Research Article

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M-hypercyclicity of C_0 -semigroup and Svp of its generator

<https://doi.org/10.1515/conop-2020-0122>

Received March 30, 2021; accepted November 12, 2021

Abstract: Let $\mathcal{T} = (T_t)_{t \geq 0}$ be a C_0 -semigroup on a separable infinite dimensional Banach space X , with generator A . In this paper, we study the relationship between the single valued extension property of generator A , and the M -hypercyclicity of the C_0 -semigroup. Specifically, we prove that if A does not have the single valued extension property at $\lambda \in i\mathbb{R}$, then there exists a closed subspace M of X , such that the C_0 -semigroup $\mathcal{T}$ is M -hypercyclic. As a corollary, we get certain conditions of the generator A , for the C_0 -semigroup to be M -hypercyclic.

Keywords: Hypercyclicity, M -Hypercyclicity, C_0 -semigroup, svp, Topologically transitive.

MSC: 47A19, 47A10, 47B48, 47D03

1 Introduction

A continuous linear operator T on a separable Banach space X is called hypercyclic if there exists a vector $x \in X$ which, means that $\text{orb}(T, x) := \{T^n x, n \in \mathbb{N}\}$ is dense in X . Such a vector is called a hypercyclic vector of T . Using Baire's theorem, it can be shown that an operator T on a separable Banach space is hypercyclic if and only if, T is topologically transitive i.e, for every two non-empty open sets U, V of X , there is a natural n such that $T^n(U) \cap V \neq \emptyset$. In [3] Ansari proved that the powers of a hypercyclic operator are also hypercyclic. Moreover, they share the same hypercyclic vector with T . Recall that Ansari[3] showed that every infinite dimensional separable complex Banach space admits a hypercyclic operator. For more details about hypercyclic operators, see [6], and [14].

A one parametre family $\mathcal{T} = (T_t)_{t \geq 0}$ of continuous linear operators in $B(X)$ is a strongly continuous semigroup (or C_0 -semigroup) of operators in $B(X)$ if:

1. $T_0 = I$, the identity operator on X ;
2. $T_t \cdot T_s = T_{t+s}$, for all $t, s \geq 0$;
3. $\lim_{t \rightarrow s} T_t x = T_s x$ for all $s \geq 0$ and $x \in X$.

In 1997, Desch, Schappacher and Webb [9] introduced the hypercyclicity of C_0 -semigroups. A C_0 -semigroup $\mathcal{T} = (T_t)_{t \geq 0}$ is said to be hypercyclic if there exists a vector $x \in X$ such that $\text{orb}(\mathcal{T}, x) := \{T_t x, t \geq 0\}$ is dense in X . Several specific examples of hypercyclic C_0 -semigroups have been studied, see for example [9, 15]. In [23], Bermudez, Bonilla and Martinon proved that every separable infinite dimensional complex Banach space admits a hypercyclic C_0 -semigroup. As for single operators, it can be shown that a C_0 -semigroup $\mathcal{T} = (T_t)_{t \geq 0}$ on a separable Banach space is hypercyclic if and only if it is topologically transitive, i.e for every two non-empty open sets U, V of X , there exists $t > 0$ such that $T_t(U) \cap V \neq \emptyset$. In [7] Conejero, Peris and Müller proved that $\mathcal{T} = (T_t)_{t \geq 0}$ is hypercyclic if and only if for all $t > 0$; T_t , is also hypercyclic.

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For an arbitrary C_0 -semigroup $\mathcal{T} = (T_t)_{t \geq 0}$ on a Banach space X , it can be shown that $Ax = \lim_{t \rightarrow 0} \frac{T_t x - x}{t}$ exists on a dense subspace of X , the set of these x is the domain of A , denoted by: $D(A) = \{x \in X / \lim_{t \rightarrow 0} \frac{T_t x - x}{t} \text{ exists}\}$. Then A is called the infinitesimal generator of $\mathcal{T}$. Moreover $T_t A x = A T_t x$, for all $x \in D(A)$ and $t \geq 0$. Another important property is proved by the point spectral mapping theorem for semigroups (see [20]), if X is a complex Banach space, then for every $x \in X, \lambda \in \mathbb{C}; Ax = \lambda x$ which implies that $T_t x = e^{t\lambda} x$ for all $t \geq 0$.

Note: $X_0(\mathcal{T}) = \{x \in X / \lim_{t \rightarrow +\infty} T_t x = 0\}$;

$X_\infty(\mathcal{T}) = \{x \in X / \forall \epsilon > 0 \exists y \in X, \exists t > 0; \|y\| < \epsilon \text{ and } \|T_t y - x\| < \epsilon\}$

$X_{\infty,0}(\mathcal{T}) = \{x \in X / \forall \epsilon > 0 \exists y \in X, \exists t > 0 \text{ such that } \|y\| < \epsilon \text{ and } T_t y = x\}$.

Observe that: $X_{\infty,0}(\mathcal{T}) \subset X_\infty(\mathcal{T})$, and in the definition of $X_\infty(\mathcal{T})$ we can take t arbitrarily large. Let us fix $x \in X_{\infty,0}(\mathcal{T})$, for any $n \in \mathbb{N}^*$, we can find $y_n \in X$ and $t_n > 0$ such that $\|y_n\| < \frac{1}{n}$ and $\|T_{t_n} y_n - x\| < \frac{1}{n}$. In [9], the following result is proved:

Theorem 1.1. *If $\mathcal{T} = (T_t)_{t \geq 0}$ is a C_0 -semigroup on a separable Banach space X , and if X_0 and X_∞ are dense in X , then $\mathcal{T}$ is hypercyclic.*

Remark 1.2. *In the theorem above is only a sufficient conditions, if $\mathcal{T}$ is hypercyclic then $X_\infty = X$, but there are hypercyclic C_0 -semigroups with $X_0 = \{0\}$ (see example 4.10 [9])*

In [9], the authors related the hypercyclicity of a C_0 -semigroup to its infinitesimal generator. A C_0 -semigroup $\mathcal{T} = (T_t)_{t \geq 0}$ generated by a linear operator A in the Banach space X is hypercyclic provided that there exists a selection of eigenvectors x_λ of A that is analytic in some open set of the complex plan which meets the imaginary axis.

Theorem 1.3. [9] *Let X be a complex separable Banach space, and $\mathcal{T} = (T_t)_{t \geq 0}$ be a C_0 -semigroup generated by A . Assume that there exists an open connected subset G of $\mathbb{C}$, and analytic function $f : G \rightarrow X$, satisfying the following conditions:*

1. $G \cap i\mathbb{R} \neq \emptyset$;
2. $f(\lambda) \in \ker(A - \lambda I)$ for all $\lambda \in G$;
3. for any $\phi \in X^*$, if $\langle f(\lambda), \phi \rangle = 0$ for all $\lambda \in G$, then $\phi = 0$.

Then $\mathcal{T}$ is hypercyclic.

And we recall the following result due to [9].

Theorem 1.4. *If $\mathcal{T} = (T_t)_{t \geq 0}$ is a hypercyclic C_0 -semigroup on a separable Banach space X , generated by A , then $\sigma_p(A^*) = \emptyset$, where $\sigma_p(A)$ is the point spectrum of A .*

2 Main Results

Throughout this work, let X be an infinite dimensional separable complex Banach space, M be a non-zero closed subspace of X , and $B(X)$ denote the set of bounded linear operator from X to X .

In 2015 the authors in [21], inspired by the corresponding notion for operators given by Madore and Martinez-Avendano [17], introduced the notion of M -hypercyclicity for C_0 -semigroups. Subspace hypercyclicity has attracted the attention of several authors, and many important results have been obtained on the topic. The references [4], [8], [13], [16], [19] and [24] contain the most relevant results on M -hypercyclicity for operators.

Definition 2.1. [21] *Let $\mathcal{T} = (T_t)_{t \geq 0}$ be a C_0 -semigroup and M is a nonzero subspace of X . $\mathcal{T}$ is called M -hypercyclic if there is a vector x of X such that $\text{Orb}(\mathcal{T}, x) \cap M$ is dense in M with $\text{Orb}(\mathcal{T}, x) = \{T_t x : t \geq 0\}$.*

The following result, inspired by proposition 4-6 in [17], was shown in [21].

Proposition 2.2. [21] If $\mathcal{T} = (T_t)_{t \geq 0}$ is M -hypercyclic, then $\ker(T_t^* - \lambda I) \subset M^\perp$ for all $t \geq 0$ and $\lambda \in \mathbb{C}$.

In [7], the authors proved that if $\mathcal{T} = (T_t)_{t \geq 0}$ is a hypercyclic C_0 -semigroup then T_t also is, and $\sigma_p(T_t^*) = \emptyset$ for all $t > 0$, but in the case of an M -hypercyclicity C_0 -semigroup, the point spectrum of T_t^* may be empty or not, (see [22]).

Let G be a connected open set of $\mathbb{C}$, and we denote by $\mathcal{A}(G, X)$ the set of analytic function from G to X . For any $f \in \mathcal{A}(G, X)$ and $z_0 \in G$ we have $f(z) = \sum_{n \geq 0} a_n(z - z_0)^n$ for all $z \in D(z_0, r)$ with $r > 0$ and $a_n = \frac{f^{(n)}(z_0)}{n!}$ for all $n \in \mathbb{N}$. We will consider the following subspaces of X :

$$\mathcal{O}_f(z_0) = \overline{\text{vect}\{a_n; n \in \mathbb{N}\}} \text{ and } \mathcal{M}_f(G) = \overline{\text{vect}\{f(z); z \in G\}}$$

Proposition 2.3. [5] Let $f \in \mathcal{A}(G, X)$, and $z_0 \in G$, then: $\mathcal{O}_f(z_0) = \mathcal{M}_f(G)$

Now we can state our main result.

Theorem 2.4. Let $\mathcal{T} = (T_t)_{t \geq 0}$ be a C_0 -semigroup generated by the operator A . Assume that there exists a connected open subset G of $\mathbb{C}$, and a non zero function $f \in \mathcal{A}(G, X)$ such that:

1. $G \cap i\mathbb{R} \neq \emptyset$;
 2. $\forall \lambda \in G; f(\lambda) \in \ker(A - \lambda I)$;
- then $\mathcal{T}$ is M -hypercyclic with $M = \mathcal{M}_f(G)$.

To prove this result, we need the following lemmas.

Lemma 2.5. Under the same assumptions of the previous theorem we have:

1. $\overline{\text{vect}\{f(z); z \in G, \text{Re}(z) < 0\}} \subseteq X_0(\mathcal{T})$
2. $\overline{\text{vect}\{f(z), z \in G; \text{Re}(z) > 0\}} \subseteq X_{\infty,0}(\mathcal{T}) \subseteq X_\infty(\mathcal{T})$

Proof: Let $z \in G$ such that $\text{Re}(z) < 0$, we have:

$\lim_{t \rightarrow +\infty} \|T_t f(z)\| = \lim_{t \rightarrow +\infty} \|e^{tz} f(z)\| = \lim_{t \rightarrow +\infty} e^{t\text{Re}(z)} \|f(z)\| = 0$ this imply that $f(z) \in X_0(\mathcal{T})$ for all $z \in G$ with $\text{Re}(z) < 0$, then $\overline{\text{vect}\{f(z); z \in G, \text{Re}(z) < 0\}} \subseteq X_0(\mathcal{T})$.

On the other hand let $x \in \overline{\text{vect}\{f(z), z \in G; \text{Re}(z) > 0\}}$, then $x = \sum_{i=1}^n c_i f(z_i)$ with $c_i \in \mathbb{C}$ and $\text{Re}(z_i) > 0$. Put

$y_t = \sum_{i=1}^n e^{-tz_i} c_i f(z_i)$ since $\|y_t\| = \left\| \sum_{i=1}^n e^{-tz_i} c_i f(z_i) \right\| \leq \sum_{i=1}^n e^{-t\text{Re}(z_i)} \|c_i f(z_i)\| \rightarrow 0$, we have $\|y_t\| \rightarrow_{t \rightarrow +\infty} 0$, then for all $\epsilon > 0$ there exists $t > 0$ such that $\|y_t\| < \epsilon$ and:

$$T_t(y_t) = T_t\left(\sum_{i=1}^n c_i e^{-tz_i} f(z_i)\right) = \sum_{i=1}^n c_i e^{-zt} T_t f(z_i) = \sum_{i=1}^n c_i e^{-zt} e^{zt} f(z_i) = \sum_{i=1}^n c_i f(z_i) = x.$$

Then for all $\epsilon > 0$ there exists $t > 0$ such that $\|y_t\| < \epsilon$ and $T_t(x) = y_t$.

Then $x \in X_{\infty,0}(\mathcal{T}) \subseteq X_\infty(\mathcal{T})$, consequently $\overline{\text{vect}\{f(z), z \in G; \text{Re}(z) > 0\}} \subseteq X_{\infty,0}(\mathcal{T}) \subseteq X_\infty(\mathcal{T})$. ■

Lemma 2.6. Under the same assumptions of the previous theorem, and suppose that $X_0(\mathcal{T}) \cap M$ and $X_\infty(\mathcal{T}) \cap M$ are dense in M , and $T_t(M) \subseteq M$ for all $t \geq 0$ then $\mathcal{T}$ is M -hypercyclic.

Proof: Since M is invariant by $\mathcal{T}$, then we denote by $\tilde{\mathcal{T}} = (\tilde{T}_t)_{t \geq 0}$ with $\tilde{T}_t := T_t|_M : M \rightarrow M$, if we prove that $\tilde{\mathcal{T}}$ is hypercyclic then $\mathcal{T}$ is M -hypercyclic, by theorem 1.1 it show that $X_0(\tilde{\mathcal{T}})$ and $X_\infty(\tilde{\mathcal{T}})$ dense in M , since $\overline{\text{vect}\{f(z); z \in G, \text{Re}(z) < 0\}} \subseteq X_0(\mathcal{T})$ and $\overline{\text{vect}\{f(z), z \in G; \text{Re}(z) > 0\}} \subseteq X_\infty(\mathcal{T})$. ■

Proof of theorem 2.4: Immediately comes from both lemmas.

Remark 2.7. $\mathcal{M}_f(G)$ is a closed subspace of X . Moreover it is different to $\{0\}$ and $\mathcal{T}$ -invariant i.e: $T_t(\mathcal{M}_f(G)) \subseteq \mathcal{M}_f(G)$ for all $t \geq 0$. Indeed it is clear that $\mathcal{M}_f(G) = \overline{\text{vect}\{f(z); z \in G\}}$ is closed, and since $f \neq 0$ then $\mathcal{M}_f(G) \neq \{0\}$, let $c_{z_i} \in \mathbb{C}$, with $i = 1, 2, \dots, n$, we have:

$$T_t\left(\sum_{i=1}^n c_{z_i} f(z_i)\right) = \sum_{i=1}^n c_{z_i} T_t f(z_i) = \sum_{i=1}^n c_{z_i} e^{z_i t} f(z_i) \in \mathcal{M}_f(G).$$

Definition 2.8. [18] Let T be a closed operator on a Banach space X . We say that T has the single valued extension property (SVEP) at $\lambda_0 \in \mathbb{C}$, if for an arbitrary open neighborhood U of λ_0 , $f = 0$ is the only analytic function $f : U \rightarrow X$ such that $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in U$. We say that T has the SVEP if T has this property at every $\lambda \in \mathbb{C}$. For more details about this property see, [1] and [12].

The SVEP of operators was first introduced by Dunford to investigate the class of spectral operators which is another important generalization of normal operators (see [11]).

Remark 2.9. [1]

1. $\sigma_p(T)$ does not cluster at $\lambda_0 \implies T$ has the SVEP at λ_0 .
2. $\text{int}(\sigma_p(T)) = \emptyset \implies T$ has SVEP.

Proposition 2.10. Let A be a generator of a C_0 -semigroup $\mathcal{T}$.

A does not have the SVEP at λ_0 if and only if there exists $r > 0$, and analytic function $f \neq 0$ such that $Af(\lambda) = \lambda f(\lambda)$ for all $\lambda \in D(\lambda_0, r)$.

Proposition 2.11. [12] Let A be a closed operator on a Banach space X .

If $\lambda_0 \in \sigma_p(A)$ and $A - \lambda_0 I$ is surjective, then A does not have the SVEP at λ_0 .

Theorem 2.12. Let $\mathcal{T}$ be a C_0 -semigroup generated by A . If A does not have the SVEP at λ_0 for some $\lambda_0 \in i\mathbb{R}$, then there exists a closed subspace M of X such that $\mathcal{T}$ is M -hypercyclic.

Proof: If A does not have the SVEP at λ_0 , then by proposition 2.10 there exists $r > 0$, and an analytic function $f \neq 0$ such that $Af(\lambda) = \lambda f(\lambda)$ for all $\lambda \in D(\lambda_0, r)$, then by theorem 2.4 $\mathcal{T}$ is M -hypercyclic with $M = M_f(D(\lambda_0, r))$. ■

Corollary 2.13. Let $\mathcal{T}$ be a C_0 -semigroup generated by A .

If $\lambda_0 \in \sigma_p(A)$, and $A - \lambda_0 I$ is surjective for some $\lambda_0 \in i\mathbb{R}$, then there exists M a closed subspace of X such that $\mathcal{T}$ is M -hypercyclic.

Remark 2.14. [2] If X is a Hilbert space, and A be a closed linear operator on X .

1. A does not have SVEP at 0 if and only if there $x_0 \in \text{Ker}(A) \setminus \{0\}$ such that $\sigma_A(x_0) = \emptyset$, with $\sigma_A(x_0)$ is the local spectrum of A at x .
2. A does not have SVEP at λ_0 if and only if $\text{Ker}(A - \lambda_0 I) \cap X_A(\emptyset) \neq \{0\}$ if and only if there $x \in \text{Ker}(A - \lambda_0 I) \setminus \{0\}$ such that $\sigma_A(x) = \{\lambda_0\}$ if and only if $\text{Ker}(A - \lambda_0) \cap K(A - \lambda_0) \neq \{0\}$, with $K(A)$ is the analytical core of A . ($K(A) = \{x \in D(A) : \exists (u_n) \subset D(A), r > 0 / u_0 = x, Ax_{n+1} = x_n, \|u_n\| \leq r^n \|x\|, \forall n \in \mathbb{N}\}$).

The following corollaries can be proved by making use of remark 2.14.

Corollary 2.15. Let $\mathcal{T}$ be a C_0 -semigroup generated by A on a Hilbert space X . If there $x_0 \in \text{Ker}(A) \setminus \{0\}$ such that $\sigma_A(x_0) = \emptyset$, then $\mathcal{T}$ is M -hypercyclic for some closed subspace M of X .

Corollary 2.16. Let $\mathcal{T}$ be a C_0 -semigroup generated by A on a Hilbert space X . If $\text{Ker}(A - \lambda_0 I) \cap X_A(\emptyset) \neq \{0\}$ or there $x \in \text{Ker}(A - \lambda_0 I) \setminus \{0\}$ such that $\sigma_A(x) = \{\lambda_0\}$ or $\text{Ker}(A - \lambda_0) \cap K(A - \lambda_0) \neq \{0\}$, then $\mathcal{T}$ is M -hypercyclic for some closed subspace M of X .

Example 2.17. Let $X = l^p(\mathbb{N}) = \{x = (x_n)_{n \geq 0} / \sum_{n \geq 0} |x_n|^p < \infty\}$ with $1 \leq p < \infty$. Put $A = \alpha B$ with $\alpha \in \mathbb{C} \setminus \{0\}$ and $B : X \rightarrow X$ defined by: $B(x_n) = (x_{n+1})$. We have A is surjective and $0 \in \sigma_p(A)$, then the C_0 -semigroup generated by A is M -hypercyclic for some closed subspace M of X .

Problem 2.18. Duggal [10], in theorem 2-6, showed that polaroid operators with the SVEP are not weakly hypercyclic. The natural question is: what can be said for C_0 -semigroups and subspace hypercyclicity, when SVEP for the generator is present at values in the imaginary axis?

Acknowledgement: The author would like to thank the anonymous reviewers who provided useful and detailed comments on a previous version of the manuscript, and especially their proposal for problem 2-18.

Conflict of interest: Author state no conflict of interest.

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