}^{m-m_0}(X) = -A\Delta_{A,B}^{m-m_0-1}(X)B + \Delta_{A,B}^{m-m_0-1}(X) \leq 0;$$

hence, since the triplet $(A, B, \Delta_{A,B}^{m-m_0}(X))$ preserves order,

$$\Delta_{A,B}^{m-m_0-1}(X) \leq A\Delta_{A,B}^{m-m_0-1}(X)B \leq \dots \leq A^n\Delta_{A,B}^{m-m_0-1}(X)B^n$$

for all integers $n \geq 0$. Since

$$\begin{aligned} A^n\Delta_{A,B}^{m-m_0-1}(X)B^n &= \sum_{j=0}^{m-m_0-2} (-1)^j \binom{m-m_0-2}{j} A^{n+j}\Delta_{A,B}(X)B^{n+j} \\ &= - \sum_{j=0}^{m-m_0-2} (-1)^j \binom{m-m_0-2}{j} A^{n+j}\tilde{\Delta}_{A,B}(X)B^{n+j}, \end{aligned}$$

letting $n \rightarrow \infty$ we have

$$\lim_{n \rightarrow \infty} A^n\Delta_{A,B}^{m-m_0-1}(X)B^n = - \sum_{j=0}^{m-m_0-2} (-1)^j \binom{m-m_0-2}{j} Q = 0.$$

Already we have

$$\lim_{n \rightarrow \infty} \Delta_{A,B}^{m-m_0-1}(X) \leq \lim_{n \rightarrow \infty} A^n\Delta_{A,B}^{m-m_0-1}(X)B^n;$$

hence

$$\Delta_{A,B}^{m-m_0-1}(X) \leq 0 \iff \Delta_{A,B}^{m-1}(P) \leq 0.$$

Starting next with $\Delta_{A,B}^{m_0+2}(P) \wedge \Delta_{A,B}^{m-1}(P) \leq 0$, and repeating the argument, we obtain $\Delta_{A,B}^{m-2}(P) \leq 0$, and consequently that $\Delta_{A,B}^r(P) \leq 0$ for all $m_0 + 1 \leq r \leq m$. $\square$

An example of operators satisfying the order preservation property of Theorem 2.2 is provided by (T^*, T) , $T \in B(\mathcal{H})$, such that $(T^*, T) \in \{(m_0, P)\text{-contractive}\} \wedge \{(m_0 + 2, P)\text{-expansive}\} \wedge \{(m, P)\text{-expansive}\}$ for some operator $P \in B(\mathcal{H})$ and integers $2 \leq m_0 + 2 < m$. Then $\Delta_{T^*, T}^r(P) \leq 0$ for all $m_0 + 1 \leq r \leq m$. In particular:

Corollary 2.3. *Given operators $T, P \in B(\mathcal{H})$, $P \geq 0$, if $(T^*, T) \in \{(2, P)\text{-expansive}\} \wedge \{(m, P)\text{-expansive}\}$ for some integer $m \geq 2$, then $(T^*, T) \in (m, P)\text{-hyperexpansive}$.*

In addition to generalising the result that 2-expansive $B(\mathcal{H})$ operators are 2-hyperexpansive [18, 25], Theorem 2.2 generalises [19, Theorem 2.1] (by dispensing with the hypothesis that $(T^*, T) \in (1, P)$ -expansive – [19] considers the case $m_0 = 0$ – and by proving that (T^*, T) is m -hyperexpansive). Observe that

$$(T^*, T) \in (m, I) - \text{expansive} \iff \sum_{j=0}^m (-1)^j \binom{m}{j} \|T^j x\|^2 \leq 0$$

for all $x \in \mathcal{H}$. A generalisation of (m, P) -expansive operators to Banach space operators is obtained (following Bayart's [7] definition of Banach space m -isometric operators) as follows: $T \in B(\mathcal{X})$ is (m, P) -expansive for some operator $P \in B(\mathcal{X})$ if

$$(4) \quad T \in \Delta_p^m(P, T, x) = \sum_{j=0}^m (-1)^j \binom{m}{j} \|PT^j x\|^p \leq 0$$

for some integer $p \geq 1$ and all $x \in \mathcal{X}$ [19, 21]. A version of Theorem 2.2 holds for (m, P) -expansive Banach space operators.

The following theorem, generalising corresponding results from [19, 21], considers the case $m_0 = 0$ of Theorem 2.2. Let $T \in \Delta_p^m(P, T, x)$ denote $T \in B(\mathcal{X})$ is (m, P) -expansive in the sense of (definition) (4).

Theorem 2.4. *If an operator $T \in B(\mathcal{X})$ is such that*

$$T \in \{\Delta_p^2(P, T, x)\} \wedge \{\Delta_p^m(P, T, x)\}, \quad m > 2,$$

for some integer $p \geq 1$ and operator $P \in B(\mathcal{X})$, then $T \in \Delta_p^r(P, T, x)$ for all $1 \leq r \leq m$ (i.e., T is (m, P) -hyperexpansive in the sense of (4)).

Proof. The proof below being similar to that of Theorem 2.2, we shall be economical. If we let $\tilde{\Delta}_p(P, T, x) = -\Delta_p(P, T, x)$, then $\Delta_p^2(P, T, x) \leq 0$ implies

$$\|PT^2 x\|^p \leq 2\tilde{\Delta}_p(P, T, x) + \|Px\|^p,$$

and hence that

$$\|PT^n x\|^p \leq n\tilde{\Delta}_p(P, T, x) + \|Px\|^p.$$

This implies

$$\tilde{\Delta}_p(P, T, x) = \lim_{n \rightarrow \infty} \tilde{\Delta}_p(P, T, x) \geq \lim_{n \rightarrow \infty} \frac{1}{n} \{\|PT^n x\|^p - \|Px\|^p\} \geq 0.$$

Since

$$\|PT^n x\|^p \leq \|PT^{n-1} x\|^p \leq \dots \leq \|PTx\|^p \leq \|PT^n x\|^p$$

for all $x \in \mathcal{X}$,

$$\lim_{n \rightarrow \infty} \|PT^n x\|^p = c$$

for some scalar $c = c(x) \geq 0$. Furthermore, $\Delta_p^2(P, T, x) \leq 0$ implies $\tilde{\Delta}_p(P, T, Tx) - \tilde{\Delta}_p(P, T, x) \leq 0$ and this in turn implies

$$(0 \leq) \tilde{\Delta}_p(P, T, x) \geq \tilde{\Delta}_p(P, T, Tx) \geq \dots \geq \tilde{\Delta}_p(P, T, T^n x),$$

where

$$\tilde{\Delta}_p(P, T, T^n x) = \|PT^{n+1} x\|^p - \|PT^n x\|^p.$$

Trivially,

$$\Delta_p^m(P, T, x) = -\Delta_p^{m-1}(P, T, Tx) + \Delta_p^{m-1}(P, T, x) (\leq 0);$$

hence,

$$\Delta_p^{m-1}(P, T, x) \leq \Delta_p^{m-1}(P, T, T^n x)$$

for integers $n \geq 1$, therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} \Delta_p^{m-1}(P, T, x) &\leq \lim_{n \rightarrow \infty} \sum_{j=0}^m (-1)^j \binom{m}{j} \|PT^{n+j}x\|^p \\ &= \sum_{j=0}^m (-1)^j \binom{m}{j} c = 0, \end{aligned}$$

i.e., $\Delta_p^{m-1}(P, T, x) \leq 0$. The proof now follows from a finite repetition of the argument. $\square$

Next, we give our example on perturbation by commuting nilpotents.

Example 2.5. (i) If $T \in B(\mathcal{H})$ is an m -expansive operator, i.e. $(T^*, T) \in (m, I)$ -expansive, and $N \in B(\mathcal{H})$ is an n -nilpotent which commutes with T , then the example of the m -isometric operators dictates that, if at all then, $T+N$ is an $(m+2n-2)$ -expansive operator. Since $[T, -N] = 0$, it follows that if $T+N$ is $(m+2n-2)$ -expansive, then $T(= T+N-N)$ is $(m+4n-4)$ -expansive also. Indeed, repeating the argument, alternating between perturbation by N and $-N$, T is $(m+2ne-2e)$ -expansive for every non-negative even integer e . Assuming now that $m=2$, it follows that if T is 2-expansive, then T is either completely hyperexpansive, or, there exists no n -nilpotent operator, $n > 1$, commuting with T . This is absurd.

(ii) Again, if T is 2-expansive, then T^n is 2-expansive for all positive integers n . Suppose that T satisfies the commuting product property, i.e. assume T^2 is $(2+2-1)=3$ -expansive and T^n is $n+1$ -expansive. Then Theorem 2.2 implies that T is $(n+1)$ -hyperexpansive. This is false.

Additional hypotheses are required for results, similar to those for (m, X) -isometries, on products and perturbation by commuting nilpotents of m -expansive operators.

Given operators $A_i, B_i, X \in B(\mathcal{X})$ such that $(A_i, B_i) \in (m_i, X)$ -expansive, $1 \leq i \leq 2$, we say in the following that:

a sequence $\{X_j\}_{j=r}^{r+k} = \{X_j(A_1, A_2, B_1, B_2, X)\}_{j=r}^{k+r}$ is a $(k+1)$ -partial expansive sequence for (A_i, B_i) , $i=1$ or 2 (exclusive or), if $\Delta_{A_i, B_i}(X_j) \leq 0$ for all $r \leq j \leq r+k$.

Partial hyperexpansive sequences occur naturally. Thus, if $(A, B) \in (m, X)$ -hyperexpansive, then, with $(A_1, B_1) = (A, B)$, $(A_2, B_2) = (I, I)$,

$$\{X_j\}_{j=0}^{m-1} = \{\Delta_{A, B}^j(X)\}_{j=0}^{m-1}$$

is an m -partial hyperexpansive sequence for (A, B) . Again, if $(A, B) \in$ completely- (m, X) -hyperexpansive, then $(\Delta_{A, B}^j(X) \leq 0$ for all $j \geq 1$, hence)

$$\{X_j\}_{j=m_0}^k = \{\Delta_{A, B}^{j-m_0}(X)\}_{j=m_0}^k$$

is a $(k-m_0+1)$ -partial hyperexpansive sequence for (A, B) for all integers $m_0 \geq 0$ and $k \geq m_0$. For (m, X) -left invertible pairs (A, B) ,

$$\{X_j\}_{j=0}^k = \{\Delta_{A, B}^{m-1+j}(X)\}_{j=0}^k$$

is a $(k-m+1)$ -partial hyperexpansive sequence for (A, B) for all non-negative integers k . Finally, if A_i, B_i and X are the operators of equation (2), then the sequence

$$\{\Delta_{A_1, B_1}^{m_1+j}(X)\}_{j=0}^{m_2-1}$$

is an m_2 -partial hyperexpansive sequence for (A_2, B_2) and

$$\{\Delta_{A_2, B_2}^{m_2+j}(X)\}_{j=0}^{m_1-1}$$

is an m_1 -partial hyperexpansive sequence for (A_1, B_1) .

The following theorem considers products of commuting expansive operators.

Theorem 2.6. Given operators $A_i, B_i, P \in B(\mathcal{X})$, $i = 1, 2$, such that $[A_1, A_2] = [B_1, B_2] = 0$ and $(A_i, B_i) \in (m_i, P)$ -expansive, if

$$\{X_j\}_{j=0}^{m_1-1} = \{(L_{A_1} R_{B_1})^{m_2+j} \triangle_{A_1, B_1}^{m_1-1-j} \triangle_{A_2, B_2}^{m_2-1+j}(P)\}_{j=0}^{m_1-1}$$

is an m_1 -partial hyperexpansive sequence for (A_2, B_2) and

$$\{Y_j\}_{j=0}^{m_2-1} = \{(R_{A_1} R_{B_1})^{m_2-1-j} \triangle_{A_1, B_1}^{m_1-1+j} \triangle_{A_2, B_2}^{m_2-1-j}(P)\}_{j=0}^{m_2-1}$$

is an m_2 -partial hyperexpansive sequence for (A_1, B_1) , then $(A_1 A_2, B_1 B_2) \in (m_1 + m_2 - 1, P)$ -expansive.

Proof. The commutativity hypotheses on A_i and B_i imply that L_{A_i} and R_{B_i} commute for all $i = 1, 2$, hence

$$\begin{aligned} & \triangle_{A_1 A_2, B_1 B_2}^{m_1+m_2-1}(P) \\ &= \{I - L_{A_1 A_2} R_{B_1 B_2}\}^{m_1+m_2-1}(P) \\ &= \{I - L_{A_1} L_{A_2} R_{B_1} R_{B_2}\}^{m_1+m_2-1}(P) \\ &= \{\triangle_{A_1, B_1} + L_{A_1} \triangle_{A_2, B_2} R_{B_1}\}^{m_1+m_2-1}(P) \\ &= \sum_{j=0}^{m_1+m_2-1} \binom{m_1+m_2-1}{j} \left[\triangle_{A_1, B_1}^j (L_{A_1} \triangle_{A_2, B_2} R_{B_1})^{m_1+m_2-1-j} \right](P) \\ &= \sum_{j=0}^{m_1-1} \binom{m_1+m_2-1}{j} (L_{A_1} R_{B_1})^{m_1+m_2-1-j} \triangle_{A_2, B_2} \left[\triangle_{A_1, B_1}^j \triangle_{A_2, B_2}^{m_1+m_2-2-j} \right](P) + \\ &+ \sum_{j=m_1}^{m_1+m_2-1} \binom{m_1+m_2-1}{j} (L_{A_1} R_{B_1})^{m_1+m_2-1-j} \triangle_{A_1, B_1} \left[\triangle_{A_1, B_1}^{j-1} \triangle_{A_2, B_2}^{m_1+m_2-1-j} \right](P). \end{aligned}$$

Defining the sequences $\{X_j\}_{j=0}^{m_1-1}$ and $\{Y_j\}_{j=0}^{m_2-1}$ as in the statement of the theorem, we have

$$\begin{aligned} \triangle_{A_2, B_2} [X_j]_{j=0}^{m_1-1} &= \left[(L_{A_1} R_{B_1})^{m_2+j} \triangle_{A_1, B_1}^{m_1-1-j} \triangle_{A_2, B_2}^{m_2+j}(P) \right]_{j=0}^{m_1-1} \\ &= \left[(L_{A_1} R_{B_1})^{m_1+m_2-1-j} \triangle_{A_1, B_1}^j \triangle_{A_2, B_2}^{m_1+m_2-1-j}(P) \right]_{j=0}^{m_1-1}, \text{ and} \\ \triangle_{A_1, B_1} [Y_j]_{j=0}^{m_2-1} &= \left[(L_{A_1} R_{B_1})^{m_2-1-j} \triangle_{A_2, B_2}^{m_2-1-j} \triangle_{A_1, B_1}^{m_1+j}(P) \right]_{j=0}^{m_2-1} \\ &= \left[(L_{A_1} R_{B_1})^{m_1+m_2-1-j} \triangle_{A_1, B_1}^j \triangle_{A_2, B_2}^{m_1+m_2-1-j}(P) \right]_{j=m_1}^{m_1+m_2-1}. \end{aligned}$$

Since the sequences $\{X_j\}_{j=0}^{m_1-1}$ and $\{Y_j\}_{j=0}^{m_2-1}$ are hyperexpansive for (A_2, B_2) and (A_1, B_1) , respectively, all the entries in the final two sums above are less than or equal to zero. Hence the result. $\square$

Observe that if A_i, B_i satisfy the commutativity hypotheses of the theorem and $(A_i, B_i) \in (m_i, P)$ -left invertible, then $\{X_j\}_{j=0}^{m_2-1}$ and $\{Y_j\}_{j=0}^{m_1-1}$ are the $\{0\}$ sequence, hence $(A_1 A_2, B_1 B_2) \in (m_1 + m_2 - 1)$ -left invertible. In particular, if $A_1 = B_1^*$, $A_2 = B_2^*$, $[B_1, B_2] = 0$, $(B_i^*, B_i) \in (m_i, P)$ -isometric, then $(B_1^* B_2^*, B_1 B_2) \in (m_1 + m_2 - 1, P)$ -isometric [8, 16, 27].

If $N \in B(\mathcal{X})$ is an n -nilpotent which commutes with $B \in B(\mathcal{X})$, then

$$\begin{aligned} \triangle_{A, B+N}^{m+n-1}(P) &= \left[(I - L_A R_B) + L_A R_N \right]^{m+n-1}(P) \\ &= \left[\sum_{j=0}^{m+n-1} \binom{m+n-1}{j} \triangle_{A, B}^j (L_A R_N)^{m+n-1-j} \right](P) \\ &= \left[\sum_{j=m}^{m+n-1} \binom{m+n-1}{j} \triangle_{A, B}^j (L_A R_N)^{m+n-1-j} \right](P) \end{aligned}$$

for all $A \in B(\mathcal{X})$ (since $N^t = 0$ for all $t \geq n$ and $j \leq m-1$ implies $m+n-1-j \geq n$). Define the sequence $\{X_j\}_{j=m}^{m+n-1}$ by setting

$$\triangle_{A,B}^{m-1-j}(L_A R_N)^{n-1-j}(P) = X_j,$$

and assume that $(A, B) \in (m, P)$ -expansive with $\{X_j\}_{j=0}^{n-1}$ an n -partial hyperexpansive sequence for (A, B) . Then $(A, B+N) \in (m+n-1, P)$ -expansive. For operators $T, N \in B(\mathcal{H})$, $[N, T] = 0$ and $N^n = 0$, such that $(T^*, T) \in (m, P)$ -expansive, this takes the following form.

Theorem 2.7. *Given operators $T, N, P \in B(\mathcal{H})$ such that $[N, T] = 0$, $N^n = 0$ and $(T^*, T) \in (m, P)$ -expansive, a sufficient condition for $(T^* + N^*, T + N) \in (m+2n-2, P)$ -expansive is that*

$$\{X_j\}_{j=0}^{2n-2} = \{\triangle_{T^*, T}^{m-1+j} S^{2n-2-j}(P)\}_{j=0}^{2n-2}, \quad S = L_{N^*} R_T + (L_{T^*} + R_{N^*}) R_N,$$

is an $(2n-1)$ -partial hyperexpansive sequence for (T^*, T) .

Proof. Defining the operator S as in the statement of the theorem, we have

$$\triangle_{T^*+N^*, T+N}(P) = (\triangle_{T^*, T} + S)(P),$$

and hence

$$\triangle_{T^*+N^*, T+N}^{m+2n-2}(P) = \left[\sum_{j=0}^{m+2n-2} \binom{m+2n-2}{j} \triangle_{T^*, T}^j S^{m+2n-2-j} \right] (P).$$

Since

$$S^{m+2n-2-j} = \sum_{k=0}^{m+2n-2-j} \binom{m+2n-2-j}{k} (L_{N^*} R_T)^{m+2n-2-j-k} (L_{T^*} + R_{N^*})^k R_N^k,$$

and $N^t = 0$ for all $t \geq n$,

$$S^{m+2n-2-j} = 0 \text{ for all } k \geq n.$$

Assume now that $k \leq n-1$, and consider $L_{N^*}^{m+2n-2-j-k}$. Since $m+2n-2-j-k \geq m+n-1-j$ and $n \leq m+n-1-j$ if and only if $j \leq m-1$,

$$\begin{aligned} \triangle_{T^*+N^*, T+N}^{m+2n-2}(P) &= \sum_{j=0}^{m+2n-2} \binom{m+2n-2}{j} \triangle_{T^*, T}^j S^{m+2n-2-j}(P) \\ &= \sum_{j=m}^{m+2n-2} \binom{m+2n-2}{j} \left[\triangle_{T^*, T}^j S^{m+2n-2-j} \right] (P). \end{aligned}$$

Since

$$\{\triangle_{T^*, T}^{m-1+j} S^{2n-2-j}\}_{j=0}^{2n-2} = \{\triangle_{T^*, T}^{j-1} S^{m+2n-2-j}\}_{j=m}^{m+2n-2},$$

and

$$\triangle_{T^*, T} \left[\triangle_{T^*, T}^{j-1} S^{m+2n-2-j}(P) \right] \leq 0$$

for all $m \leq j \leq m+2n-2$, all the entries in the sum are less than or equal to zero. This completes the proof. $\square$

It is clear that

$$\triangle_{T^*, T}^j S^{m+2n-2-j}(P) = S^{m+2n-2-j} \triangle_{T^*, T}^j(P),$$

hence if T is (m, P) -isometric, then $T + N$ is $(m+2n-2, P)$ -isometric.

3 Drazin invertible (m, P) -expansive operators

A straightforward induction argument proves that

$$(a-1)^m = a^m - \sum_{j=0}^{m-1} \binom{m}{j} (a-1)^j.$$

If we let $a = L_A R_B$ (and think of 1 as the identity operator), then

$$\begin{aligned} (5) \quad \tilde{\Delta}_{A,B}^m(P) &= (L_A R_B - I)^m(P) \\ &= (L_A R_B)^m - \sum_{j=0}^{m-1} \binom{m}{j} \tilde{\Delta}_{A,B}^j(P) \\ &= (-1)^m \Delta_{A,B}^m(P) \end{aligned}$$

for all $P \in B(\mathcal{X})$. Evidently,

$$\tilde{\Delta}_{A,B}^j = L_A \tilde{\Delta}_{A,B}^{j-1} R_B - \tilde{\Delta}_{A,B}^{j-1}.$$

Hence, for all integers $n \geq m$,

$$\begin{aligned} &\sum_{j=0}^{m-1} \binom{n}{j} L_A \tilde{\Delta}_{A,B}^j R_B \\ &= \sum_{j=0}^{m-1} \binom{n}{j} \tilde{\Delta}_{A,B}^{j+1} + \sum_{j=0}^{m-1} \binom{n}{j} \tilde{\Delta}_{A,B}^j \\ &= \binom{n}{m-1} \tilde{\Delta}_{A,B}^m + \sum_{j=0}^{m-1} \binom{n+1}{j} \tilde{\Delta}_{A,B}^j. \end{aligned}$$

Consider now $(L_A R_B)^t \tilde{\Delta}_{A,B}^m(P)$, where $A, B, P \in B(\mathcal{H})$. If $t = 1$, then equation (5) implies

$$\begin{aligned} L_A R_B \tilde{\Delta}_{A,B}^m(P) &= (L_A R_B)^{m+1}(P) - \sum_{j=0}^{m-1} \binom{m}{j} L_A R_B \tilde{\Delta}_{A,B}^j(P) \\ &= (L_A R_B)^{m+1}(P) - \left[\sum_{j=0}^{m-1} \binom{m}{j} \tilde{\Delta}_{A,B}^{j+1}(P) + \sum_{j=0}^{m-1} \binom{m}{j} \tilde{\Delta}_{A,B}^j(P) \right] \\ &= (L_A R_B)^{m+1}(P) - \left[\binom{m}{m-1} \tilde{\Delta}_{A,B}^m(P) + \sum_{j=0}^{m-1} \binom{m+1}{j} \tilde{\Delta}_{A,B}^j(P) \right], \end{aligned}$$

$$\begin{aligned} (L_A R_B)^2 \tilde{\Delta}_{A,B}^m(P) &= (L_A R_B)^{m+2}(P) - \binom{m}{m-1} L_A R_B \tilde{\Delta}_{A,B}^m(P) \\ &\quad - \sum_{j=0}^{m-1} \binom{m+1}{j} \tilde{\Delta}_{A,B}^{j+1}(P) - \sum_{j=0}^{m-1} \binom{m+1}{j} \tilde{\Delta}_{A,B}^j(P) \\ &= (L_A R_B)^{m+2}(P) - \left[\sum_{k=0}^1 \binom{m+1-k}{m-1} (L_A R_B)^k \right] \tilde{\Delta}_{A,B}^m(P) - \\ &\quad - \sum_{j=0}^{m-1} \binom{m+2}{j} \tilde{\Delta}_{A,B}^j(P), \end{aligned}$$

$$\begin{aligned}
(L_A R_B)^3 \tilde{\Delta}_{A,B}^m(P) &= (L_A R_B)^{m+3}(P) - \left[\sum_{k=0}^1 \binom{m+1-k}{m-1} (L_A R_B)^{k+1} \right] \tilde{\Delta}_{A,B}^m(P) - \\
&\quad - \sum_{j=0}^{m-1} \binom{m+2}{j} L_A R_B \tilde{\Delta}_{A,B}^j(P) \\
&= (L_A R_B)^{m+3}(P) - \left[\sum_{k=0}^2 \binom{m+1-k}{m-1} (L_A R_B)^k \right] \tilde{\Delta}_{A,B}^m(P) - \\
&\quad - \sum_{j=0}^{m-1} \binom{m+3}{j} \tilde{\Delta}_{A,B}^j(P),
\end{aligned}$$

and by a finite induction argument that

$$\begin{aligned}
(L_A R_B)^{n-m} \tilde{\Delta}_{A,B}^m(P) &= (L_A R_B)^n(P) - \left[\sum_{k=0}^{n-m-1} \binom{n+1-k}{m-1} (L_A R_B)^k \right] \tilde{\Delta}_{A,B}^m(P) - \\
&\quad - \sum_{j=0}^{m-1} \binom{n}{j} \tilde{\Delta}_{A,B}^j(P)
\end{aligned}$$

for all integers $n \geq m$.

Proposition 3.1. *Given operators $A, B, P \in B(\mathcal{H})$, if the triplet $(A, B, \tilde{\Delta}_{A,B}^m(P))$ preserves order and $\tilde{\Delta}_{A,B}^m(P) \leq 0$, then*

$$(6) \quad (L_A R_B)^n(P) \leq \sum_{j=0}^{m-1} \binom{n}{j} \tilde{\Delta}_{A,B}^j(P)$$

for all integers $n \geq m$.

Proof. If the triplet $(A, B, \tilde{\Delta}_{A,B}^m(P))$ preserves order and $\tilde{\Delta}_{A,B}^m(P) \leq 0$, then inequality (6) follows from

$$\begin{aligned}
&(L_A R_B)^n(P) - \sum_{j=0}^{m-1} \binom{n}{j} \tilde{\Delta}_{A,B}^j(P) \\
&= (L_A R_B)^{n-m} \tilde{\Delta}_{A,B}^m(P) + \sum_{k=0}^{n-m-1} \binom{n+1-k}{m-1} \tilde{\Delta}_{A,B}^m(P) \\
&\leq 0.
\end{aligned}$$

□

If we now further assume in Proposition 3.1 that the triplet (A, B, P) , $P \geq 0$, preserves order, then inequality (6) implies

$$\begin{aligned}
(0 \leq) (L_A R_B)^n(P) &\leq \sum_{j=0}^{m-1} \binom{n}{j} \tilde{\Delta}_{A,B}^j(P) \\
&= \binom{n}{m-1} \tilde{\Delta}_{A,B}^{m-1}(P) + \sum_{j=0}^{m-2} \binom{n}{j} \tilde{\Delta}_{A,B}^j(P).
\end{aligned}$$

Hence

$$(0 \leq) \frac{1}{n^{m-1}} (L_A R_B)^n(P) \leq \frac{1}{n^{m-1}} \left[\binom{n}{m-1} \tilde{\Delta}_{A,B}^{m-1}(P) + \sum_{j=0}^{m-2} \binom{n}{j} \tilde{\Delta}_{A,B}^j(P) \right].$$

Since $\binom{n}{m-1}$ is of the order n^{m-1} and $\binom{n}{m-2}$ is of the order n^{m-2} for large n , letting $n \rightarrow \infty$ we have

$$(7) \quad \tilde{\Delta}_{A,B}^{m-1}(P) \geq 0 \iff (-1)^m \Delta_{A,B}^{m-1}(P) \leq 0.$$

Given invertible operators $A, B \in B(\mathcal{H})$, we say that the triplet (A, B, X_s) is doubly order preserving if the triplets (A, B, X_s) and (A^{-1}, B^{-1}, X_s) are order preserving.

It is clear from inequality (7) that if m is an odd integer and $\Delta_{A,B}^m(P) \geq 0$, i.e. $(A, B) \in (m, P)$ -contractive, then $\tilde{\Delta}_{A,B}^m(P) \leq 0$ and $\tilde{\Delta}_{A,B}^{m-1}(P) = \Delta_{A,B}^{m-1}(P) \geq 0$. The following theorem says that for invertible (m, P) -expansive pairs (A, B) satisfying a suitable order preserving property, either $(A, B) \in (m-1, P)$ -isometric or $(A, B) \in (m-1, P)$ -contractive.

Theorem 3.2. Given operators $A, B, P \in B(\mathcal{H})$ such that A and B are invertible, $P \geq 0$ and $(A, B) \in (m, P)$ -expansive, if the triplet (A, B, X) doubly preserves order for $X = \Delta_{A,B}^t(P)$, $t = 0, m-1$ and m , then

$$(A, B) \text{ and } (A^{-1}, B^{-1}) \in (m-1, P) \text{ - isometric if } m \text{ is even,}$$

$$(A, B) \text{ and } (A^{-1}, B^{-1}) \in (m-1, P) \text{ - contractive if } m \text{ is odd.}$$

Proof. We consider m even and m odd cases separately.

m even. If m is even, then

$$\Delta_{A,B}^m(P) \leq 0 \iff \tilde{\Delta}_{A,B}^m(P) \leq 0,$$

and this, since $(A, B, \Delta_{A,B}^m(P))$ and (A, B, P) preserve order, implies

$$\tilde{\Delta}_{A,B}^{m-1}(P) \geq 0 \iff \Delta_{A,B}^{m-1}(P) \leq 0$$

(see above). Since A, B are invertible and $(A^{-1}, B^{-1}, \Delta_{A,B}^m(P))$ also preserves order,

$$\Delta_{A,B}^m(P) \leq 0 \iff [(L_A R_B)^{-m} \Delta_{A,B}^m](P) \leq 0 \iff \tilde{\Delta}_{A^{-1}, B^{-1}}^m(P) \leq 0.$$

Consequently, since $(A^{-1}, B^{-1}, \Delta_{A,B}^{m-1}(P))$ preserves order,

$$\tilde{\Delta}_{A^{-1}, B^{-1}}^{m-1}(P) \geq 0 \iff [(L_{A^{-1}} R_{B^{-1}})^{m-1} \tilde{\Delta}_{A,B}^{m-1}](P) \geq 0 \iff \Delta_{A,B}^{m-1}(P) \geq 0.$$

Combining, $\Delta_{A,B}^{m-1}(P) = 0$ (which then implies $(A, B) \in (n, P)$ -isometric for all integers $n \geq m-1$).

m odd. If m is odd, and both $(A, B, \Delta_{A,B}^m(P))$ and (A, B, P) doubly preserves order, then

$$\Delta_{A,B}^m(P) \leq 0 \iff \tilde{\Delta}_{A^{-1}, B^{-1}}^m(P) \leq 0,$$

and this, since $m-1$ is even and $(A, B, \Delta_{A,B}^{m-1}(P))$ doubly preserves order, in turn implies

$$\begin{aligned} \tilde{\Delta}_{A^{-1}, B^{-1}}^{m-1}(P) \geq 0 &\iff \Delta_{A^{-1}, B^{-1}}^{m-1}(P) \geq 0 \\ &\iff \tilde{\Delta}_{A,B}^{m-1}(P) \geq 0 \iff \Delta_{A,B}^{m-1}(P) \geq 0. \end{aligned}$$

This completes the proof. □

Drazin Invertible Pairs. Recall [29] that $A \in B(\mathcal{X})$ is Drazin invertible, with Drazin inverse A_d (of order p for some positive integer p) if

$$[A_d, A] = 0, A_d^2 A = A_d, A^{p+1} A_d = A^p.$$

he Drazin inverse A_d of A , whenever it exists, is unique. The Drazin invertibility of A implies that 0 is an isolated point of the spectrum of A and there exists a direct sum decomposition

$$\mathcal{X} = A^p(\mathcal{X}) \oplus A^{-p}(\mathcal{X}) = \mathcal{X}_1 \oplus \mathcal{X}_2$$

of $\mathcal{X}$ and a direct sum decomposition

$$A = A|_{\mathcal{X}_1} \oplus A|_{\mathcal{X}_2} = A_1 \oplus A_2$$

of A , where A_1 is invertible and A_2 is p -nilpotent. It is easily seen that

$$A_d = A_1^{-1} \oplus 0 \in B(\mathcal{X}_1 \oplus \mathcal{X}_2).$$

The following theorem extends Theorem 3.2 to Drazin invertible pairs (A, B) to prove that if $A = A_1 \oplus A_2 \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$, $B = B_1 \oplus B_2 \in B(\mathcal{H}'_1 \oplus \mathcal{H}'_2)$, $(A, B) \in (m, P)$ -expansive and the triplet (A_1, B_1, X) , $X = \tilde{\Delta}_{A_1, B_1}^t(P_{11})$ for $t = m, m-1, 0$ and $P_{11} = P|_{(\mathcal{H}'_1 \rightarrow \mathcal{H}_1)}$, is doubly order preserving, then either $(A, B) \in (m-1, P)$ -isometric or $(A, B) \in (m-1, P)$ -contractive.

Theorem 3.3. *Given operators $A, B, P \in B(\mathcal{H})$ such that $P \geq 0$, A and B are Drazin invertible with Drazin inverses A_d and B_d , if $(A, B) \in (m, P)$ -expansive and the triplets (A, B, X) and (A_d, B_d, X) , $X = \tilde{\Delta}_{A, B}^t(P)$ for $t = m, m-1$ and 0, are order preserving, then $(A, B) \wedge (A_d, B_d) \in (m-1, P)$ -isometric if m is even and $(A, B) \wedge (A_d, B_d) \in (m-1, P)$ -contractive if m is odd.*

Proof. Letting A, B, A_d, B_d have the (canonical) representations

$$\begin{aligned} A &= A_1 \oplus A_2 \in B(\mathcal{H}_1 \oplus \mathcal{H}_2), \quad A_1 \text{ invertible and } A_2^p = 0, \\ B &= B_1 \oplus B_2 \in B(\mathcal{H}'_1 \oplus \mathcal{H}'_2), \quad B_1 \text{ invertible and } B_2^q = 0, \\ A_d &= A_1^{-1} \oplus 0 \in B(\mathcal{H}_1 \oplus \mathcal{H}_2), \quad B_d = B_1^{-1} \oplus 0 \in B(\mathcal{H}'_1 \oplus \mathcal{H}'_2) \end{aligned}$$

for some positive integers p and q , let $P \in B(\mathcal{H}'_1 \oplus \mathcal{H}'_2, \mathcal{H}_1 \oplus \mathcal{H}_2)$ have the matrix representation $P = [P_{ij}]_{i,j=1}^2$. Then

$$\Delta_{A, B}^m(P) = \left[\sum_{k=0}^m (-1)^k \binom{m}{k} A_1^k P_{ij} B_1^k \right]_{i,j=1}^2 \leq 0.$$

We prove that $P_{ij} = 0$ for all $i = j \neq 1$, and to this end we start by considering

$$\sum_{k=0}^m (-1)^k \binom{m}{k} A_2^k P_{22} B_2^k \leq 0.$$

Let $r = \max(p, q)$; then $A_2^r = B_2^r = 0$. Since $(A, B, \tilde{\Delta}_{A, B}^m(P))$ is order preserving, multiplying $\Delta_{A, B}^m(P)$ by A^{r-t} on the left and by B^{r-t} on the right, we have:

if $t = 1$, then

$$A_2^{r-1} \left(\sum_{k=0}^m (-1)^k \binom{m}{k} A_2^k P_{22} B_2^k \right) B_2^{r-1} \leq 0 \implies A_2^{r-1} P_{22} B_2^{r-1} \leq 0;$$

if $t = 2$, then

$$\begin{aligned} A_2^{r-2} P_{22} B_2^{r-2} - \binom{m}{1} A_2^{r-1} P_{22} B_2^{r-1} &\leq 0 \\ \implies A_2^{r-2} P_{22} B_2^{r-2} &\leq m A_2^{r-1} P_{22} B_2^{r-1} \leq 0; \end{aligned}$$

if $t = 3$, then

$$A_2^{r-3} P_{22} B_2^{r-3} - \binom{m}{1} A_2^{r-2} P_{22} B_2^{r-2} + \binom{m}{2} A_2^{r-1} P_{22} B_2^{r-1} \leq 0$$

$$\begin{aligned}
&\Rightarrow A_2^{r-3} P_{22} B_2^{r-3} \leq \binom{m}{1} A_2^{r-2} P_{22} B_2^{r-2} - \binom{m}{2} A_2^{r-1} P_{22} B_2^{r-1} \\
&\Rightarrow A_2^{r-3} P_{22} B_2^{r-3} \leq \left(m^2 - \frac{m(m-1)}{2} \right) A_2^{r-1} P_{22} B_2^{r-1} = \\
&= \binom{m+1}{2} A_2^{r-1} P_{22} B_2^{r-1} \leq 0
\end{aligned}$$

and more generally, by an induction argument, that if $t = s$, then

$$A_2^{r-s} P_{22} B_2^{r-s} \leq \binom{m+s-2}{s-1} A_2^{r-1} P_{22} B_2^{r-1} \leq 0.$$

Conclusion:

$$P_{22} \leq 0.$$

Since $P \geq 0$ implies $P_{22} \geq 0$,

$$P_{22} = 0.$$

Recall that a $B(\mathcal{H})$ operator $P = [P_{ij}]_{i,j=1}^2 \geq 0$ if and only if $P_{11}, P_{22} \geq 0$ and $P_{12}^* = P_{21} = P_{22}^{\frac{1}{2}} C P_{11}^{\frac{1}{2}}$ for some contraction C [30]. This, since $P_{22} = 0$, implies

$$P_{12} = P_{21} = 0 \Rightarrow P = P_{11} \oplus 0$$

and

$$\triangle_{A,B}^m(P) = \triangle_{A_1,B_1}^m(P_{11}) \oplus 0 \leq 0 \Rightarrow \triangle_{A_1,B_1}^m(P_{11}) \leq 0.$$

Evidently, (A, B, X) and (A_d, B_d, X) order preserving, X as in the statement of the theorem, implies (A_1, B_1, X_1) and $(A_1^{-1}, B_1^{-1}, X_1)$, where $X_1 = \tilde{\Delta}_{A_1,B_1}^t(P_{11})$ for $t = m, m-1, 0$, are order preserving. Hence (A_1, B_1, X_1) is doubly order preserving, Theorem 3.2 applies to the invertible pair $(A_1, B_1) \in (m, P_{11})$ -expansive to prove that

$$(A_1^{-1}, B_1^{-1}) \wedge (A_1, B_1) \in \begin{cases} (m-1, P_{11}) - \text{isometric if } m \text{ is even} \\ (m-1, P_{11}) - \text{contractive if } m \text{ is odd.} \end{cases}$$

Consequently, since $P = P_{11} \oplus 0$,

$$(A_d, B_d) \wedge (A, B) \in \begin{cases} (m-1, P) - \text{isometric if } m \text{ is even} \\ (m-1, P) - \text{contractive if } m \text{ is odd.} \end{cases}$$

This completes the proof. $\square$

A particularly important case of Theorem 3.2 and 3.3 is the one in which $A^* = B = T$: $(T^*, T, \tilde{\Delta}_{T^*,T}^m(P))$ then preserves order (resp., doubly preserves order if T is invertible) for all values of m . We have:

Corollary 3.4. *Let $T, P \in B(\mathcal{H})$ be such that $P \geq 0$ and $\triangle_{T^*,T}^m(P) \leq 0$. If T is invertible, respectively Drazin invertible, then*

$$(T^{*-1}, T^{-1}) \wedge (T^*, T) \in \begin{cases} (m-1, P) - \text{isometric if } m \text{ is even} \\ (m-1, P) - \text{contractive if } m \text{ is odd,} \end{cases}$$

respectively

$$(T_d^*, T_d) \wedge (T^*, T) \in \begin{cases} (m-1, P) - \text{isometric if } m \text{ is even} \\ (m-1, P) - \text{contractive if } m \text{ is odd.} \end{cases}$$

In particular, if $m = 2$ and T is Drazin invertible, then $\|P^{\frac{1}{2}} T x\| = \|P^{\frac{1}{2}} x\| = \|P^{\frac{1}{2}} T_d x\|$.

Here, the proof of the final assertion of the corollary is an immediate consequence of the fact that if $m = 2$, then $T^*PT = P = T_d^*PT_d$.

Remark 3.5. (i). The argument of the proof of Theorem 3.3 leading to the conclusion $P_{12} = P_{21} = P_{22} = 0$ for order preserving triplets $(A, B, \tilde{\Delta}_{A,B}^m(P))$ implies that $(A, B) \notin (m, I)$ -expansive for any pair (A, B) of Drazin invertible operators (satisfying this order preserving property). In particular, there does not exist a Drazin invertible operator $T \in B(\mathcal{H})$ such that $\Delta_{T,T}^m(I) \leq 0$.

(ii) If a Banach space operator $T \in B(\mathcal{X})$ is expansive in the sense of (the definition in) equality (4), i.e., if $\Delta_p^m(P, T, x) \leq 0$ for all $x \in \mathcal{X}$, then

$$(8) \quad \|PT^n x\|^p \leq \sum_{j=0}^{m-1} \binom{n}{j} \tilde{\Delta}_p^j(P, T, x)$$

and if also T is invertible, then

$$(9) \quad \Delta_p^m(P, T, x) \leq 0 \implies \begin{cases} \Delta_p^{m-1}(P, T, x) = \Delta_p^{m-1}(P, T^{-1}, x) = 0 \text{ if } m \text{ is even} \\ \Delta_p^{m-1}(P, T, x) \wedge \Delta_p^{m-1}(P, T^{-1}, x) \geq 0 \text{ if } m \text{ is odd.} \end{cases}$$

Here, for the proof of (8) one argues as in the proof of Proposition 2.1 of Bayart [7] – simply replace $\|T^k x\|^p$ by $\|PT^k x\|^p$ to prove

$$(10) \quad \tilde{\Delta}_p^m(P, T, x) = \|PT^n x\|^p - \sum_{j=0}^{m-1} \binom{n}{j} \tilde{\Delta}_p^j(P, T, x)$$

for all integers $n \geq 0$. Evidently

$$(11) \quad \begin{aligned} \sum_{j=0}^{m-1} \binom{n}{j} \tilde{\Delta}_p^j(P, T, Tx) &= \\ &= \binom{n}{m-1} \tilde{\Delta}_p^m(P, T, x) + \sum_{j=0}^{m-1} \binom{n+1}{j} \tilde{\Delta}_p^j(P, T, x) \end{aligned}$$

Combine (10) with (11), and use a limit argument similar to that used to reach inequality (7), to obtain $\tilde{\Delta}_p^{m-1}(P, T, x) \geq 0$. Since $\tilde{\Delta}_p^{m-1}(P, T, x) = (-1)^{m-1} \Delta_p^{m-1}(P, T, x)$ and $\tilde{\Delta}_p^{m-1}(P, T^{-1}, x) = (-1)^{m-1} \tilde{\Delta}_p^{m-1}(P, T, T^{-m+1}x)$, we also have that

$$(-1)^{m-1} \tilde{\Delta}_p^{m-1}(P, T^{-1}, x) = \Delta_p^{m-1}(P, T^{-1}, x) \geq 0.$$

This proves (9). The case $P = I$ of (8) and (9) was first proved by Gu [21].

The following theorem proves that Theorem 3.3 extends to Drazin invertible Banach space operators T satisfying $\Delta_p^m(P, T, x) \leq 0$.

Theorem 3.6. Given operators $P, T \in B(\mathcal{X})$ such that T Drazin invertible with Drazin inverse T_d ,

$$\Delta_p^m(P, T, x) \leq 0 \implies \begin{cases} \Delta_p^{m-1}(P, T, x) \wedge \Delta_p^{m-1}(P, T_d, x) = 0 \text{ if } m \text{ is even} \\ \Delta_p^{m-1}(P, T, x) \wedge \Delta_p^{m-1}(P, T_d, x) \geq 0 \text{ if } m \text{ is odd.} \end{cases}$$

Proof. If T is Drazin invertible with Drazin inverse T_d of index q for some integer $q \geq 1$, then there exists a decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ of $\mathcal{X}$ such that

$$\mathcal{X}_1 = T^q(\mathcal{X}), \quad \mathcal{X}_2 = T^{-q}(\mathcal{X}), \quad T = T|_{\mathcal{X}_1} \oplus T|_{\mathcal{X}_2} = T_1 \oplus T_2,$$

where T_1 is invertible and $T_2^q = 0$. Furthermore

$$T_d = T_1^{-1} \oplus 0 \in B(\mathcal{X}_1 \oplus \mathcal{X}_2).$$

Letting $P \in B(\mathcal{X}_1 \oplus \mathcal{X}_2)$ have the matrix representation $P = [P_{ik}]_{i,k=1}^2$, and letting $x = 0 \oplus x_2 \in \mathcal{X}_1 \oplus \mathcal{X}_2$, it then follows from $(\Delta_p^m(P, T, x) \leq 0$ implies) $\Delta_p^m(P, T, T^{q-1}x) \leq 0$ that

$$\begin{aligned} & \sum_{j=0}^m (-1)^j \binom{m}{j} \|PT^{j+q-1}x\|^p = \|PT^{q-1}x\|^p \\ &= \|P_{12}T_2^{q-1}x_2 \oplus P_{22}T_2^{q-1}x_2\|^p \leq 0 \\ &\iff P_{12}T_2^{q-1}x_2 = P_{22}T_2^{q-1}x_2 = 0. \end{aligned}$$

Considering next $\Delta_p^m(P, T, T^{q-2}x)$, and then repeating the process, it is seen that

$$\begin{aligned} & \|Px\|^p = \|P_{12}x_2 \oplus P_{22}x_2\|^p = 0 \\ &\iff P_{12}x_2 = P_{22}x_2 = 0. \end{aligned}$$

Since this is true for all $x_2 \in \mathcal{X}_2$, we must have

$$P_{12} = P_{22} = 0 \iff P = \begin{pmatrix} P_{11} & 0 \\ P_{21} & 0 \end{pmatrix} \in B(\mathcal{X}_1 \oplus \mathcal{X}_2).$$

Consequently, for all $x \in \mathcal{X}$,

$$\begin{aligned} \Delta_p^m(P, T, x) &= \sum_{j=0}^m (-1)^j \binom{m}{j} \|PT^jx\|^p \leq 0 \\ &\iff \sum_{j=0}^m (-1)^j \binom{m}{j} \|P(T_1 \oplus 0)^jx\|^p \leq 0 \\ &\iff \sum_{j=0}^m (-1)^j \binom{m}{j} \|P(T_1 \oplus I_2)^jx\|^p \leq 0 \\ &\quad (I_2 \text{ the identity of } B(\mathcal{X}_2)) \\ &\iff \sum_{j=0}^m (-1)^j \binom{m}{j} \|P(T_1^{-1} \oplus I_2)^{m-j}x\|^p \leq 0 \\ &\quad (\text{replace } x \text{ by } (T_1 \oplus I_2)^{-m}x) \\ &\iff \tilde{\Delta}_p^m(P, T_d, x) = \sum_{j=0}^m (-1)^j \binom{m}{j} \|PT_d^{m-j}x\|^p \leq 0 \\ &\implies \tilde{\Delta}_p^{m-1}(P, T_d, x) = (-1)^{m-1} \Delta_p^{m-1}(P, T_d, x) \geq 0 \\ &\quad (\text{see above}) \\ &\iff \tilde{\Delta}_p^{m-1}(P, T_1^{-1} \oplus I_2, x) \geq 0 \\ &\iff (-1)^{m-1} \tilde{\Delta}_p^{m-1}(P, T_1 \oplus I_2, x) \geq 0 \\ &\quad (\text{replace } x \text{ by } (T_1 \oplus I_2)^m x) \\ &\iff \Delta_p^{m-1}(P, T, x) \geq 0. \end{aligned}$$

Thus, if m is even, then

$$\Delta_p^{m-1}(P, T, x) \geq 0, \quad \Delta_p^{m-1}(P, T_d, x) \leq 0,$$

and if m is odd, then

$$\{\Delta_p^{m-1}(P, T, x)\} \wedge \{\Delta_p^{m-1}(P, T_d, x)\} \geq 0.$$

Already we have (from (7) and (8) that)

$$\begin{aligned}\tilde{\Delta}_p^{m-1}(P, T, x) \geq 0 &\iff \tilde{\Delta}_p^{m-1}(P, T_1 + I_2, x) \geq 0 \\ &\iff \Delta_p^{m-1}(P, T_1^{-1} + I_2, x) \geq 0 \\ &\iff \Delta_p^{m-1}(P, T_d, x) \geq 0,\end{aligned}$$

i.e., if m is even, then

$$\Delta_p^{m-1}(P, T, x) \leq 0, \Delta_p^{m-1}(P, T_d, x) \geq 0$$

and if m is odd, then

$$\Delta_p^{m-1}(P, T, x) \wedge \Delta_p^{m-1}(P, T_d, x) \geq 0.$$

Putting together the even, and the odd, cases, the proof follows. $\square$

Acknowledgements: It is my pleasure to thank a referee for his notes on the original version of the manuscript. His remarks asking for clarity on the “preservation of order property” have led to the current formulation of “order preserving triplets”.

Conflict of interest: No conflict of interested reported.

References

- [1] J. Agler, Hypercontractions and subnormality, *J. Operator Th.* **13**(1985), 203-217.
- [2] J. Agler and M. Stankus, m -Isometric transformations of Hilbert space I, *Integral Equat. Oper. Theory* **21**(1995), 383-420.
- [3] J. Agler and M. Stankus, m -Isometric transformations of Hilbert space II, *Integral Equat. Oper. Theory* **23**(1995), 1-48.
- [4] J. Agler and M. Stankus, m -Isometric transformations of Hilbert space III, *Integral Equat. Oper. Theory* **24**(1996), 379-421.
- [5] O.A.M. Sid Ahmed, Some properties of m -isometries and m -invertible operators in Banach spaces, *Acta Math. Sci. Ser. B English Ed.* **32**(2012), 520-530.
- [6] F. Bayart, m -isometries on Banach Spaces, *Math. Nachr.* **284**(2011), 2141-2147.
- [7] F. Botelho and J. Jamison, Isometric properties of elementary operators, *Linear Alg. Appl.* **432**(2010), 357-365.
- [8] T. Bermúdez, A. Martínón and J.N. Noda, *Products of m -isometries*, *Linear Alg. Appl.* **408**(2013) 80-86.
- [9] T. Bermúdez, A. Martínón and J.N. Noda, An isometry plus a nilpotent operator is an m -isometry, *Applications, J. Math. Anal Appl.* **407**(2013) 505-512.
- [10] T. Bermúdez, A. Martínón, V. Müller and J.N. Noda, Perturbation of m -isometries by nilpotent operators, *Abstract and Applied Analysis*, Volume 2014, Article ID 745479(6pages).
- [11] B.P. Duggal and I.H. Kim, Structure of elementary operators defining m -left invertible, m -selfadjoint and related classes of operators, *J. Math Anal Appl.* **495**(2021), 124718.
- [12] B.P. Duggal and I.H. Kim, Structure of n -quasi left m -invertible and related classes of operators, *Demonstratio Mathematica* **53**(2020), 249-268.
- [13] B. P. Duggal, Tensor product of n -isometries, *Linear Alg. Appl.* **437**(2012), 307-318.
- [14] B.P. Duggal, Tensor product of n -isometries II, *Functional Anal. Approx. and Computation* **4:1**(2012), 27-32.
- [15] B.P. Duggal, Tensor product of n -isometries III, *Functional Anal. Approx. and Computation* **4:2**(2012), 61-67.
- [16] C. Gu, Structure of left n -invertible operators and their applications, *Studia Math.* **226**(2015), 189-211.
- [17] C. Gu, Elementary operators which are m -isometric, *Linear Alg. Appl.* **451**(2014), 49-64.
- [18] A. Athavale, On completely hyperexpansive operators, *Proc. Amer. Math. Soc.* **124**(1996), 3745-3752.
- [19] O.A.M. Sid Ahmed, On $A(m, p)$ -expansive and $A(m, p)$ -hyperexpansive operators on Banach spaces II, *J. Math. Comput. Sci.* **5**(2015), 123-128.
- [20] G. Exner, I. B. Jung and C. Li, On k -hyperexpansive operators, *J. Math. Anal. Appl.* **323**(2006), 569-582.
- [21] C. Gu, On (m, p) -expansive and (m, p) -contractive operators on Hilbert and Banach spaces, *J. Math. Anal. Appl.* **426**(2015), 893-916.
- [22] S. Jung, Y. Kim, E. Ko and J.E. Lee, On (A, m) -expansive operators, *Studia Math.* **213**(2012), 3-23.
- [23] V.M. Sholapurkar and A. Athavale, *Completely and alternatively hyperexpansive operators*, *J. Operator Th.* **43**(2000), 43-68.
- [24] B.P. Duggal and I.H. Kim, Left m -invertibility by the adjoint of Drazin inverse and m -selfadjointness of Hilbert space operators, *Lin. Multilin. Algebra*. (DOI: 10.1080/03081087.2020.1793880)
- [25] S. Richter, Invariant subspaces of the Dirichlet shift, *J. Reine Angew. Math.* **81**(1988), 265-220.
- [26] C. Gu, Functional calculus for m -isometrise and related operators on Hilbert spaces and Banach spaces, *Acta Sci. Math. (Szeged)* **81**(2015), 605-641.

- [27] B.P. Duggal and V. Müller, Tensor product of left n -invertible operators, *Studia Math.* **215**(2)(2013), 113-125.
- [28] Trieu Le, Algebraic properties of operator roots of polynomials, *J. Math. Anal. Appl.* **421**(2015), 1238-1246.
- [29] D.S. Djordjevic and V. Rakocevic, *Lectures on Generalized Inverses*, Faculty of Sciences and Mathematics, University of Nis, 2008.
- [30] T. Ando, *Topics on Operator Inequalities*, Division of Applied Mathematics, Research Institute of Applied Electricity, Hokkaido University, Sapporo, Japan (1978).