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Paatero's $V(k)$ space II

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Abstract: In this article we continue our investigation of the Paatero space. We prove that the intersection of every Paatero class $V(k)$ with every Hardy space H^p is closed in that H^p and associate singular continuous measures to elements of $V(k)$. There have been no examples in the literature of functions in $V(k)$ with zeros in the unit disk other than the one at the origin. We close this gap in the literature. We derive a representation of the measure associated to a function in $V(k)$ for functions whose derivatives are rational, or algebraic, or transcendental functions in the unit disk.Finally, we consider the notion of regulated domains, introduced by Pommerenke and show that there are regulated domains whose boundary is not of bounded boundary rotation.

Keywords: Paatero class, H^p spaces, geometric function theory

MSC: 30C45, 30C15, 30H10

1 Introduction.

Let f be an analytic function in the unit disk $\mathbb{D}$ and let $f'(z) \neq 0$ for all $z \in \mathbb{D}$, $f(0) = 0$ and $f'(0) = 1$. Let

$$\left\{1 + z\frac{f''(z)}{f'(z)}\right\} = q_f(z) = q(z) \tag{1.1}$$

and

$$h(z) \equiv \Re q(z).$$

The function $h(z)$ represents the direction angle of the forward tangent to the curve C_r, where C_r is the image of each circle $|z| = r < 1$ under f [9, p.43 and p.27]. Hence, if there is a $k \geq 2$ such that

$$\int_{\mathbb{T}} |h(re^{it})|\, dt \leq k\pi$$

for all $0 < r < 1$, $\mathbb{T} = \partial\mathbb{D}$, then f is said to be a function of bounded boundary rotation, and the set of such functions is known as $V(k)$.

Paatero [14] has shown that if f is in some $V(k)$ then f' can be expressed as

$$f'(z) = \exp\left\{-\frac{1}{\pi}\int_{\mathbb{T}} \log(1 - ze^{-it})d\mu(t)\right\}, \tag{1.2}$$

$z \in \mathbb{D}$, μ is a function of bounded variation, $d\mu$ is a real, signed measure and is called "the measure associated to f", satisfying

$$\int_{\mathbb{T}} d\mu(t) = 2\pi$$

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and

$$\|\mu\| = \int_{\mathbb{T}} |d\mu(t)| \le k\pi.$$

Notice that by differentiation of both sides of (1.2) with respect to z one obtains that $h(z)$ is the Poisson integral of $d\mu$.

Paatero [14] proved that $g \in V(k)$ if and only if $h \in$ harmonic h^1. (Harmonic h^1 is the space of harmonic functions h for which

$$\sup_{0<r<1} \int_{\mathbb{T}} |h(re^{it})| dm(t) < \infty.$$

For h in this space it is known $h + i\tilde{h}$ is in Bergman space L^1_a.) Positive and negative harmonic functions on the unit disk are in h^1. It is known that if $g \in V(k)$ then g has finite valence, $\mathrm{Val}(g) = 1$ if $k = 2$ and $\mathrm{Val}(g) < k/2$ if $k > 2$ [4, Theorem 1]. We refer to the classical books by P. Duren [8] and J. Garnett [10] and the more recent book by J. Mashreghi [12] for definitions and results on Hardy spaces on the unit disk.

Many early studies of the properties of $V(k)$ classes were focused on coefficient estimates and distortion properties. As early as 1917 Loewner [11] obtained the sharp distortion theorem

$$\frac{(1-r)^{k/2-1}}{(1+r)^{k/2+1}} \le |f'(z)| \le \frac{(1+r)^{k/2-1}}{(1-r)^{k/2+1}}, \tag{1.3}$$

$|z| = r < 1$, for $f \in V(k)$ with equality for rotations of

$$F_k(z) = \frac{1}{k}\left\{\left[\frac{1+z}{1-z}\right]^{k/2} - 1\right\}. \tag{1.4}$$

In later years the interest shifted towards studies of $V(k)$ based on measure-theoretic methods, e. g., [4], [13], [15], as well as properties of $V(k)$ as a subset of Banach spaces, such as H^p and Hornich space [6], [1], [17].

Note. The normalization of the functions in $V(k)$ above precludes $V(k)$ being a normed vector space under the usual point-wise operations.

2 Properties of $V(k)$ as subsets of H^p.

In [2] we showed, in particular, that the H^p factorization of every f in $V(k)$ consists only of a finite Blaschke product and an outer function. Here we show that for k and p fixed each set $V(k)$ is closed in H^p and associate singular continuous measures to elements of $V(k)$.

Theorem 2.1. *Assume k and p are fixed, $f_n \in V(k) \cap H^p$, and f_n converges in H^p to a function F not identically zero. Then $F \in V(k)$.*

Proof. Let $|z| < 1$ and $L(z,t) = \log(1 - ze^{-it})$. Assume that $d\mu_n(t)$ are the Paatero measures associated to f_n by (1.2) with the corresponding properties. We may assume that $\|f_n\|_p \le 1$, $f_n \to_p F$, and that f'_n is also a normal family and therefore that it converges to F' uniformly on compacta. Further, there is a real measure $d\mu(t)$ and a subsequence of f_n, which we shall continue to label as the subsequence of f_n, with

$$d\mu_n(t) \to_{\text{weak}^\star} d\mu(t)$$

and since $L(z,t)$ is a continuous function for all z, $|z| < 1$ with $L_n(0,t) = 0$,

$$\int L(z,t)d\mu_n(t) \to_{n\to\infty} \int L(z,t)d\mu(t)$$

uniformly on compact subsets of $|z| < 1$. It follows that $\int d\mu(t) = 2\pi$ and $\|\mu\| \le k\pi$. Hence

$$\exp\{-\frac{1}{2\pi}\int L(z,t)d\mu_n(t)\} \to_{n\to\infty} \exp\{-\frac{1}{2\pi}\int L(z,t)d\mu(t)\} =_{\text{def}} G$$

uniformly on compacta. By assumption, the sequence f_n' converges uniformly to F', and a subsequence of the sequence f_n' converges uniformly to G, hence F' is identical with G. □

An interesting set of elements of $V(k)$ can be constructed by associating singular measures to elements of $V(k)$. Let α_j, $j = 1, \ldots,$ be a countable number of disjoint arcs on the unit circle $\mathbb{T}$, such that $\cup_j \alpha_j = \mathbb{T}$. Let μ_j, $j = 1, \ldots,$ be a countable number of singular finite positive measures with supports α_j, respectively, such that

$$\sum_{J=1}^{\infty} \mu_j(\alpha_j) = k\pi$$

for $2 \le k$. We also assume that the set $\{\mu_j(\alpha_j) : j = 1, \ldots\}$ can be divided into two subsets $\{\mu_{j'}(\alpha_{j'}) : j' = 1, \ldots\}$ and $\{\mu_{j"}(\alpha_{j"}) : j" = 1, \ldots\}$ such that

$$\sum_{j'} \mu_{j'}(\alpha_{j'}) - \sum_{j"} \mu_{j"}(\alpha_{j"}) = 2\pi.$$

The analytic functions

$$u_j(z) = \exp\left[-\int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z} d\mu_j(t)\right],$$

$|z| < 1, j = 1, \ldots,$ are singular analytic functions, $|u_j(z)| < 1$ for $|z| < 1$ and $|u_j(e^{it})| = 1$ a.e. on $\mathbb{T}$. In particular, $\log u_j(0) = -\mu_j(\alpha_j)$.

Define an analytic function f as the solution of the differential equation

$$1 + z\frac{f''(z)}{f'(z)} = \frac{\sum_{j'} \log u_{j'}(z) - \sum_{j"} \log u_{j"}(z)}{2\pi}$$

for $|z| < 1$ with the properties that $f(0) = f'(0) - 1 = 0$. By the construction, the function f belongs to the class $V(k)$. Explicitly, f is given by

$$f(z) = \int_0^z \exp\left\{\int_0^\zeta \frac{1}{w}\left[\frac{\sum_{j'} \log u_{j'}(w) - \sum_{j"} \log u_{j"}(w)}{2\pi} - 1\right] dw\right\} d\zeta.$$

The integral is well-defined since $\frac{1}{2\pi}[\sum_{j'} \log u_{j'}(0) - \sum_{j"} \log u_{j"}(0)] - 1 = 0$ and $\lim_{w\to 0} \frac{1}{w}\{\frac{1}{2\pi}[\sum_{j'} \log u_{j'}(w) - \sum_{j"} \log u_{j"}] - 1\} = const.$

3 A valence example.

At this time we have not seen any examples in the literature of functions f in $V(k)$ with zeros in the unit disk other that the one at the origin. In the light of the valence condition [4], [2], it is interesting to produce examples with the factorization of f having a finite Blaschke product term in its factorization. Example 1 below provides a satisfactory solution to this gap in the literature.

Example 1. Let $f(z) = \frac{e^{2\pi i \ell z}-1}{2\pi i \ell}$, where ℓ is a positive integer, thus $f(0) = f'(0)) - 1 = 0$. If s is an integer which divides ℓ without remainder and z_s is a real rational number, $z_s = \lambda/s$, where λ is an arbitrary integer such that $|z_s| < 1$ then z_s is a zero of f for all such λ. Thus f has finitely many zeros in $\mathbb{D}$, two on $\partial\mathbb{D}$, and infinitely many other zeros in $\mathbb{C} \setminus \bar{\mathbb{D}}$. Notice that since q_f is the analytic completion of the Poisson integral of $d\mu$, one obtains that

$$1 + z\frac{f''(z)}{f'(z)} = 1 + z2\pi i\ell = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+z\bar{w}}{1-z\bar{w}} d\mu(t),$$

where $w = e^{it}$ and

$$\frac{1+z\bar{w}}{1-z\bar{w}} = 1 + 2\sum_{k+1}^{\infty} z^k e^{-ikt}.$$

We want $\int_0^{2\pi} d\mu(t) = 2\pi$ and the measure $d\mu(t)$ to be real. It is clear that for $d\mu(t)$ we need $dt + \pi i\ell e^{it}dt$ in order to obtain $1 + z2\pi i\ell$, which is not real but can be adjusted to become a real measure:

$$d\mu(t) = dt + \pi i\ell(e^{it} - e^{-it})dt = dt + \pi\ell\, \mathrm{d}(e^{it} + e^{-it}) = dt + 2\pi\ell\, \mathrm{d}(\cos t).$$

Hence, using $d\mu(t) = dt + 2\pi\ell\, \mathrm{d}(\cos t)$, $\int d\mu(t) = 2\pi$ and

$$\int |d\mu(t)| = 2\pi + 2\pi\ell[| - 1 - 1| + (1 - (-1))] = 2\pi(1 + 4\ell)$$

with $k \le 2(1 + 4\ell)$. In the other direction,

$$\begin{aligned} f'(z) &= \exp\Big\{ -\frac{1}{\pi}\int_0^{2\pi} \log(1 - ze^{-it})d\mu(t)\Big\} \\ &= \exp\Big\{ -\frac{1}{\pi}\int_0^{2\pi} \log(1 - ze^{-it})[dt + \pi i\ell(e^{it} - e^{-it})dt\Big\} \\ &= \exp\Big\{ -\frac{1}{\pi}[-zi\pi\ell(2\pi)]\Big\} \\ &= e^{z2\pi i\ell}. \end{aligned}$$

Notice that by letting $\ell := e^{it_0}\ell$ and $z_j = e^{-it_0}\imath/\jmath$, $0 \le t_0 < 2\pi$, we can place the zeros along any line through the origin.

It is well-known [13], [15] that $\mu(t) = \nu(t) - \sigma(t)$, where ν and σ are nondecreasing. Also, for $f \in V(k)$ we have the representation [4, Theorem 3.1]

$$f'(z) = \frac{(s_1(z)/z)^{c_1}}{(s_2(z)/z)^{c_2}},$$

where s_1 and s_2 are normalized and starlike, and

$$c_1 = \frac{1}{2\pi}\int_0^{2\pi} d\nu(t) = \frac{k + 2}{4},$$

$$c_2 = \frac{1}{2\pi}\int_0^{2\pi} d\nu(t) = \frac{k - 2}{4}.$$

It is of interest to determine the functions s_1 and s_2 for the functions f in Example 1 with $k = 2(1+4\ell)$. Clearly,

$$c_1 = 1 + 2\ell \quad \text{and} \quad c_2 = 2\ell.$$

In order to obtain the decomposition $\mu(t) = \nu(t) - \sigma(t)$, we notice that the function $\mu(t) = t + 2\pi\ell\cos t$ has derivative $\mu'(t) = 1 - 2\pi\ell\sin t$, which is positive for $t \in [0, t_0] \cup [\pi - t_0, 2\pi]$ and is negative for $t \in [t_0, \pi - t_0]$, where $t_0 = \sin^{-1}\frac{1}{2\pi\ell}$. Thus the functions $\nu(t) = t + 2\pi\ell\cos t$ is increasing for $t \in [0, t_0] \cup [\pi - t_0, 2\pi]$ and $\sigma(t) = -(t + 2\pi\ell\cos t)$ is increasing for $t \in [t_0, \pi - t_0]$.

Therefore

$$\begin{aligned} d\mu &= d\nu - d\sigma \\ &= \{\chi_{[0,t_0]}[dt + 2\pi\ell d\cos t] + \chi_{[\pi-t_0,2\pi]}[dt + 2\pi\ell d\cos t]\} \\ &- \chi_{[t_0,\pi-t_0]}[-dt - 2\pi\ell d\cos t] \\ &= \{\chi_{[0,t_0]}[dt + \pi i\ell(e^{it} - e^{-it})dt] + \chi_{[\pi-t_0,2\pi]}[dt + \pi\ell(e^{it} - e^{-it})dt]\} \\ &- [\chi_{[t_0,\pi-t_0]}[-dt - \pi i\ell(e^{it} - e^{-it})dt], \end{aligned}$$

where χ denotes the characteristic function of an interval. Then

$$\frac{(s_1(z)/z)^{2\ell+1}}{(s_2(z)/z)^{2\ell}} = \frac{\exp\{-\frac{1}{\pi}\int_0^{2\pi}\log(1-ze^{-it})d\nu(t)\}}{\exp\{-\frac{1}{\pi}\int_0^{2\pi}\log(1-ze^{-it})d\sigma(t)\}}$$

and

$$s_1(z) = z\exp\Big\{-\frac{1}{\pi(2\ell+1)}\int_0^{2\pi}\log(1-ze^{-it})\{\chi_{[0,t_0]}[dt + \pi i\ell(e^{it}-e^{-it})dt] + \chi_{[\pi-t_0,2\pi]}[dt+\pi\ell(e^{it}-e^{-it})dt]\}\Big\},$$

$$s_2(z) = z\exp\Big\{-\frac{1}{2\ell\pi}\int_0^{2\pi}\log(1-ze^{-it})[\chi_{[t_0,\pi-t_0]}[-dt-\pi i\ell(e^{it}-e^{-it})dt]\Big\}.$$

4 Functions whose derivatives are rational, or algebraic, or transcendental functions on the unit circle.

In [2] we showed that the famous Suffridge polynomials belong to $V(k)$ and discussed some properties of the measures associated to them. We also proved that locally-univalent functions whose derivative is a rational function outside the open unit disk belong to some $V(k)$. The following is a generalization of these results.

Theorem 4.1. *If f is analytic in $\mathbb{D}$, $f(0)=0$ and $f'(0)=1$, $f'(z)\neq 0$ for all $z\in\mathbb{D}$, and such that*

$$f'(z) = C\Big\{\prod_{j=1}^{N}(z-\zeta_j)^{k_j}\Big\}\Big\{\prod_{l=1}^{M}(z-\xi_l)^{-s_l}\Big\},$$

where $|\zeta_j|=1$, $|\xi_l|=1$, and k_j and s_l are nonnegative reals such that $k_1+\ldots+k_N=n$, $l_1+\ldots+l_M=m$ and

$$C\Big\{\prod_{j=1}^{N}(-\zeta_j)^{k_j}\Big\}\Big\{\prod_{l=1}^{M}(-\xi_l)^{-s_l}\Big\} = 1.$$

since $f'(0)=1$, then $f\in V(k)$ for some $k\geq 2$, and the measure $d\mu$ associated to f is given by

$$d\mu(t) = \frac{2+n-m}{2}dt - \pi\sum_{j=1}^{J}k_j d\delta_{t_j}(t) + \pi\sum_{l=1}^{M}s_l d\delta_{t_l}(t). \tag{4.1}$$

Furthermore, (a) $k=2(n+1)$ if $2+n>m$, (b) $k=2(n+1)=2(m-1)$ if $1+n=m-1$, and (c) $k=2(m-1)$ if $m>n+2$.

Remark 1. As pointed out in Theorem 4.1 of [2], a finite number of zeros or singularities of f' outside the closed unit disk may only affect to which $V(k)$ the function belongs, since if $F'(z)=f'(z)G(z)$, where f' is as defined in the theorem and G is the term containing a finite number of zeros and/or singularities outside the closed unit disk, then

$$\frac{F''(z)}{F'(z)} = \frac{f''(z)}{f'(z)} + \frac{G'(z)}{G(z)}$$

and the term $\frac{G'(z)}{G(z)}$ consists of a finite sum of terms of the form $a/(z-b)$, $|b|>1$, which is bounded on $\bar{\mathbb{D}}$ and thus its real part is in h^1.

Remark 2. If f' is a rational function,

$$f'(z) = \frac{a_1 + \ldots + a_n z^n}{b_0 + \ldots + b_m z^m}$$

then $a_1 = b_0$ since $f'(0) = 1$, and $C = b_m/a_n$.

Proof. The function q_f defined by equation (1.1) of [2] is given by

$$q_f = 1 + z\Big\{\sum_{j=1}^{N} \frac{k_j}{z - \zeta_j} - \sum_{l=1}^{M} \frac{s_l}{z - \xi_l}\Big\}$$

for $z \in \mathbb{D}$. It follows that $f \in V(k)$ for some $k \geq 2$ by well-known argument, e. g. [2]. Similarly to our previous calculations, one obtains that

$$\begin{aligned} 1 + z\Big\{\sum_{j=1}^{N} \frac{k_j}{z - \zeta_j} - \sum_{l=1}^{M} \frac{s_l}{z - \xi_l}\Big\} \\ = 1 + \frac{n-m}{2} - \sum_{j=1}^{N} \frac{k_j}{2}\Big(\frac{1 + z\bar{\zeta}_j}{1 - z\bar{\zeta}_j}\Big) + \sum_{l=1}^{M} \frac{s_l}{2}\Big(\frac{1 + z\bar{\xi}_l}{1 - z\bar{\xi}_l}\Big). \end{aligned} \tag{4.2}$$

But the last expression can be written as

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1 + z\bar{w}}{1 - z\bar{w}} d\mu(t),$$

where $|w| = 1$ and the real measure $d\mu(t)$ is

$$d\mu(t) = \frac{2 + n - m}{2} dt - \pi \sum_{j=1}^{N} k_j d\delta_{t_j}(t) + \pi \sum_{l=1}^{M} s_l d\delta_{t_l}(t).$$

Notice that

$$\int_0^{2\pi} d\mu(t) = (2 + n - m)\pi - n\pi + m\pi = 2\pi.$$

Furthermore,

(a) if $2 + n > m$, then

$$\int_0^{2\pi} |d\mu(t)| = (2 + n - m)\pi + n\pi + m\pi = 2(n + 1)\pi;$$

(b) if $2 + n = m$ or $1 + n = m - 1$, then

$$\int_0^{2\pi} |d\mu(t)| = n\pi + m\pi = 2(n + 1)\pi = 2(m - 1)\pi;$$

(c) if $m > n + 2$. then

$$\int_0^{2\pi} |d\mu(t)| = (m - n - 2)\pi + n\pi + m\pi = 2(m - 1)\pi.$$

□

Remark 1. Of particular importance is the case when $m = 2 + n$, m and n are positive integers, and $M = N = 1$,

$$d\mu(t) = -\pi k_1 d\delta_{t_j}(t) + \pi s_1 d\delta_{t_l}(t),$$

since the functions f obtained from these measures by means of (1.2) are extreme points for the classes $V(k)$ in Hornich space [3].

Remark 2. If $n = 0$ and $m = s_1 + \ldots + s_M = 2$, then we obtain the famous Schwarz–Christoffel formula for the conformal map from the unit disk onto a polygon.

Example 1. If $f(z) = \frac{z}{1-z}$, then $f'(z) = \frac{1}{(1-z)^2}$; thus $n = 0$, $m = 2$, and $d\mu(t) = 2\pi d\delta_0(t)$.

Example 2. If $f(z) = \frac{z}{(1-z)^2}$, then $f'(z) = -\frac{1+z}{(z-1)^3}$ and $a_1/b_3 = -1$, thus $d\mu(t) = -\pi d\delta_\pi(t) + 3\pi d\delta_0(t)$.

It is natural to ask how one recovers f' from (1.1)

$$\begin{aligned} f'(z) &= \exp\Big\{-\frac{1}{\pi}\int_0^{2\pi}\log(1-ze^{-it})d\mu(t)\Big\} \\ &= \exp\Big\{-\frac{1}{\pi}[-\pi\sum_{j=1}^{N}k_j\log(1-ze^{-it_j}) + \pi\sum_{l=1}^{M}s_l\log(1-ze^{-it_l})]\Big\} \\ &= \prod_{j=1}^{N}(1-ze^{-it_j})^{k_j}\prod_{l=1}^{M}(1-ze^{-it_l})^{-s_l} \\ &= \frac{\prod_{j=1}^{N}(z-e^{it_j})^{k_j}}{\prod_{j=1}^{N}(-e^{it_j})^{k_j}}\frac{\prod_{l=1}^{M}(z-e^{it_l})^{-s_l}}{\prod_{l=1}^{M}(-e^{it_l})^{-s_l}} \\ &= \frac{a_n}{b_m}\prod_{j=1}^{N}(z-e^{it_j})^{k_j}\prod_{l=1}^{M}(z-e^{it_l})^{-s_l} \\ &= f'(z). \end{aligned}$$

5 Regulated Domains.

In [16] Pommerenke introduced the notion of *regulated domains* for a simply connected domain Ω in $\mathbb{C}$ with locally connected boundary $C = \partial\Omega$. Let f map $\mathbb{D}$ conformally onto Ω. Let $C: w(t) = f(e^{it})$, $0 \le t \le 2\pi$. Then Ω is called a regulated domain if each point on C is attained only finitely many times by f and if

$$\beta(t) = \lim_{\tau\to t^+}\arg[w(\tau) - w(t)]$$

for $w(t) \neq \infty$,

$$\beta(t) = \lim_{\tau\to t^+}\arg w(\tau) + \pi$$

for $w(t) = \infty$, exists for all t and defines a *regulated function*. A function β is regulated on $[a, b]$ if and only if [7, p.145] it can be uniformly approximated by finitely valued step functions: for every ϵ there exist $a = t_0 < t_1 < \ldots < t_n = b$ and constants $\gamma_1, \ldots, \gamma_n$ such that $|\beta(t) - \gamma_k| < \epsilon$ for $t_{v-1} < t < t_v$, $v = 1, \ldots, n$. The function β is the direction angle of the forward half tangent of C at $w(t)$.

It is known that the intersection of $V(k)$ with the set of regulated domains is non-empty [16]. Also, $V(k)$ contains non-univalent functions for $k > 4$. We will give an example of regulated domains for which C is not of bounded boundary rotation.

Example 1. Let Ω_m be a domain in $\mathbb{C}$ bounded by the curves $g_m(x) = x^m \sin\frac{\pi}{2x}$, $m > 0$, for $0 < x \le 1$, $g_m(1) = 1$, then by the segment from (0,0) to (-1,0) on the real axis, a segment from (-1,0) to (-1,1), and finally by the segment connecting the points (-1,1) and (1,1). The curves are transversed in positive direction. Also, $\lim_{x\to 0} g_m(x) = 0$. By the definition, β is the direction angle of the forward half tangent to the curves. Clearly β is regulated for all the pieces bounding Ω_m, except possibly for the part bounded by g_m. We shall show that Ω_m is regulated for all $m > 2$ and that C_m is of bounded boundary rotation for all $m > 3$.

We want to estimate the total variation of the direction angle from above and below. Let $x_{1,n} = \frac{1}{2(n+\frac{1}{2})}$ and $x_{2,n} = \frac{1}{2(n+1)}$, $n = 0, 1, \ldots$. It is clear that the maximal increase of the direction angle happens between two consecutive points $x_{1,n}$ and $x_{2,n}$ where $\sin\frac{\pi}{2x_{1,n}} = \pm 1$ and $\sin\frac{\pi}{2x_{2,n}} = 0$, in which case $\cos\frac{\pi}{2x_{1,n}} = 0$ and $\cos\frac{\pi}{2x_{2,n}} = \pm 1$.

Let $\theta_m(x)$ denote the direction angle for $g_m(x)$ at x. Then $\tan\theta_m(x) = g'_m(x) = mx^{m-1}\sin\frac{\pi}{2x} - \frac{\pi}{2}x^{m-2}\cos\frac{\pi}{2x}$. Hence $\tan\theta_m(x_{1,n}) = \frac{\pm m}{(2(n+\frac{1}{2}))^{m-1}}$ and $\tan\theta_m(x_{2,n}) = \frac{\pm\pi}{2(2n)^{m-2}}$, or $\theta_m(x_{1,n}) = \arctan\frac{\pm m}{(2(n+\frac{1}{2}))^{m-1}}$ and $\theta_m(x_{2,n}) = \arctan\frac{\pm\pi}{2(2n)^{m-2}}$.

Since $\arctan w = w - \frac{w^3}{3} + \dots$, $\arctan w < w$ and $\arctan w > w - \frac{w^3}{3}$ for $0 < w$, and with reversed inequalities when $w < 0$. It follows that an upper bound on the increase of $\theta_m(x)$ from $x_{2,n}$ to $x_{1,n}$ for $n > 0$ is

$$\begin{aligned}\left|\frac{m}{(2(n+\frac{1}{2}))^{m-1}} + \frac{\pi}{2(2n)^{m-2}}\right| &< \left|\frac{m}{(2n)^{m-1}} + \frac{\pi}{2(2n)^{m-2}}\right| \\ &= \frac{1}{(2n)^{m-2}}\left|\frac{m}{2n} + \frac{\pi}{2}\right| \\ &< \frac{1}{(2n)^{m-2}}\left|\frac{m}{2} + \frac{\pi}{2}\right|.\end{aligned}$$

Hence an upper bound for the total variation of the direction angle over g_m is

$$\left|\frac{m}{2} + \frac{\pi}{2}\right|\sum_{n=1}^{\infty}\frac{1}{(2n)^{m-2}},$$

which is finite for $m > 3$.

For the lower bound we have

$$\begin{aligned}&\left|\frac{m}{(2(n+\frac{1}{2}))^{m-1}} - \frac{1}{3}\left(\frac{m}{(2(n+\frac{1}{2}))^{m-1}}\right)^3 - \frac{\pi}{2(2n)^{m-2}}\right| \\ &= \frac{1}{(2n)^{m-2}}\left|\frac{m(2n)^{m-2}}{(2(n+\frac{1}{2}))^{m-1}} - \frac{1}{3}\left(\frac{m(2n)^{m-2}}{(2(n+\frac{1}{2}))^{m-1}}\right)^3 - \frac{\pi}{2}\right| \\ &> \frac{\pi}{2}\frac{1}{(2n)^{m-2}}.\end{aligned}$$

Hence we again require $m > 3$. Thus the boundary of Ω_m has bounded boundary rotation for $m > 3$.

The proof of the claim about regulated domains is based on Theorem 3.14 of [16], which states that a domain is regulated if and only if for every $\epsilon > 0$ there are finitely many points $0 = t_0 < t_1 < \dots < t_n = 2\pi$ such that $w(t) \neq \infty$, $|\arg[w(\tau) - w(t)] - \gamma_\nu| < \epsilon$ for $t_{\nu-1} < t < \tau < t_\nu$ and for $\nu = 1, \dots, n$ with real constants γ_ν.

Since $\arg(w(x_1) - w(x_2))$ for $x_{2,n} < x_1 < x_2 < x_{1,n}$, $n = 0, \dots$, represents the slope of a secant, it is between $\theta_m(x_{1,n})$ and $\theta_m(x_{2,n})$. It is easy to verify that for each m there exists an $N_1 = N(m)$ such that for all $n \geq N_1$ the inequality $\frac{m}{(2(n+\frac{1}{2}))^{m-1}} < \frac{\pi}{2(2n)^{m-2}}$ holds. Hence

$$|\arg(w(x_1) - w(x_2)| < \frac{\pi}{2(2n)^{m-2}},$$

holds. Clearly for every $\epsilon > 0$ there is an $N_2 = N(\epsilon)$ such that for all x, $0 < x < x_{2,n}$ with $n \geq N_2$ the inequality $x_{2,n} = \frac{\pi}{2(2n)^{m-2}} < \epsilon$ hold. Hence, the inequality

$$|\arg(w(x_1) - w(x_2)| < \epsilon$$

holds for all $n \geq \max\{N_1, N_2\}$ and for all x, $0 < x < x_{2,n}$. The remaining part of the boundary of Ω_m satisfies the hypothesis of Pommerenke's theorem. Hence Ω_m is a regulated domain for all $m > 2$.

As Pommerenke points out, the boundary curves must be quite flat. The distance between two successive zeros of g_m is

$$x_{2,n} - x_{2,n+1} = \frac{1}{2(n+1)} - \frac{1}{2(n+2)} = \frac{1}{2(n+1)(n+2)},$$

while the rise between these two zeros is $|g(x_{1,n})| = \frac{1}{(2(n+1.5))^m}$, for $n = 0, \dots$. Thus the ratio of the rise over run is

$$\frac{2(n+1)(n+2)}{(2(n+1.5))^m} < \frac{1}{2^{m-1}(n+1.5)^{m-2}}$$

for $m > 2$.

Question 1. It would be interesting to find an example of a univalent function in $V(k)$, whose image is not a regulated domain.

Question 2. It would be interesting to find (characterize) functions $f \in V(k)$ for which the range, $f(\mathbb{D})$, is not simply connected. (At this point all examples in the literature have simply connected ranges.).

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