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Weighted composition operators on Hardy–Smirnov spaces

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Abstract: Operators of type $f \to \psi f \circ \varphi$ acting on function spaces are called weighted composition operators. If the weight function ψ is the constant function 1, then they are called composition operators. We consider weighted composition operators acting on Hardy–Smirnov spaces and prove that their unitarily invariant properties are reducible to the study of weighted composition operators on the classical Hardy space over a disc. We give examples of such results, for instance proving that Forelli's theorem saying that the isometries of non–Hilbert Hardy spaces over the unit disc need to be special weighted composition operators extends to all non–Hilbert Hardy–Smirnov spaces. A thorough study of boundedness of weighted composition operators is performed.

Keywords: weighted composition operators, spaces of analytic functions

MSC: Primary 47B33, Secondary 47B32

1 Introduction

Consider a nonempty set D and a complex linear space $\mathcal{H}$ consisting of complex–valued functions on D. Given a function $\psi : D \to \mathbb{C}$, not necessarily belonging to $\mathcal{H}$, and another function φ with property

$$\varphi(D) \subseteq D\,, \tag{1.1}$$

the transform

$$T_{\psi,\varphi} f = \psi f \circ \varphi \qquad f \in \mathcal{H} \tag{1.2}$$

is necessarily linear.

We call $T_{\psi,\varphi}$ the *weighted composition operator induced by* ψ *and* φ and refer to ψ as the *weight symbol* or *weight function* inducing $T_{\psi,\varphi}$ and to φ as the *composition symbol* of $T_{\psi,\varphi}$. That composition symbol and in general, any map φ with property (1.1) will be referred to as a *selfmap* of D.

Now, if ψ is the null function, then $T_{\psi,\varphi}$ is the null operator no matter what composition symbol is used. To avoid triviality, all weighted composition operators are assumed nonzero.

The particular case $T_{1,\varphi}$ deserves attention. These operators are called *composition operators with symbol* φ (or *induced* by φ), and denoted C_φ rather than $T_{1,\varphi}$, since they act by composition to the right with the fixed selfmap φ; that is

$$C_\varphi f = f \circ \varphi \qquad f \in \mathcal{H}.$$

An important remark is noting that a composite (or "product" in operator theory terminology) of weighted composition operators is again: a weighted composition operator. More formally:

Remark 1. *Assume T_{ψ_1,φ_1} and T_{ψ_2,φ_2} are weighted composition operators and the range of T_{ψ_2,φ_2} is contained in the domain of T_{ψ_1,φ_1}, then*

$$T_{\psi_1,\varphi_1} T_{\psi_2,\varphi_2} = T_{\psi_1(\psi_2\circ\varphi_1),\varphi_2\circ\varphi_1} \tag{1.3}$$

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The operator $T_{\psi,\varphi}$ might be bounded or not. Nevertheless, in this paper, we will be interested only in the case when such operators act between complete normed spaces $\mathcal{H}$ and $\mathcal{K}$ consisting of functions, where the point evaluation functionals, that is functionals of type

$$\hat{x}(f) = f(x) \qquad f \in \mathcal{H} \tag{1.4}$$

are bounded. It is an easy exercise left to the reader to prove that, in such spaces, norm-convergence implies pointwise convergence. The consequence is:

Remark 2. *If $T_{\psi,\varphi}$ acts from $\mathcal{H}$ into $\mathcal{K}$ with the above property, then $T_{\psi,\varphi}$ is bounded. In particular, $T_{\psi,\varphi}$ is bounded when acting on $\mathcal{H}$, that is when $T_{\psi,\varphi}\mathcal{H} \subseteq \mathcal{H}$.*

Indeed, since norm–convergence implies pointwise convergence in both $\mathcal{H}$ and $\mathcal{K}$ then, if $T_{\psi,\varphi}\mathcal{H} \subseteq \mathcal{K}$, $f_n \to f$, and $T_{\psi,\varphi}f_n \to g$ in $\mathcal{K}$, it follows that

$$f_n(z) \to f(z) \quad \text{and hence} \quad \psi(z)f_n \circ \varphi(z) \to \psi(z)f \circ \varphi(z)$$

for all values of the variable z, and so, $g = \psi f \circ \varphi$ i.e. $T_{\psi,\varphi}$ has closed graph and is therefore bounded.

Hilbert spaces $\mathcal{H}$ consisting of functions on some set D are called *reproducing kernel Hilbert spaces* (RKHS) if having property (1.4). The reason for this denomination is the existence of the special functions k_z, $z \in D$, which have property

$$f(z) = \langle f, k_z \rangle \quad z \in D,\ f \in \mathcal{H}, \tag{1.5}$$

called the *reproducing property*. The functions k_z are referred to as the *reproducing kernel functions* of $\mathcal{H}$.

Weighted composition operators can be studied on any space of functions defined on the same set but in this paper, we consider only Hardy–Smirnov spaces. Here is what we mean.

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc. Let γ be a conformal isomorphism of $\mathbb{U}$ onto G, a nonempty simply connected domain other than $\mathbb{C}$. Riemann's conformal equivalence theorem guarantees the existence of such γ and for that reason, we will call γ a *Riemann map* of $\mathbb{U}$ onto G. For each $0 < p < \infty$, the Hardy–Smirnov space $H^p(G)$ is by definition the collection of all functions f analytic in G which satisfy the condition

$$\sup_{0<r<1} \left(\int_{\Gamma_r} |f(z)|^p \, |dz| \right)^{1/p} < +\infty, \tag{1.6}$$

where, for each r, Γ_r is the image under γ of the circle of radius r about the origin. Although condition (1.6) seems to produce spaces that depend on the conformal isomorphism γ, it is shown in [10, Theorem 10.1] that, for each $0 < p < +\infty$, $H^p(G)$ depends only on G. For $p \geq 1$, $H^p(G)$ is a Banach space where the functionals in (1.4) are bounded, even a Hilbert space if $p = 2$. The notation $H^\infty(G)$ denotes the space of bounded analytic functions on G, a Banach space when endowed with the supremum norm

$$\|f\|_\infty = \sup\{|f(z)| : z \in G\}.$$

If $G = \mathbb{U}$, condition (1.6) looks as follows

$$\|f\|_p := \sup_{0<r<1} \left(\int_{\mathbb{T}} |f(ru)|^p \, dm(u) \right)^{1/p} < \infty\,, \tag{1.7}$$

where dm denotes the normalized arclength measure $d\theta/2\pi$ on the unit circle $\mathbb{T} = \partial\mathbb{U}$.

Norm $\|\cdot\|_2$ is a a Hilbert norm computable by the alternate formula

$$\|f\|_2 = \sqrt{\sum_{n=0}^{\infty} |c_n|^2} < +\infty, \quad f(z) = \sum_{n=0}^{\infty} c_n z^n. \tag{1.8}$$

Given equality (1.8) (see [10] for a proof), $H^2(\mathbb{U})$ that is the Hilbert Hardy space of $\mathbb{U}$, can be viewed as the collection of all analytic functions in $\mathbb{U}$ having square summable Maclaurin coefficients. A theorem of Fatou, eventually extended by F. and M. Riesz (see [10]) says that $H^p(\mathbb{U})$–functions have radial limits a.e. on the unit circle. The radial limit–function of any $f \in H^p(\mathbb{U})$ will be denoted by f as well. It is known that $f \in L^p_{\mathbb{T}}(dm)$ and the L^p–norm is the same as the $H^p(\mathbb{U})$–norm of the original function f. As customary, we will rely on the context to distinguish between $f \in H^p(\mathbb{U})$ and its radial limit–function.

A last fact we want to recall about the space $H^2(\mathbb{U})$ is that it is a RKHS with kernel functions $K_z(w) = \frac{1}{1-\bar{z}w}$, $w \in \mathbb{U}$. Recall also, that a *nonnegative kernel* in the sense of Aronszajn–Moore, (or positive semidefinite *reproducing kernel*, called sometimes positive semidefinite *Moore matrix*) is a complex valued map $K(x, y)$ on some set S with the property

$$\sum_{i,j=1}^{n} K(x_i, x_j)c_i\overline{c_j} \geq 0 \qquad n \geq 1, c_i \in \mathbb{C}, x_i \in S, i = 1, 2, \dots, n. \tag{1.9}$$

A basic reference for the terminology nonnegative kernels and the properties of those maps is [2], a paper where one can see that nonnegative kernels are related in a unique way to RKHS; namely, if H is a RKHS consisting of functions on some set S and having kernel functions k_x, $x \in S$, then $K(x, y) = \langle k_y, k_x\rangle$, $x, y \in S$ is a *nonnegative kernel* on S called the *nonnegative kernel of H* and all nonnegative kernels are generated that way. The notation $K \circ \varphi$, where φ is any selfmap of S, designates the nonnegative kernel $K \circ \varphi(x, y) := K(\varphi(x), \varphi(y))$, $x, y \in S$. Finally, if $K_1(x, y)$ and $K_2(x, y)$ are nonnegative kernels on some set S, the notation $K_1 \leq K_2$, means that $K_2 - K_1$ is a nonnegative kernel on S. We recalled all these statements for the benefit of Subsection 3.3 where the currently known general boundedness criteria for weighted composition operators on $H^2(\mathbb{U})$ are reviewed. The adjoints of bounded weighted composition operators map kernel functions of RKHS to scaled kernel functions as follows

$$T^\star_{\psi,\varphi} k_w = \overline{\psi(w)} k_{\varphi(w)}. \tag{1.10}$$

Equation (1.10) will be used multiple times in this paper and referred to as the *Caughran–Schwartz* equation [23, Theorem 5].

This paper contains original results, easier proofs of known results, and monographic presentation of some results on weighted composition operators. Its structure is as follows.

In the current section, we set up the notation, present the main concepts, and outline the content of the next sections. In Section 2, we prove that Hardy – Smirnov spaces are isometrically equivalent to the classical Hardy spaces over the open unit disc, via invertible isometries which are weighted composition operators (Theorem 1). We exemplify the utility of this remark by showing that Forelli's theorems saying that the isometries of non–Hilbert Hardy spaces over the open unit disc, must be special composition operators, extend to arbitrary Hardy–Smirnov spaces (Theorems 2 and 3). The characterization of isometric weighted composition operators on Hilbert Hardy–Smirnov spaces is also contained by Theorems 2 and 3. The conclusion of Section 2 is that several isometrically invariant properties of weighted composition operators on one Hardy–Smirnov space or another, are reducible to the study of the same properties on the classical Hardy spaces over a disc. To exemplify, we characterize the invertible and Fredholm weighted composition operators acting on Hilbert Hardy–Smirnov spaces (Theorem 4). Theorems 2–4 are original. Theorem 1 was first proved in [19, Proposition 1]. It was also formulated and proved on particular simply connected domains in [34] and [21]. In Section 3 we perform a study of boundedness in $H^2(\mathbb{U})$, the classical Hilbert Hardy space. We review old results, namely Theorems 6, 7, giving them new proofs based on Theorem 8 (which is also known, but is included with a new proof). We prove that the invariance of Beurling subspaces (i.e. the invariant subspaces of M_z the multiplication operator induced by the coordinate function), under composition operators is equivalent to the boundedness of certain, associated, weighted composition operators (Theorem 12). Condition (3.15) in Theorem 12 is new. The importance of such results is, among other things, the fact that classical theorems of function theory can be understood “operatorially" by the action of composition operators on special Beurling subspaces (see [9], [25], and the comments in Subsection 3.2 in this paper). We also discuss the case of multiplication operators on arbitrary Hilbert Hardy–Smirnov spaces in Theorem 5, obtaining results known in the case of particular spaces, but not in the frame of arbitrary Hilbert Hardy–Smirnov spaces. Theorem 9 is original. It relates the norm of a weighted composition operator on H^2 to the Denjoy–Wolff point of its composition symbol. The Denjoy–Wolff point of an analytic selfmap of the disc is a point where the angular derivative exists. The interesting fact that the weight symbol of a bounded weighted composition operator whose composition symbol has a finite angular derivative at some $\omega \in \mathbb{T}$ is bounded in all angular approach regions with vertex at ω is known. We included it with a new proof in Theorem 11 for the sake of producing a full collection of results on the boundedness of weighted composition operators on H^2. Section 3 contains two known results where the boundedness of weighted composition operators is characterized in terms of Carleson measures Theorem 13, respectively in terms of nonnegative kernels Theorem 14. They were included without proofs, to keep this paper reasonably short. The same is the case of Theorem 16 which contains a known characterization of the boundedness of weighted composition operators on H^2 in terms of integral transforms associated to their symbols. The case when the composition symbol is an inner function is considered in Theorem 15, which is known. The boundedness condition contained by that theorem is combined with Theorem 16 to prove Theorem 17 where condition (3.26) is new.

2 Sufficience of the Hardy spaces over discs

We will use the short notation H^p rather that $H^p(\mathbb{U})$ for the Hardy–Smirnov spaces over $\mathbb{U}$, calling them the classical Hardy spaces. They are sufficient to understand many properties of weighted composition operators on all Hardy–Smirnov spaces by the following result.

Theorem 1. *Given any Hardy–Smirnov space $H^p(G)$, $p \geq 1$, over any simply connected domain $G \neq \mathbb{C}$, a fixed Riemannn map γ of G onto $\mathbb{U}$, any analytic map $\tilde{\psi}$ on G, and any analytic selfmap ϕ of G, there exist surjective isometries V and V^{-1}, inverse to each other, acting between spaces $H^p(G)$ and H^p so that*

$$T_{\tilde{\psi},\phi} = V^{-1} T_{\psi,\varphi} V \tag{2.1}$$

where

$$\varphi = \gamma^{-1} \circ \phi \circ \gamma \tag{2.2}$$

is the conformal conjugate of ϕ via the Riemann map γ and

$$\psi = \tilde{\psi} \circ \gamma \left(\frac{\gamma'}{\gamma' \circ \varphi} \right)^{1/p}. \tag{2.3}$$

Proof. By [10], the weighted composition operator

$$V = T_{(\gamma')^{1/p},\gamma}$$

is a surjective isometry of H^p onto $H^p(G)$. A routine calculation left to the reader shows that

$$V^{-1} = T_{1/((\gamma')^{1/p} \circ \gamma^{-1}), \gamma^{-1}}.$$

and hence (2.1) holds, as the reader can easily check by using (1.3). □

Two Banach space operators T and S are called similar if there is some invertible operator V so that $V^{-1}SV = T$. If V is a surjective isometry, we will say that T and S are isometrically *similar*. Theorem 1 emphasizes the fact that all properties of weighted composition operators which are invariant under isometric similarity (also known as unitary equivalence in the case of Hilbert spaces), can be transferred from H^p to $H^p(G)$ with reasonable effort.

To exemplify, let us note that Forelli's theorems [11] identifying the isometries of H^p, $p \neq 2$, as special weighted composition operators extend to all Hardy–Smirnov spaces. More formally, let $P(z, u)$ denote the usual Poisson kernel that is

$$P(z, u) = \operatorname{Re} \frac{u + z}{u - z} = \frac{1 - |z|^2}{|u - z|^2} \qquad u \in \mathbb{T}, z \in \mathbb{U}.$$

Denote by $\mathcal{L}$ the σ–algebra of Lebesgue measurable subsets of $\mathbb{T}$ and recall [11, Theorem 1], saying that, for $1 \le p < \infty$, $p \ne 2$, the isometries of H^p are the operators V of the form $V = T_{\psi,\varphi}$ where φ is an inner function and ψ is an H^p–function with the property

$$\int_X |\psi|^p \, dm = \int_X \frac{dm(u)}{P(\varphi(0), \varphi(u))} \qquad X = \varphi^{-1}(Y) \qquad Y \in \mathcal{L}. \tag{2.4}$$

An *inner function* is a selfmap of $\mathbb{U}$ whose radial limits are unimodular a.e. on $\mathbb{T}$. This result combines with Theorem 1 into proving the following.

Theorem 2. *An operator W on $H^p(G)$, $p \ne 2$ is isometric if and only it it is a weighted composition operator of a special type; namely let the symbols ψ, $\tilde{\psi}$, ϕ, φ, γ, and V be as in Theorem 1, then W is an isometry of $H^p(G)$ if and only if $W = T_{\tilde{\psi},\phi}$ where $\psi \in H^p$, φ is inner, and ψ and φ satisfy condition (2.4).*

If $p = 2$ and so, $H^2(G)$ is a Hilbert space, then the above requirements on φ and ψ (i.e. $\psi \in H^2$ and φ is inner), plus condition (2.4) with $p = 2$ characterize the isometric weighted composition operators on $H^2(G)$, but there are of course isometries of $H^2(G)$ which are not weighted composition operators.

Proof. The conclusion follows immediately by Theorem 1, [11, Theorem 1] (respectively [24, Theorem 5] in the case $p = 2$), and the fact that two isometrically equivalent operators are simultaneously isometric (meaning that one of those operators is an isometry if and only if the other one is isometric). □

In [11], Forelli also addressed the case of invertible isometries on H^p, $p \ne 2$, and Bourdon and Narayan [6] completed that work by treating the case H^2, see also [24].

Theorem 3. *Use the same notation as in the previous theorem. For $1 \le p < \infty$, $p \ne 2$, the surjective isometries of $H^p(G)$ are the operators $W = T_{\tilde{\psi},\phi}$ induced by conformal automorphisms ϕ of G, paired with weight symbols $\tilde{\psi}$ with property*

$$\psi = \lambda \left(\varphi'\right)^{\frac{1}{p}} \tag{2.5}$$

where $|\lambda| = 1$. If $p = 2$, then the above conditions with $p = 2$ characterize the unitary weighted composition operators on $H^2(G)$.

Proof. According to [11, Theorem 2], for $1 \le p < \infty$, $p \ne 2$, the surjective isometries of H^p are the operators V of the form

$$V = T_{\lambda(\varphi')^{\frac{1}{p}},\varphi} \tag{2.6}$$

where φ is a disc automorphism and $|\lambda| = 1$. The above statement combines with Theorem 1, and the fact that isometricaly equivalent operators are simultaneously isometric, respectively invertible, into establishing the conclusion of the current theorem when $p \ne 2$.

In the case $p = 2$ the authors of [6] actually proved that $T_{\psi,\varphi}$ is unitary on H^2 if and only if φ is a disc automorphism and $\psi = cK_q/\|K_q\|_2$. This author observed and proved in [24] that the above characterization is equivalent to taking $p = 2$ in (2.6).

□

Now, the exact form of the above isometries in the case of a specific simply connected domain G, requires the choice of a suitable Riemann map transforming $\mathbb{U}$ into G and some extra computational work. To exemplify, please see this job done by this author in the case when G is a half–plane [28].

Most of the results about weighted composition operators, valid in $H^2(G)$, are provable and valid in $H^p(G)$, $p \ge 1$. For simplicity, in the following, we will take $p = 2$.

Since invertible isometries were mentioned and it turned out that their composition symbols need to be conformal automorphisms, let us note that this is the case of all invertible weighted composition operators on all Hardy–Smirnov spaces. More formally, using the notation in Theorem 1:

Theorem 4. *A weighted composition operator $T_{\tilde{\psi},\phi}$ on some Hardy–Smirnov space $H^2(G)$ is invertible if and only if ϕ is a conformal automorphism of G and the analytic map $\psi = \tilde{\psi} \circ \gamma \left(\frac{\gamma'}{\gamma' \circ \varphi}\right)^{1/2}$ is both bounded and bounded away from zero on G. A weighted composition operator on some Hardy–Smirnov space $H^2(G)$ is Fredholm if and only if it is invertible.*

Proof. The invertibility criterion above was proved in the case of the space H^2 in [14, Theorem 2.0.1]. For an arbitrary Hardy–Smirnov space $H^2(G)$ and weighted composition operator $T_{\tilde{\psi},\phi}$ on that space, we resort to Theorem 1 and, using the same notation, observe that the similar operator $T_{\psi,\varphi}$ is invertible if and only if $T_{\tilde{\psi},\phi}$ is invertible, a fact that happens if and only if ψ is both bounded and bounded away from zero, and φ is a disc automorphism. The map φ is a disc automorphism if and only if the conformally conjugated map ϕ is an analytic automorphism of G, which ends the discussion of invertibility.

The fact that a composition operator on $H^2(G)$ is Fredholm if and only if it is invertible was proved by this author [28, Theorem 2.3] in the case when G is a half–plane. That fact is valid in all Hardy–Smirnov spaces by Theorem 1 and the fact that, if two operators are unitarily equivalent, the first operator is Fredholm if and only if the second is Fredholm. □

The class of (unweighted) composition operators on H^2 transfers via Theorem 1 to $H^2(G)$ producing a class of weighted composition operators on $H^2(G)$ and proving that, all Hilbert Hardy–Smirnov spaces support weighted composition operators which are: bounded, compact, belong to the Schatten–vonNeumann ideals, are invertible, Fredholm, or normal, since composition operators with the aforementioned properties exist on H^2.

Nevertheless, occasionally unweighted composition operators may fail to belong to one or more of the operator classes of Hilbert space operators we mentioned above; for instance, if G is a half–plane, then $H^2(G)$ does not support compact composition operators [22] but compact weighted composition operators on that space do exist [28].

3 Boundedness

In this section we summarize some well known results relating to the following question: "When is a weighted composition operator on H^2 bounded?", that is [26, Problem 18]. We put together both known facts, most of the times with possibly new, short proofs, and new facts. We acknowledge, the best we can, everybody's contribution in chronological order. In select cases, we mention known results without proof, if the original one is too large to include and we could not produce a shorter proof. The purpose of this section is to serve as a minimal primer for researchers interested in the boundedness of weighted composition operators on H^2. As outlined in the previous section, the study of boundedness of weighted composition operators on H^2 transfers in a standard way to the boundedness of weighted composition operators on any Hardy–Smirnov space $H^2(G)$.

3.1 Multipliers approach

All composition operators on H^2 are bounded, a fact that can be established by considering Littlewood's subordination principle [10], which says that all composition operators C_φ induced by analytic selfmaps φ of $\mathbb{U}$ fixing the origin are contractions (that is have property $\|C_\varphi\| \le 1$), then extend this property with little effort [32] to

$$\|C_\varphi\| \le \sqrt{\frac{1 + |\varphi(0)|}{1 + |\varphi(0)|}} \tag{3.1}$$

an upper bound valid for all φ, attained by all φ fixing the origin, by Littlewood's principle, and the simple fact that $C_\varphi 1 = 1$. In the case $\varphi(0) \neq 0$, the upper bound in (3.1) is attained if and only if φ is inner [33].

Then, the standard upper norm bound for a weighted composition operator on H^2 is

$$\|T_{\psi,\varphi}\| \le \|\psi\|_\infty \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}. \tag{3.2}$$

The above norm estimate is an immediate consequence of (3.1) and the fact that a multiplication operator

$$M_\psi f = \psi f \qquad f \in H^2$$

is bounded if and only if $\|\psi\|_\infty < \infty$ in which case $\|M_\psi\| = \|\psi\|_\infty$.

Now, multiplication operators exist on all Hardy–Smirnov spaces $H^2(G)$ and are the particular weighted composition operators $T_{\psi,\varphi}$ induced by ψ analytic on G and $\varphi(z) = z$, $z \in G$. As we mentioned, in the case of H^2, they are bounded if and only if $\|\psi\|_\infty < \infty$. This fact and many other known properties of multiplication operators on H^2 can be extended to arbitrary Hardy–Smirnov spaces by Theorem 1 as follows:

Theorem 5. *Let $H^2(G)$ be a Hilbert Hardy–Smirnov space and $\tilde{\psi}$ an analytic map on G, then $M_{\tilde{\psi}}$, the multiplication operator on $H^2(G)$, has the following properties.*

$M_{\tilde{\psi}}$ is bounded if and only if $\|\tilde{\psi}\|_\infty < \infty$, and in that case, $\|M_{\tilde{\psi}}\| = \|\tilde{\psi}\|_\infty$

The spectrum $\sigma(M_{\tilde{\psi}})$ of $M_{\tilde{\psi}}$ is the closure $\overline{\tilde{\psi}(G)}$ of the range of $\tilde{\psi}$ and hence the spectral radius is $r(M_{\tilde{\psi}}) = \|\tilde{\psi}\|_\infty$

The numerical range of $M_{\tilde{\psi}}$ is the singleton $\{p\}$ if $\tilde{\psi} = p$ is constant, respectively the interior of the convex hull of $\overline{\tilde{\psi}(G)}$ if ψ is not constant.

Proof. The properties above are known in the case $G = \mathbb{U}$ [35]. For the lesser known numerical range, see [18]. Now, using the same notations as in Theorem 1, one has that φ is the identity map of G and hence ϕ is the identity map of $\mathbb{U}$. Thus $M_{\tilde{\psi}}$ is unitarily equivalent with a multiplication operator on H^2, namely M_ψ and so, all statements in this theorem follow by that fact. □

Returning to the boundedness of $T_{\psi,\varphi}$ on H^2, let us recall that a necessary condition for it, is $\psi \in H^2$, an immediate consequence of the obvious equality $T_{\psi,\varphi}1 = \psi$. It is convenient to assume $\|\psi\|_\infty < \infty$, since in that case, $T_{\psi,\varphi}$ would be bounded for all selfmaps φ of $\mathbb{U}$, yet that assumption is too restrictive, since, to give a popular example, not all isometric weighted composition operators on H^2 would be described that way.

On the other hand, Littlewood's Subordination Principle will not extend from unweighted to general weighted composition operators since, given any analytic selfmap φ of $\mathbb{U}$, there are choices of ψ making operator $T_{\psi,\varphi}$ on H^2, a contraction $\left(\text{for instance } \psi = \sqrt{\frac{1-|\varphi(0)|}{1+|\varphi(0)|}}\right)$. Furthermore, if $T_{\psi,\varphi} \neq 0$ is bounded, one can scale ψ and make the scaled operator a weighted composition contraction.

Indeed, the following equation is obvious

$$\lambda T_{\psi,\varphi} = T_{\lambda\psi,\varphi} \qquad \lambda \in \mathbb{C}.$$

Thus, take $\tilde{\psi} = \frac{1}{\|\psi\|_2}\sqrt{\frac{1-|\varphi(0)|}{1+|\varphi(0)|}}\,\psi$. By (3.2), one has that $\|T_{\tilde{\psi},\varphi}\| \le 1$.

On the other hand, multiplication operators on H^2 have unitary equivalent copies in $H^2(G)$ via Theorem 1, which are multiplication operators on $H^2(G)$, as one can see in the proof of Theorem 5, but composition operators on $H^2(G)$ do not necessarily have unitary equivalent copies in H^2 which are composition operators, but weighted composition operators with non–constant weight symbol. This fact explains why all composition operators on H^2 are bounded, but only few analytic selfmaps of a half–plane G induce bounded composition operators on $H^2(G)$ [23, Theorem 15]. By the same argument, one understands why H^2 supports compact composition operators, but $H^2(G)$ does not [22, Theorem 3.1].

Given a space $\mathbb{S}$ of analytic functions on a set, the space of multipliers of $\mathbb{S}$ consists of all functions ψ inducing bounded multiplication operators M_ψ on $\mathbb{S}$ and so, with this terminology, $H^\infty(G)$ is the space of multipliers of $H^2(G)$. This terminology can be easily and naturally extended to the space of multipliers of a space into another. For the benefit of understanding when weighted composition operators are bounded, note that $T_{\psi,\varphi}$ is bounded on H^2 if and only if ψ is a multiplier of the range $R(C_\varphi)$ of C_φ into H^2. This is a consequence of the closed range principle and a valuable observation of the author of [3].

More formally, consider:

$$\mathcal{M}(C_\varphi) := \{\psi \in H^2 : \psi R(C_\varphi) \subseteq H^2\}.$$

By Remark 2, given an analytic map $\varphi : \mathbb{U} \to \mathbb{U}$, we have that $T_{\psi,\varphi}$ is bounded on H^2 if and only if $\psi \in \mathcal{M}(C_\varphi)$. By our considerations above,

$$H^\infty \subseteq \mathcal{M}(C_\varphi) \subseteq H^2. \tag{3.3}$$

The case when the first containment in (3.3) is an equality is found in [3], and reproved in [8] and [23].

Theorem 6. *The equality $\mathcal{M}(C_\varphi) = H^\infty$ occurs if and only if φ is a finite Blaschke product.*

By finite Blaschke product, we mean a product of finitely many disc automorphisms. The author of [3] takes care of the second containment in (3.3) as follows:

Theorem 7. *The equality $\mathcal{M}(C_\varphi) = H^2$ holds if and only if $\|\varphi\|_\infty < 1$.*

The above result is reproved in [12, Theorem 2.2]. It can be obtained as a consequence of a more general fact, namely the big O–version of a little o–principle proved in [23].

In order to establish that fact, a first step we take, is to note that a norm–bounded family of H^2–functions is uniformly bounded on compacta (a property shared by all RKHS whose kernel functions are uniformly bounded on compacta). More formally, what we mean is:

Lemma 1. *Let $\mathcal{F} \subseteq H^2$ be such that, there is $M > 0$ so that*

$$\|f\|_2 \le M \qquad f \in \mathcal{F}.$$

Given K a compact subset of $\mathbb{U}$, there is some $M_K > 0$ so that

$$\sup\{|f(z)| : z \in K\} \le M_K \qquad f \in \mathcal{F}.$$

Proof. If $K = \{0\}$, one can take $M_K = M$, since $\sup\{|f(z)| : z \in K\} = |f(0)| \le \|f\|_2 \le M$, for all $f \in \mathcal{F}$. Otherwise, consider $z \in K$ arbitrary and fixed. By the Cauchy–Schwarz inequality and the reproducing property, one can write

$$|f(z)| = |\langle f, K_z\rangle| \le \|f\|_2 \, \|K_z\|_2 =$$
$$\frac{\|f\|_2}{\sqrt{1-|z|^2}} \le \frac{M}{\sqrt{1-|\mathrm{d}(K,\mathbb{T})|^2}} := M_K,$$

where $\mathrm{d}(K,\mathbb{T}) = \inf\{|z-u| : z \in K,\ u \in \mathbb{T}\}$. □

For any measurable $E \subseteq \mathbb{T}$ and any $\psi \in H^2$, the notation $\|\psi|E\|_\infty$ designates the essential supremum of the restriction of ψ to E. By E^c we mean $\mathbb{T} \setminus E$. Given a fixed $0 < r < 1$ and an analytic selfmap φ of $\mathbb{U}$, let $E_r(\varphi) := \{u \in \mathbb{T} : |\varphi(u)| \le r\}$. With this notation we prove:

Theorem 8. *If there are $0 < r < 1$, φ an analytic selfmap of $\mathbb{U}$, and $\psi \in H^2$ so that $\|\psi|E_r^c(\varphi)\|_\infty < \infty$, then $T_{\psi,\varphi}$ is bounded.*

Proof. In Lemma 1, take $\mathcal{F}$ to be the closed unit ball of H^2 and $K = r\overline{\mathbb{U}}$. Then

$$|f(\varphi(u))| \le \frac{1}{\sqrt{1-r^2}} \qquad \|f\|_2 \le 1, u \in E_r(\varphi).$$

So for fixed $f \in H^2$, $\|f\|_2 \le 1$, one can write

$$\|T_{\psi,\varphi}f\|_2^2 = \int\limits_{E_r(\varphi)} |\psi(u)|^2 |f(\varphi(u))|^2 \, dm(u) + \int\limits_{E_r^c(\varphi)} |\psi(u)|^2 |f(\varphi(u))|^2 \, dm(u) \le$$

$$\frac{1}{\sqrt{1-r^2}}\int\limits_{E_r(\varphi)}|\psi(u)|^2\,dm(u)+\|\psi|E_r^c(\varphi)\|_\infty^2\int\limits_{E_r^c(\varphi)}|f(\varphi(u))|^2\,dm(u)\le$$

$$\frac{1}{\sqrt{1-r^2}}\|\psi\|_2^2+\|\psi|E_r^c(\varphi)\|_\infty^2\|C_\varphi\|^2,$$

which ends the proof, since it establishes the inequality

$$\|T_{\psi,\varphi}f\|_2\le\sqrt{\frac{1}{\sqrt{1-r^2}}\|\psi\|_2^2+\|\psi|E_r^c(\varphi)\|_\infty^2\|C_\varphi\|^2}.$$

□

As announced:

Remark 3. *Theorem 7 follows from Theorem 8.*

Indeed, if $\|\varphi\|_\infty<1$, then there is $0<r<1$ so that $E_r(\varphi)$ is the whole unit circle, and so $\|\psi|E_r^c(\varphi)\|_\infty=0<\infty$ for all $\psi\in H^2$.

The little-o version of the condition in Theorem 8 was proved first [23, Theorem 12]. The authors of [12] reproved it, as part of [12, Theorem 28]. That little-o condition implies compactness. The corresponding big-O condition contained by Theorem 8 was proved first in [12, Theorem 2.8]. We chose to include it with proof for the sake of completeness.

By the Denjoy–Wolff theorem, if φ is an analytic selfmap of $\mathbb{U}$ without fixed points, there is a remarkable point $\omega\in\mathbb{T}$, so that $\varphi_n\to\omega$ uniformly on the compact subsets of $\mathbb{U}$, where for all n, φ_n denotes the n–fold iterate of φ, that is $\varphi_n=\varphi\circ\cdots\circ\varphi$, n times. We refer to ω as the *Denjoy–Wolff point* of φ. The norm of a weighted composition operator relates to the Denjoy–Wolff point of its composition symbol as follows.

Theorem 9. *Let φ an analytic selfmap of $\mathbb{U}$ without fixed points and ω its Denjoy–Wolff point. If $T_{\psi,\varphi}$ is bounded on H^2, then* $\liminf_{z\to\omega}|\psi(z)|\le\|T_{\psi,\varphi}\|$.

Proof. Assume $\|T_{\psi,\varphi}\|:=A<\infty$. For any $\epsilon>0$ denote $N_\epsilon:=\{z\in\mathbb{U}:|z-\omega|<\epsilon\}$. Consider now any fixed $c>A$, and arguing by contradiction, assume that, for some arbitrary, fixed $\epsilon>0$ one has that $|\psi(z)|\ge c$, for all $z\in N_\epsilon$. By the Denjoy–Wolff theorem, there is a positive integer n_0 so that

$$\varphi_n(r\mathbb{T})\subseteq N_\epsilon\qquad n\ge n_0.$$

On the other hand

$$\|T_{\psi,\varphi}^{n+1}(1)\|\le A^{n+1}$$

and

$$T_{\psi,\varphi}^{n+1}(1)=\psi\psi\circ\varphi\psi\circ\varphi_2\ldots\psi\circ\varphi_n\qquad n=1,2,3,\ldots$$

Therefore, one has that

$$c^{n-n_0}\sqrt{\int\limits_{\mathbb{T}}|\psi\psi\circ\varphi\psi\circ\varphi_2\ldots\psi\circ\varphi_{n_0-1}(ru)|^2\,dm(u)}\le$$

$$\|T_{\psi,\varphi}^{n+1}(1)\|\le A^{n+1}\qquad n\ge n_0,$$

which leads to the inequality

$$\left(\frac{c}{A}\right)^{n-n_0}\sqrt{\int\limits_{\mathbb{T}}|\psi\psi\circ\varphi\psi\circ\varphi_2\ldots\psi\circ\varphi_{n_0-1}(ru)|^2\,dm(u)}\le A^{n_0+1}\qquad n\ge n_0.$$

The above inequality produces a contradiction. Indeed, let $n\to\infty$ above. One gets $\infty\le A^{n_0+1}$.

Thus, we proved by contradiction that

$$\liminf_{z\to\omega}|\psi(z)| = \sup_{\epsilon>0}\inf_{z\in N_\epsilon}|\psi(z)| \le \|T_{\psi,\varphi}\|. \tag{3.4}$$

□

Some authors, call Denjoy–Wolff point of an analytic selfmap φ of $\mathbb{U}$ its fixed point $\omega \in \mathbb{U}$ as well, if φ (not the identity map) has one, since the sequence of iterates $\{\varphi_n\}$, tends uniformly on compacts to ω (a consequence of the Schwarz lemma in classical complex analysis), with the exception of the case when φ is an elliptic disc automorphism. Therefore, the proof of Theorem 9 works in that case too, proving that inequality

$$|\psi(\omega)| \le \|T_{\psi,\varphi}\| \tag{3.5}$$

holds (since ψ is continuous at ω), in the case when φ, an analytic selfmap of $\mathbb{U}$, other than the identity or an elliptic disc automorphism, has a fixed point $\omega \in \mathbb{U}$. That fact has a simple short proof though, and the case of elliptic disc automorphisms is no exception, as we show in the following:

Proposition 1. *Let φ be an analytic selfmap of $\mathbb{U}$ other than the identity and $\psi \in H^2$. If there is $\omega \in \mathbb{U}$ so that $\varphi(\omega) = \omega$, then* (3.5) *holds.*

Proof. In the interesting case when $\|T_{\psi,\varphi}\| < \infty$, note that, by the Caughran–Schwartz equation, one can write

$$\|T^\star_{\psi,\varphi}K_\omega\|_2 = |\psi(\omega)|\|K_\omega\|_2 \le \|T_{\psi,\varphi}\|\|K_\omega\|_2.$$

□

As the authors of [12, Theorem 2.7] note, the equality in Theorem 7 is specific to a class of analytic selfmaps φ of $\mathbb{U}$ inducing trace–class composition operators. Their proof is based on intermediate results taken from [16]. So, for the benefit of the reader and rendering this paper self contained, we want to note that the fact that $T_{\psi,\varphi}$ is trace class if $\psi \in H^2$ and $\|\varphi\|_\infty < 1$ has the following short, direct proof, which also provides a trace norm estimate:

Remark 4 ([17, Lemma 5.1]). *If $\psi \in H^2$ and $\|\varphi\|_\infty < 1$, then both C_φ and $T_{\psi,\varphi}$ are trace–class* (*hence compact*) *operators, since*

$$\|T_{\psi,\varphi}\|_1 \le \frac{\|\psi\|_2}{1-\|\varphi\|_\infty}, \tag{3.6}$$

where, for all Hilbert space operators T, we use the notation $\|T\|_1 = \mathrm{tr}(\sqrt{T^\star T})$, for their trace norm.

Indeed, recall that

$$\|T\|_1 \le \sum_{n=0}^{\infty}\|Te_n\|$$

where $\{e_n\}$ is a complete orthonormal basis. Therefore one has

$$\|T_{\psi,\varphi}\|_1 \le \sum_{n=0}^{\infty}\|T_{\psi,\varphi}(z^n)\|_2 = \sum_{n=0}^{\infty}\|\psi\varphi^n\|_2 \le \sum_{n=0}^{\infty}\|\psi\|_2\|\varphi\|_\infty^n = \frac{\|\psi\|_2}{1-\|\varphi\|_\infty}.$$

It is worth remarking that equality $\mathcal{M}(C_\varphi) = H^2$ is not specific to all trace class weighted composition operators. Indeed:

Remark 5. *There exist analytic selfmaps φ of $\mathbb{U}$ inducing trace–class composition operators, with property $\|\varphi\|_\infty = 1$ and so, for such φ one has that $\mathcal{M}(C_\varphi) \ne H^2$.*

Indeed, by [31], one can consider a map φ whose range is contained in a polygon inscribed in the unit circle and such that $\|\varphi\|_\infty = 1$. By Riemann's conformal equivalence theorem in classical complex analysis, there are univalent such φ.

It is interesting to mention here the following result from the recent paper [20, Theorem 6.2]:

Theorem 10. *Given an analytic selfmap φ of $\mathbb{U}$, there are functions $\psi \in H^2$ so that $T_{\psi,\varphi}$ is bounded but not compact if and only if $\|\varphi\|_\infty = 1$.*

Relative to Theorem 6 now. It was originally proved in [3], by a rather complicated argument, simplified in [7] and [23]. For the sake of completeness and because the following result is interesting in its own right, we produce a new proof based on the following:

Proposition 2. *An analytic selfmap φ of $\mathbb{U}$ is a finite Blaschke product if and only if* $\sup\left\{\frac{1-|\varphi(z)|}{1-|z|} : z \in \mathbb{U}\right\} < \infty$.

Proof. The necessity in the equivalence above is known [23, Lemma 1]. For the sufficiency, assume $\sup\left\{\frac{1-|\varphi(z)|}{1-|z|} : z \in \mathbb{U}\right\} < \infty$, hence $\sup\left\{\frac{1-|\varphi(z)|^2}{1-|z|^2} : z \in \mathbb{U}\right\} < \infty$, and note that φ must be an inner function. Also note that, for all $\lambda \in \mathbb{T}$, one has that $\liminf_{z\to\lambda} \frac{1-|\varphi(z)|}{1-|z|} < \infty$ that is, by the Julia–Carathéodory Theorem, [23, Theorem 2], the inner function φ has a finite angular derivative at λ for all $\lambda \in \mathbb{T}$. Then, by that same theorem, φ has an angular limit $\eta \in \mathbb{T}$ at λ and

$$\sup\left\{\frac{|\eta-\varphi(z)|^2(1-|z|^2)}{1-|\varphi(z)|^2|\lambda-z|^2} : z \in \mathbb{U}\right\} < \infty \quad \text{[23, Theorem 2].} \tag{3.7}$$

Given the boundedness of $\frac{1-|\varphi(z)|^2}{1-|z|^2}$, there is some $M > 0$ so that

$$|\eta-\varphi(z)|^2 \le M|\lambda-z|^2 \qquad z \in \mathbb{U},$$

the consequence being that φ extends by continuity at λ for all $\lambda \in \mathbb{T}$ and is therefore a finite Blaschke product. □

As the reader can easily see, Theorem 6 follows as an immediate consequence of Proposition 2 and the fact that a necessary condition for the boundedness of $T_{\psi,\varphi}$ is

$$\sup\left\{\frac{|\psi(z)|^2(1-|z|^2)}{1-|\varphi(z)|^2} : z \in \mathbb{U}\right\} < \infty \quad \text{[23, Corollary 2].} \tag{3.8}$$

Given that quantity $\frac{1+|\varphi(z)|}{1+|z|}$ is both bounded and bounded away from 0, condition (3.8), is equivalent to

$$\sup\left\{\frac{|\psi(z)|^2(1-|z|)}{1-|\varphi(z)|} : z \in \mathbb{U}\right\} < \infty\,, \tag{3.9}$$

which holds in the case of a finite Blaschke product if and only if $\|\psi\|_\infty < \infty$.

Relative to all that, we want to review two facts. First, given $\omega \in \mathbb{T}$, and $M > 1$, the set

$$\Gamma_M(\omega) = \left\{z \in \mathbb{U} : \frac{|\omega-z|}{1-|z|} < M\right\} \tag{3.10}$$

is called an angular approach region with vertex at ω. A classical result of F. Riesz, says that H^2–functions ψ, have restricted limits at the vertices of almost all angular approach regions. This means that for almost all $\omega \in \mathbb{T}$ and all $M > 1$, the limit of the restriction of ψ to $\Gamma_M(\omega)$ exists. Of course, the consequence is that ψ is bounded in $\Gamma_M(\omega)$ for all $M > 1$, a fact that holds for almost all $\omega \in \mathbb{T}$. Now, H^2–functions which are not bounded in some boundary approach regions, do exist. Nevertheless:

Theorem 11. *If $T_{\psi,\varphi}$ is bounded and φ has a finite angular derivative at ω, then ψ must be bounded in all angular approach regions with vertex at ω.*

Proof. Let $\omega \in \mathbb{T}$ be a point where the angular derivative of φ exists. Then by [1], φ has a limit $\eta \in \mathbb{T}$ as $z \to \omega$ in all nontangential approach regions and there is $M_1 > 0$ so that

$$\frac{|\eta-\varphi(z)|^2}{1-|\varphi(z)|^2}\frac{1-|z|^2}{|\omega-z|^2} \le M_1 \qquad z \in \mathbb{U}. \tag{3.11}$$

Assume $z \in \Gamma_M(\omega)$. Then, by conditions (3.8), (3.10), and (3.11), there is some constant $N > 0$, so that

$$\frac{|\eta - \varphi(z)|^2}{1 - |\varphi(z)|^2} \frac{1 - |z|^2}{|\omega - z|^2} \leq M_1 \qquad z \in \mathbb{U},$$

$$\frac{|\omega - z|^2}{(1 - |z|)^2} < M^2 \qquad z \in \Gamma_M(\omega),$$

and

$$\frac{|\psi(z)|^2(1 - |z|^2)}{1 - |\varphi(z)|^2} \leq N \qquad z \in \mathbb{U}.$$

Thus, if $z \in \Gamma_M(\omega)$, one can multiply the above inequalities getting that

$$|\psi(z)|^2 \leq M_1 M^2 N \left(\frac{1 - |\varphi(z)|^2}{|\eta - \varphi(z)|} \right)^2 \leq 4 M_1 M^2 N \qquad z \in \Gamma_M(\omega).$$

□

The above theorem was initially proved in [3], and reproved in [12]. The original proofs are close to the one presented in this paper. We included Theorem 11 with proof for the sake of completeness.

3.2 Beurling subspaces

The (necessarily closed) subspaces of type uH^2, where u is inner are called Beurling subspaces due to Arne Beurling's paper [4], where it is shown that the invariant subspace lattice of a unilateral forward shift of multiplicity 1 represented as M_z, that is as the operator of multiplication with the coordinate function acting on H^2, consists of all spaces of type uH^2 and the null space.

In papers [9] and [25], it is observed and proved, that classical theorems in function theory, for instance the Julia–Carathéodory theorem, can be understood in terms of the action of composition operators on Beurling spaces.

That theorem addresses the existence of angular derivatives in the sense of Constantin Carathéodory, and the authors of [9] and [25] observed and proved that the existence of such an angular derivative of some analytic selfmap φ of the unit disc is equivalent to the fact that C_φ maps certain Beurling spaces induced by some atomic singular inner functions into similar spaces. In particular, $\omega \in \mathbb{T}$ is the Denjoy–Wolff point of some fixed points free φ if and only if $C_\varphi S_{\delta_\omega} \subseteq S_{\delta_\omega}$, where S_{δ_ω} is the atomic, singular inner function $S_{\delta_\omega}(z) = e^{-\frac{\omega+z}{\omega-z}}$, $z \in \mathbb{U}$.

Recently, the authors of [5] answered the question: *When is a Beurling space invariant under a composition operator?*, by proving that:

$$C_\varphi(uH^2) \subseteq uH^2 \iff \frac{u \circ \varphi}{\varphi} \in H^\infty \quad \text{and} \quad \left\| \frac{u \circ \varphi}{\varphi} \right\|_\infty \leq 1. \tag{3.12}$$

[5, Theorem 3.2]. We wish to note, that the above result, although correct, is understated and so, we prove the following version:

Theorem 12. *Let u be an inner function and φ an analytic selfmap of the open unit disc. Then, the following statements are equivalent:*

$$C_\varphi uH^2 \subseteq uH^2. \tag{3.13}$$

$$\frac{u \circ \varphi}{\varphi} \in H^2. \tag{3.14}$$

The weighted composition operator $T_{u\circ\varphi/u,\varphi}$ is bounded. (3.15)

If any of the above holds, then

$$\frac{u \circ \varphi}{\varphi} \in H^\infty \quad \textit{and} \quad \left\| \frac{u \circ \varphi}{\varphi} \right\|_\infty \leq 1. \tag{3.16}$$

Proof. First we prove that (3.13)$\Longleftrightarrow$(3.14) and if any of these statements holds then (3.16) follows. Indeed, If $C_\varphi(uH^2) \subseteq uH^2$, then $C_\varphi(u) \in uH^2$, that is $u \circ \varphi / u = f$ for some $f \in H^2$. Given the fact that Hardy space functions have radial limits a.e. on the unit circle, and the fact that u is inner, one can write

$$|f(e^{it})| = \frac{|u \circ \varphi(e^{it})|}{|u(e^{it})|} = |u \circ \varphi(e^{it})| \leq 1 \quad a.e.$$

and deduce that $f \in H^\infty$ and $\|f\|_\infty \leq 1$.

Conversely, if $f = u \circ \varphi / u \in H^2$ then, by our considerations above, $u \circ \varphi / u \in H^\infty$ and $\|u \circ \varphi / u\|_\infty \leq 1$. Then $fH^2 \subseteq H^2$ and one can write

$$C_\varphi(uH^2) = ufR(C_\varphi) \subseteq uH^2.$$

Now (3.14)$\Longrightarrow$(3.15) is trivial and if (3.15) holds, then by the Caughran–Schwartz equation, one has that

$$T^\star_{\psi,\varphi} K_w = \overline{\psi(w)} K_{\varphi(w)} \qquad w \in \mathbb{U},$$

which implies that $T^\star_{u \circ \varphi / u, \varphi}$ is the extension by linearity and continuity of the map

$$A(\overline{\varphi(w)} K_w) = \overline{u \circ \varphi(w)} K_w \quad w \in \mathbb{U}.$$

Given that A has such an extension, it follows by [5, Corollary 2.4] that (3.16) holds, hence (3.14) holds. □

Our interest in the above theorem is demonstrating that the boundedness of special weighted composition operators is equivalent to the invariance of Beurling spaces under composition operators, which in turn is the operatorial description of the Julia–Carathéodory theorem, in select cases. This illustrates the statement saying that: *the theory of (weighted) composition operators is a beautiful interplay of function theory and functional analysis*.

As a last statement, we wish to acknowledge that, posterior to proving the above theorem, we noted that, in the (seemingly unpublished) preprint [29], some of the authors of [5] realize that their initial result (3.12), can be improved to (3.13)$\Leftrightarrow$(3.14).

3.3 General Boundedness Criteria

Composition operators on various Hardy–Smirnov spaces (e.g $H^p(G)$, where G is a half–plane), are subject to *Carleson measure* criteria (see [22]), whose main utility (in the opinion of this author), is to establish that the set of analytic selfmaps of G inducing bounded composition operators on $H^p(G)$ is the same for all $1 \leq p < \infty$.

Carleson measure boundedness criteria for weighted composition operators were proved in [7]. They are one of the few kinds of general boundedness criteria but may be hard to use, since proving that a measure is Carleson is often as hard as proving that the norm of the associated weighted composition operator is finite. Here is, more formally, what we mean.

For all $u \in \mathbb{T}$ and $r > 0$, $S(u, r) := \{z \in \overline{\mathbb{U}} : |z - u| \leq r\}$ is called the *Carleson window* determined by u and r. A nonnegative Borel measure μ on $\overline{\mathbb{U}}$ is called a Carleson measure if there is some $M > 0$ so that

$$\mu(S(u, r)) \leq Mr \qquad u \in \mathbb{T},\ r > 0.$$

For $\psi \in H^2$ and an analytic selfmap φ of $\mathbb{U}$ consider the Borel measure $\mu_{\varphi,\psi}$ on $\overline{\mathbb{U}}$:

$$\mu_{\varphi,\psi}(E) = \int\limits_{\varphi^{-1}(E)\cap\mathbb{T}} |\psi|^2\, dm.$$

The first announced “general" boundedness criterion is:

Theorem 13 ([8, Theorem 2.2]). *The operator $T_{\psi,\varphi}$ is bounded in H^2 if and only if $\mu_{\varphi,\psi}$ is a Carleson measure; i.e. if and only if there is $M > 0$ so that*

$$\int\limits_{\varphi^{-1}(S(u,r))\cap\mathbb{T}} |\psi|^2\, dm \leq Mr \qquad u \in \mathbb{T},\ r > 0.$$

A second “general" boundedness criterion for weighted composition operators, this time in terms of nonnegative kernels is:

Theorem 14 ([13, Theorem 5]). *Let $\mathcal{H}$ be a RKHS consisting of scalar valued functions on some set Ω, having nonnegative kernel $\mathcal{K}$, and the property that the set of all kernel functions of $\mathcal{H}$ is a linearly independent set. Then a weighted composition operator $T_{\psi,\varphi}$ is a bounded operator on $\mathcal{H}$ if and only if there is some $M \geq 0$ such that*

$$M^2\mathcal{K}(x,y) \geq \overline{\psi(x)}\psi(y)\mathcal{K}\circ\varphi(x,y), \quad x,y \in \Omega. \tag{3.17}$$

Furthermore, the following equality holds:

$$\min\{M \geq 0 : (3.17) \text{ holds}\} = \|T_{\psi,\varphi}\|. \tag{3.18}$$

If $M^2 = \sup\{|\psi(x)|^2\mathcal{K}\circ\varphi(x,x)/\mathcal{K}(x,x) : x \in S\}$ is finite and satisfies (3.17)*, then $T_{\psi,\varphi}$ is bounded and $\|T_{\psi,\varphi}\| = M$.*

Clearly, $T_{1,\varphi} = C_\varphi$ and so, condition (3.17) *looks as follows in this particular case:*

$$\mathcal{K}\circ\varphi \leq M^2\mathcal{K}. \tag{3.19}$$

Remark 6. *In the case of the* RKHS H^2*, condition* (3.17) *has the particular form*

$$M^2\frac{1}{1-\overline{w}z} - \overline{\psi(z)}\psi(w)\frac{1}{1-\overline{\varphi(w)}\varphi(z)} \geq 0 \tag{3.20}$$

meaning that the function above is a nonnegative kernel on the open unit disc. By our considerations, that fact is valid if φ is any analytic selfmap of $\mathbb{U}$, $\psi \in H^\infty$, and

$$M \geq \|\psi\|_\infty\sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}.$$

The above criterion has applications (see [27]), but is equally hard to use in concrete situations as is Theorem 13.

Finally, if the composition symbol of a weighted composition operator is inner, then a general criterion for boundedness exists [23, Theorem 7] in terms of a certain pull–back measure and its Nykodim derivative. Here are the details.

For any Borel measure μ on $\mathbb{T}$ and any inner function φ, $\mu\varphi^{-1}$ denotes the pull–back measure of μ under φ; that is the measure $\mu\varphi^{-1}(E) = \mu(\varphi^{-1}(E))$, for each measurable set $E \subseteq \mathbb{T}$. Given $f \in L^1_{\mathbb{T}}(d\mu)$ $f \geq 0$ a.e., the notation μ_f denotes the measure described by the equality

$$\mu_f(E) = \int_E f\,d\mu.$$

The following is known.

Theorem 15 ([23, Theorem 7]). *If φ is inner and $\psi \in H^2$, then $T_{\psi,\varphi}$ is bounded if and only if the Nykodim derivative $dm_{|\psi|^2}\varphi^{-1}/dm$ is essentially bounded and then, the following equality holds:*

$$\|T_{\psi,\varphi}\| = \sqrt{\left\|\frac{dm_{|\psi|^2}\varphi^{-1}}{dm}\right\|_\infty}.$$

As one can easily note, if φ is inner and $\psi \in H^2$, then $\mu_{\varphi,\psi} = m_{|\psi|^2}\varphi^{-1}$, and so, $\mu_{\varphi,\psi}$ is Carleson if and only if that measure has an essentially bounded Nykodim derivative. The supremum norm of that derivative, gives the operator norm of $T_{\psi,\varphi}$.

Relative to the above theorem, recall that condition (3.8) can be obtained by writing $\sup_{z\in\mathbb{U}}\|T^\star_{\psi,\varphi}(K_z/\|K_z\|_2)\|_2 \leq \|T^\star_{\psi,\varphi}\|$, and using (1.10), (an easy exercise left to the reader; see [23] for details).

Obviously, one can also write the estimate $\sup_{w\in\mathbb{U}} \|T_{\psi,\varphi}(K_w/\|K_w\|_2)\|_2 \le \|T_{\psi,\varphi}\|$, getting

$$\sup_{w\in\mathbb{U}} \left\| \frac{(1-|w|^2)\psi(z)}{1-\overline{w}\varphi(z)} \right\|_2 \le \|T_{\psi,\varphi}\|, \tag{3.21}$$

that is the integral transform

$$\mathcal{I}_{\psi,\varphi}(w) := \left\| \frac{(1-|w|^2)\psi(z)}{1-\overline{w}\varphi(z)} \right\|_2^2 = \int_{\mathbb{T}} \frac{(1-|w|^2)|\psi(u)|^2}{|1-\overline{w}\varphi(u)|^2}\, dm(u)$$

needs to be bounded if $T_{\psi,\varphi}$ is bounded, since by(3.21), one has that

$$\left\|\mathcal{I}_{\psi,\varphi}\right\|_\infty \le \|T_{\psi,\varphi}\|^2, \tag{3.22}$$

proving that the boundedness of the integral transform $\mathcal{I}_{\psi,\varphi}$ is necessary for the boundedness of $T_{\psi,\varphi}$. This proves one implication in the equivalence of the following integral transform general criterion for the boundedness of weighted composition operators:

Theorem 16. *Given $\psi \in H^2$ and φ an analytic selfmap of $\mathbb{U}$, the weighted composition operator $T_{\psi,\varphi}$ is bounded if and only if the integral transform $\mathcal{I}_{\psi,\varphi}$ is bounded and estimate* (3.22) *holds.*

The other implication above, that is the fact that the boundedness of the integral transform $\mathcal{I}_{\psi,\varphi}$ is sufficient for the boundedness of $T_{\psi,\varphi}$, follows by [15, Theorem 3.3], as noted in [12].

Theorems 15 and 16 combine to prove the following:

Theorem 17. *Let φ be inner and $\psi \in H^2$; then the following are equivalent:*

$$\|T_{\psi,\varphi}\| < \infty \tag{3.23}$$

$$\left\|\mathcal{I}_{\psi,\varphi}\right\|_\infty < \infty \tag{3.24}$$

$$\left\| \frac{dm_{|\psi|^2}\varphi^{-1}}{dm} \right\|_\infty < \infty \tag{3.25}$$

$$\left\| P_{m_{|\psi|^2}\varphi^{-1}} \right\|_\infty < \infty \tag{3.26}$$

where $P_{m_{|\psi|^2}\varphi^{-1}}$ denotes the Poisson integral of measure $m_{|\psi|^2}\varphi^{-1}$. In that case, the following norm formula holds:

$$\|T_{\psi,\varphi}\| = \sqrt{\left\| \frac{dm_{|\psi|^2}\varphi^{-1}}{dm} \right\|_\infty} = \sqrt{\left\|\mathcal{I}_{\psi,\varphi}\right\|_\infty} = \sup_{w\in\mathbb{U}} \|T_{\psi,\varphi}(K_w/\|K_w\|_2)\|_2. \tag{3.27}$$

Proof. We claim that the integral transform $\mathcal{I}_{\psi,\varphi}$ equals the Poisson integral of the measure $m_{|\psi|^2}\varphi^{-1}$. Indeed, for arbitrary inner φ and $\psi \in H^2$, one can write

$$\mathcal{I}_{\psi,\varphi}(w) := \left\| \frac{(1-|w|^2)\psi(z)}{1-\overline{w}\varphi(z)} \right\|_2^2 = \int_{\mathbb{T}} \frac{(1-|w|^2)|\psi(u)|^2}{|1-\overline{w}\varphi(u)|^2}\, dm(u) =$$

$$\int_{\mathbb{T}} \frac{(1-|w|^2)|\psi(u)|^2}{|\varphi(u)-w|^2}\, dm(u) = \int_{\mathbb{T}} P(w,\varphi(u))|\psi(u)|^2\, dm(u) =$$

$$\int_{\mathbb{T}} P(w,u)\, dm_{|\psi|^2}\varphi^{-1}(u),$$

where $P(w, u)$, $w \in \mathbb{U}$, $u \in \mathbb{T}$ is the usual Poisson kernel.

The Poisson integral of a positive measure μ is bounded if and only if that measure is absolutely continuous with respect to Lebesgue measure and has essentially bounded Nykodim derivative f, [30, Theorem 11.30]. It is easy to see that $\|P_\mu\|_\infty = \|f\|_\infty$, given the obvious estimate

$$P_\mu(z) = \int_{\mathbb{T}} P(z, u) f(u)\, d\mu \le \int_{\mathbb{T}} P(z, u) \|f\|_\infty\, d\mu = \|f\|_\infty$$

and the known fact that the radial limit function of $P_\mu(z)$ which exists a.e. on $\mathbb{T}$, equals f a.e.

By our previous considerations, the proof is over. □

As we mentioned before, the set of analytic selfmaps of G inducing bounded composition operators on $H^p(G)$ is the same for all $p \ge 1$. This principle fails if "composition" is substituted by "weighted composition", and for a simple reason. Indeed, we would expect that the pairs ψ, φ inducing bounded weighted composition operators on $H^p(G)$ be the same for all values of p. This is not true, as we observe in the following:

Example 1. *Assume $G = \mathbb{U}$ and $1 \le p < q < \infty$. Choose $\psi \in H^q \setminus H^p$ and φ, an analytic selfmap of $\mathbb{U}$, with the property that $\|\varphi\|_\infty < 1$. Then $T_{\psi,\varphi}$ is bonded on H^q (by Theorem 7), but not on H^p (since $\psi \in H^q \setminus H^p$).*

As a concluding remark, it may be challenging in certain situations to prove or disprove the boundedness of a weighted composition operator. The main issue is that the spaces $\mathcal{M}(C_\varphi)$ coincide with Hardy spaces H^p, $2 \le p \le \infty$ only if $p = 2$ or $p = \infty$ [3, Proposition 10], being otherwise hard to describe.

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