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Erratum to

C^*-simplicity of locally compact Powers groups

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In [5, Lemma 5.1] we made the following claim. We claimed that if G is a locally compact group and $K \leq G$ is a compact open subgroup, then for all $g \in G$ we have

$$p_K u_g^* p_K u_g p_K \geq \left[K : K \cap gKg^{-1}\right]^{-2} p_K.$$

As previously reported in [4], Yuhei Suzuki pointed out that this claim is not correct and provided the counterexample $\mathbb{Z}_2 \times \{e\} \leq (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, where the action of $\overline{1} \in \mathbb{Z}_2$ flips the two copies of $\mathbb{Z}_2$. The mistake in the proof of [5, Lemma 5.1] lies in the second equality of the first displayed equation, which exchanges a square of a sum with the sum of the squares.

It is a priori not clear whether the invalidity of [5, Lemma 5.1] affects any of the main results of [5]. In this erratum, we show that the main example of a non-discrete C^*-simple group obtained in Theorem G persists, while Theorem B is not correct in the stated generality.

In [5, Theorem G] we claimed that given $2 \leq |m| \leq n$ the relative profinite completion $G(m,n)$ of the Baumslag–Solitar group $BS(m,n)$ is C^*-simple. We sketch an alternative proof for this statement, which does avoid the use of [5, Lemma 5.1].

Proof for Theorem G. Since $G(m,n) \cong \mathbb{Z}_m * \mathbb{Z}$ is a discrete Powers group if $|m| = n$, it suffices to consider the case $|m| \neq n$. Fix such m and n and write $G = G(m,n)$. Denote by $\Delta : G \to \mathbb{R}_{>0}$ the modular function and by G^0 the kernel of Δ. Denote by $t \in G$ the image of the free letter. Then $G = G^0 \rtimes \langle t \rangle$. If we show that $C^*_{red}(G^0)$ is simple and has a unique tracial weight, then work of Archbold and Spielberg [1, Corollary, p. 122] shows that also $C^*_{red}(G) \cong C^*_{red}(G^0) \rtimes \mathbb{Z}$ is simple, since the action $\mathrm{Ad}\, t$ on $C^*_{red}(G^0)$ scales the Plancherel trace by $\left|\frac{n}{m}\right| \neq 1$.

Denote by $K = \overline{\langle a \rangle} \leq G$ the elliptic subgroup of G, and recall that G is an HNN-extension of K by $\mathrm{Ad}\, t$. Denote by $\sigma_t : G \to \mathbb{N}$ the associated t-exponent sum, as used for example in [3]. Denote by

$$h_t(w) = \max\{|\sigma_t(v)| \mid w = uv \text{ reduced decomposition}\}$$

the t-height of $w \in G^0$. It is the absolute value of the maximal t-exponent sum of a terminal piece of w. We put

$$G_k = \{w \in G^0 \mid h_t(w) \leq k\}.$$

Then G_k is a an open subgroup of G^0 and $K_k = \overline{\langle a^{m^k n^k} \rangle}$ is a normal compact open subgroup of G_k. Since $\bigcup_{k \in \mathbb{N}} G_k = G^0$, it suffices to show that $\Gamma_k = G_k / K_k$ is C*-simple by Suzuki's result [6].

Denote by $T = \mathrm{BS(m,n)}/\langle \mathrm{a} \rangle = \mathrm{G/K}$ the Bass–Serre tree of G, and write ρ for its root. Write $T_k = G_k \cdot \rho$ and observe that $T_k \leq T$ is a subtree. Note that the action of K_k on T_k is trivial, so that we obtain a well-defined action of Γ_k on T_k. We observe that for every non-empty open subset $O \subset \partial T_k$ there are infinitely many hyperbolic elements in G_k with pairwise different endpoints, which all lie in O. So by the arguments of [2], the group Γ_k is a Powers group and thus C*-simple, if we can exhibit for every finite subset $F \subset \Gamma_k \setminus \{e\}$ a non-empty open subset $O \subset \partial T_k$ such that $FO \cap O = \emptyset$. To this end, it suffices to show that the set of fix points of every non-trivial element of Γ_k is meager in ∂T_k. Hyperbolic elements have at most two fixed points, while the action of any non-trivial elliptic element in Γ_k is conjugate to one of the elements $a, a^2, \dots, a^{m^k n^k - 1}$, whose fixed points lie in $\partial T_{k-1} \subset \partial T_k$. Since the latter subset is meager, we can conclude that G is C*-simple. □

As explained above, [5, Theorem B] is not correct as stated. A counterexample is provided by the Burger–Mozes group $U(F)$ associated with the permutation group $\mathbb{Z}_2 \wr \mathrm{S}_3$ with its imprimitive action on a set with six elements. This counterexample as well as the following criterion for non-C*-simplicity were suggested to us by Pierre-Emmanuel Caprace. We thank him for allowing us to include them into this piece. An action of a group G on a space X is called *micro-supported* if for all non-empty open subsets $U \subset X$ there is some $g \in G$ acting non-trivially on X and fixing $X \setminus U$ pointwise. Note that the action of any Burger–Mozes group on the boundary of its tree is micro-supported, thanks to Tits' independence property.

Proposition 1. *Let G be a locally compact group acting on a compact space X. Assume that the action of G has some open amenable point stabiliser and is micro-supported. Then G is not* C**-simple.*

Proof. We may assume that X is non-trivial, since amenable groups are not C*-simple. Let $x \in X$ have an open and amenable point stabiliser. Then $G \curvearrowright \ell^2(Gx)$ is unitarily equivalent with the quasi-regular representation on G/G_x and thus weakly contained in the left-regular representation. Denote by $\pi : \mathrm{C}^*_{\mathrm{red}}(G) \to \mathcal{B}(\ell^2(Gx))$ the associated $*$-homomorphism. Since the inclusion $\mathrm{C}^*_{\mathrm{red}}(G) \subset \mathrm{M}(\mathrm{C}^*_{\mathrm{red}}(G))$ is essential, it suffices to show that the extension $\tilde{\pi}$ of π to the multiplier algebra $\mathrm{M}(\mathrm{C}^*_{\mathrm{red}}(G))$ is not injective. Let $U, V \subset X$ be two disjoint open subsets. Let $g, h \in G$ be elements acting non-trivially on X such that g fixes $X \setminus U$ pointwise and h fixes $X \setminus V$ pointwise. Then $(1-u_g)(1-u_h) \in \ker \tilde{\pi}$. Indeed, calculating with the orthogonal basis $(\delta_y)_{y \in Gx}$ of $\ell^2(Gx)$, we find $(1-u_h)\delta_y = 0$ if $y \in X \setminus V$ and $(1-u_h)\delta_y = \delta_y - \delta_{hy}$ if $y \in V$. In the latter case, we have $y, hy \in V \subset X \setminus U$. So that $(1-u_g)(1-u_h)\delta_y$ follows for all $y \in Gx$. □

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