Ample line bundles and generation time

By Noah Olander at Amsterdam

Abstract. We prove that if \( X \) is a regular quasi-projective variety of dimension \( d \), the set of line bundles \( \{\mathcal{O}_X(n)\}_{n \in \mathbb{Z}} \) generates the bounded derived category of \( X \) in \( d \) steps. This proves new cases of a conjecture of Orlov as well as a conjecture of Elagin and Lunts.

1. Introduction

Let \( \mathcal{T} \) be a triangulated category. For a set \( S \) of objects of \( \mathcal{T} \), one denotes by \( (S)_{n+1} \) the full (not necessarily triangulated) subcategory of \( \mathcal{T} \) generated from \( S \) using direct sums, shifts, direct summands, and at most \( n \) cones, see [13, Section 3.1]. The Rouquier dimension of \( \mathcal{T} \), denoted \( \text{Rdim}(\mathcal{T}) \), is the smallest integer \( d \) such that there exists a single object \( G \) of \( \mathcal{T} \) such that \( \mathcal{T} = \langle G \rangle_{d+1} \). In this paper, we consider a variant of this notion defined by Dmitrii Pirozhkov called the countable Rouquier dimension, which replaces the single object \( G \) with a countable set of objects, see Definition 6. We study this notion for the triangulated category \( \text{Dbcoh}(X) \) with \( X \) a Noetherian scheme. This can be defined either as the full subcategory of the derived category of \( \mathcal{O}_X \)-modules consisting of objects with bounded, coherent cohomology; or equivalently, as the bounded derived category of the category of coherent sheaves on \( X \), see [14, Tag 09T4]. Our main result, Theorem 4, implies that if \( X \) is a Noetherian regular scheme with an ample line bundle, then its bounded derived category of coherent sheaves \( \text{Dbcoh}(X) \) has countable Rouquier dimension at most \( \dim(X) \). This should be compared to Orlov’s conjecture [9, Conjecture 10] that if \( X \) is a smooth quasi-projective scheme over a field, then \( \text{Rdim}(\text{Dbcoh}(X)) = \dim(X) \).

We give three applications of Theorem 4. First, we prove the above mentioned conjecture of Orlov in the quasi-affine case, see Corollary 5. Second, we prove Theorem 12 which says that if \( \text{Dbcoh}(X) \) is an admissible subcategory of \( \text{Dbcoh}(Y) \) for smooth projective varieties \( X \) and \( Y \), then \( \dim(X) \leq \dim(Y) \). This was a folklore conjecture known to follow from the projective case of [9, Conjecture 10]. It was explicitly stated as an open problem by Elagin and Lunts in [4, Section 5.3]. Third, we give a quick proof that if \( X \) is a Noetherian regular scheme of finite Krull dimension which possesses an ample line bundle, then \( \text{Dbcoh}(X) \) has a strong generator. This was conjectured by Bondal and Van den Bergh in [2, Section 3.1] and proved in greater generality by Neeman in [7, Theorem 0.5] – in particular, he does not assume \( X \) has an ample line bundle. We were pleasantly surprised to find a much shorter proof under this additional hypothesis.
The proof of Theorem 4 goes through Theorem 2 which is interesting in its own right. It says that if \( X \) is a Noetherian regular scheme of dimension \( d \) and \( K_0 \to K_1 \to \cdots \to K_{d+1} \) are morphisms in \( D^b_{\text{coh}}(X) \) which are zero on cohomology sheaves, then the composition \( K_0 \to K_{d+1} \) is zero. This is proved via a spectral sequence argument which will be familiar to some topologists, see [3, Proposition 4.5] for instance.

2. Main result

Let \( \mathcal{A} \) be an abelian category and \( \varphi : K \to L \) a morphism in \( D(\mathcal{A}) \). We would like to know if \( \varphi = 0 \). An obvious necessary condition is that \( H^n(\varphi) : H^n(K) \to H^n(L) \) be zero for all \( n \), but this is not sufficient: Consider any nonzero morphism \( A \to B[1] \) with \( A, B \in \mathcal{A} \). In fact, it is just the first in an infinite string of necessary conditions which together are sufficient.

**Proposition 1.** Let \( \mathcal{A} \) be an abelian category with enough injectives. Then for each \( K, L \in D^b(\mathcal{A}) \) the group \( \text{Hom}_{D(\mathcal{A})}(K, L) \) carries a functorial decreasing filtration \( F \) which satisfies for \( K, L, M \in D^b(\mathcal{A}) \):

1. \( F^0\text{Hom}_{D(\mathcal{A})}(K, L) = \text{Hom}_{D(\mathcal{A})}(K, L) \) and \( F^p\text{Hom}_{D(\mathcal{A})}(K, L) = 0 \) for \( p \gg 0 \).
2. If \( f \in F^p\text{Hom}_{D(\mathcal{A})}(K, L) \) and \( g \in F^q\text{Hom}_{D(\mathcal{A})}(L, M) \), then \( g \circ f \in F^{p+q}\text{Hom}_{D(\mathcal{A})}(K, M) \).
3. \( F^p\text{Hom}_{D(\mathcal{A})}(K, L)/F^{p+1}\text{Hom}_{D(\mathcal{A})}(K, L) \) is a subquotient of \( \prod_{n \in \mathbb{Z}} \text{Ext}^p_\mathcal{A}(H^n(K), H^{n-p}(L)) \).
4. \( F^1\text{Hom}_{D(\mathcal{A})}(K, L) = \{ \varphi \in \text{Hom}_{D(\mathcal{A})}(K, L) : H^n(\varphi) = 0 \text{ for all } n \in \mathbb{Z} \} \).

**Proof.** The filtration is the one induced by the spectral sequence

\[ E_1^{p,q} = \prod_{n \in \mathbb{Z}} \text{Ext}^{p+q}_\mathcal{A}(H^n(K), H^{n-p}(L)) \implies \text{Ext}^{p+q}(K, L), \]

of [1, Equation 3.1.3.4] obtained by endowing \( K \) and \( L \) with their canonical filtrations. Part (3) follows immediately from this description (note that \( p + q = 0 \) implies \( 2p + q = p \)). Since negative Ext groups vanish in \( \mathcal{A} \), (3) implies the inclusions \( \cdots \supset F^{-2} \supset F^{-1} \supset F^0 \) are equalities. Convergence of the spectral sequence gives \( F^p = \text{Hom}_{D(\mathcal{A})}(K, L) \) for \( p \ll 0 \) (hence for all \( p \leq 0 \) by the previous sentence) and \( F^p = 0 \) for \( p \gg 0 \), proving (1). Next, note that all differentials going into an \( E_r^{0,0} \) are zero for degree reasons, and hence \( E_r^{0,0} \subset E_1^{0,0} \). But the composition \( F^0 \to F^0/F^1 = E_\infty^{0,0} \to E_1^{0,0} \) is a map

\[ \text{Hom}_{D(\mathcal{A})}(K, L) \to \prod_{n \in \mathbb{Z}} \text{Hom}_\mathcal{A}(H^n(K), H^n(L)) \]

which one can show using the definition of the spectral sequence is just the map taking a morphism to its associated morphisms on cohomology. Hence (4) follows. Finally, (2) holds because the spectral sequence is compatible with composition in the usual sense of a spectral sequence with products. We refer to [8, Appendix A] for all omitted details. \( \Box \)
Theorem 2. Let $X$ be a Noetherian regular scheme of dimension $d < \infty$. Let

$$K_0 \to K_1 \to \cdots \to K_{d+1}$$

be morphisms in $D^b_{\text{coh}}(X)$ whose induced morphisms on cohomology sheaves vanish. Then the composition $K_0 \to K_{d+1}$ is zero.

Proof. Note that $D^b_{\text{coh}}(X)$ is a full subcategory of the bounded derived category of the category of $\mathcal{O}_X$-modules, which is an abelian category with enough injectives. Thus we may use the filtration

$$\text{Hom}(K_0, K_{d+1}) = F^0 \supset F^1 \supset F^2 \supset \cdots$$

of Proposition 1. By [14, Tag 0FZ3] we have $\text{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$ for $i > d$ and $\mathcal{F}, \mathcal{G}$ coherent sheaves on $X$. Therefore by (3) of Proposition 1, $F^{d+1} = F^{d+2} = \cdots$. Since $F^p = 0$ for $p \gg 0$, in fact $F^{d+1} = 0$. Then by (4) each $K_i \to K_{i+1}$ is in $F^1\text{Hom}(K_i, K_{i+1})$, so that by (2) the composition $K_0 \to K_{d+1}$ is in $F^{d+1}\text{Hom}(K_0, K_{d+1}) = 0$ and we are done. $\square$

Lemma 3. Let $X$ be a Noetherian scheme with an ample invertible sheaf $\mathcal{L}$ and let $K \in D^b_{\text{coh}}(X)$. Then there exist a finite set $I$ and a morphism $\bigoplus_{i \in I} \mathcal{L}^\otimes m_i[n_i] \to K$ with $n_i, m_i \in \mathbb{Z}$, which is surjective on cohomology sheaves.

Proof. Represent $K$ by a bounded complex of coherent sheaves

$$\ldots \to \mathcal{F}^k \xrightarrow{d^k} \mathcal{F}^{k+1} \to \cdots.$$ 

For $n$ sufficiently negative there is a surjection $\bigoplus_i \mathcal{L}^{\otimes n} \to \ker(d^k)$ with the sum finite. This gives rise to a morphism $\bigoplus_i \mathcal{L}^{\otimes n}[-k] \to K$ which is surjective on $H^k$. Putting together these morphisms for every $k$ proves the result. $\square$

Theorem 4. Let $X$ be a Noetherian regular scheme of dimension $d < \infty$. Assume $X$ has an ample invertible sheaf $\mathcal{L}$. Then $D^b_{\text{coh}}(X) = \langle \{\mathcal{L}^{\otimes n}\}_{n \in \mathbb{Z}} \rangle_{d+1}$.

Proof. Let $K = K_0 \in D^b_{\text{coh}}(X)$. Choose a finite set $I$ and a morphism $\bigoplus_{i \in I} \mathcal{L}^{\otimes m_i}[n_i] \to K$ as in Lemma 3 and let $K_1$ be the cone. Note that $K_0 \to K_1$ is zero on cohomology sheaves by construction, and its cone is in $\langle \{\mathcal{L}^{\otimes n}\}_{n \in \mathbb{Z}} \rangle_1$. Now repeat the process with $K = K_1$ and so forth to obtain a sequence

$$K_0 \to K_1 \to \cdots \to K_{d+1}$$

such that each $K_i \to K_{i+1}$ is zero on cohomology sheaves and has cone in $\langle \{\mathcal{L}^{\otimes n}\}_{n \in \mathbb{Z}} \rangle_1$. Thus $K_0 \to K_{d+1}$ is zero by Theorem 2. We will prove by induction that the cone of $K_0 \to K_i$ is in $\langle \{\mathcal{L}^{\otimes n}\}_{n \in \mathbb{Z}} \rangle_i$. For $i = 1$ this is known and for $i = d + 1$ this proves the theorem. Since $K = K_0 \to K_{d+1}$ is zero, it follows that the cone is isomorphic to $K_{d+1} \oplus K[1]$ and the category $\langle \{\mathcal{L}^{\otimes n}\}_{n \in \mathbb{Z}} \rangle_{d+1}$ is closed under direct summands and shifts.

So assume known that the cone of $K_0 \to K_i$ is in $\langle \{\mathcal{L}^{\otimes n}\}_{n \in \mathbb{Z}} \rangle_i$. Then by the octahedral axiom there is a distinguished triangle

$$C \to D \to E \to C[1]$$
with $C$ a cone of $K_0 \to K_1$, $D$ a cone of $K_0 \to K_i$, and $E$ a cone of $K_i \to K_{i+1}$. Since $C \in \{\{\mathcal{L}^n\}_{n \in \mathbb{Z}}\}_i$ and $E \in \{\{\mathcal{L}^n\}_{n \in \mathbb{Z}}\}_1$, it follows that $D \in \{\{\mathcal{L}^n\}_{n \in \mathbb{Z}}\}_{i+1}$, as needed.

**Corollary 5.** Let $X$ be a Noetherian regular scheme of dimension $d < \infty$. If $X$ is quasi-affine, then $D^b_{\text{coh}}(X) = \langle \mathcal{O}_X \rangle_d$ and hence $R\text{dim}(D^b_{\text{coh}}(X)) \leq d$. If $X$ is also of finite type over a field, then $R\text{dim}(D^b_{\text{coh}}(X)) = d$.

**Proof.** The structure sheaf is ample on a quasi-affine scheme so the first part follows from Theorem 4. The reverse inequality when $X$ is of finite type over a field is [13, Proposition 7.16].

### 3. Countable Rouquier dimension

We will now show how Theorem 12 follows from Theorem 4. The key is the following definition due to Pirozhkov.

**Definition 6.** Let $\mathcal{T}$ be a triangulated category. The **countable Rouquier dimension** of $\mathcal{T}$, denoted $\text{CRdim}(\mathcal{T})$, is the smallest $n$ such that there exists a countable set $\{E_i\}_{i \in I}$ of objects of $\mathcal{T}$ such that $\mathcal{T} \supseteq \langle \{E_i\}_{i \in I} \rangle_n$, or infinity if no such $n$ exists.

**Lemma 7.** Let $F : \mathcal{T} \to \mathcal{T}'$ be an exact functor between triangulated categories which is essentially surjective. Then $\text{CRdim}(\mathcal{T}') \leq \text{CRdim}(\mathcal{T})$.

**Proof.** If $\mathcal{T} = \langle \{E_i\}_{i \in I} \rangle_{n+1}$, then $\mathcal{T}' = \langle \{F(E_i)\}_{i \in I} \rangle_{n+1}$.

**Example 8.** Let $X$ be a Noetherian regular scheme with an ample line bundle. Then by Theorem 4, $\text{CRdim}(D^b_{\text{coh}}(X)) \leq \dim(X)$.

The reverse inequality is proved below for varieties over an uncountable field. Note that if $X$ is a variety over a countable field, then $D^b_{\text{coh}}(X)$ has countably many objects up to isomorphism, hence $\text{CRdim}(D^b_{\text{coh}}(X)) = 0$.

**Proposition 9.** Let $k$ be an uncountable field. Let $X$ be a reduced scheme of finite type over $k$. Then $\text{CRdim}(D^b_{\text{coh}}(X)) \geq \dim(X)$.

**Proof.** Compare to the proof of [13, Proposition 7.16]. Let $n = \text{CRdim}(D^b_{\text{coh}}(X))$ and let $\{E_i\}_{i \in I}$ be a countable family of objects such that $D^b_{\text{coh}}(X) = \langle \{E_i\}_{i \in I} \rangle_{n+1}$. Consider the set of closed points $x \in X$ such that for every $i \in I$, the cohomology modules of $(E_i)_x$ are free $\mathcal{O}_{X,x}$-modules. Since $X$ is not a countable union of closed subsets with dense complement (see [6, Exercise 2.5.10]), the set contains a closed point $x$ such that $\dim(\mathcal{O}_{X,x}) = \dim(X)$. We have $(E_i)_x \in \langle \mathcal{O}_{X,x} \rangle_1$ for each $i$ since a complex with projective cohomology modules is decomposable, hence

$$\kappa(x) \in \langle \{((E_i)_x)_{i \in I}\}_{i \in I} \rangle_{n+1} \subseteq \langle \mathcal{O}_{X,x} \rangle_{n+1},$$

hence $n \geq \dim(X)$ by [13, Proposition 7.14].


Corollary 10. Let $X$ be a regular, quasi-projective scheme over an uncountable field. Then $\text{CRdim}(D^b_{\text{coh}}(X)) = \text{dim}(X)$.


Remark 11. Since the countable Rouquier dimension only gives the expected answer for varieties over a sufficiently large field, Orlov suggests an alternative notion which makes sense for a $k$-linear, pre-triangulated dg-category $\mathcal{A}$ with $k$ a field: Take the smallest $n$ such that there exists a countable family $\{E_i\}_i$ of objects of $\mathcal{A}$ such that $\mathcal{A}_K = \{(E_i)_K\}_i$ for every field extension $K/k$.

Theorem 12. Let $k$ be a field. Let $X, Y$ be smooth projective varieties over $k$. Assume there exists a fully faithful, exact, $k$-linear functor $F : D^b_{\text{coh}}(X) \rightarrow D^b_{\text{coh}}(Y)$. Then $\text{dim}(X) \leq \text{dim}(Y)$.

Proof. Let us choose any uncountable extension field $K/k$. By [11, Theorem 2.2] and [2, Theorem 1.1], $F$ is the Fourier–Mukai transform with respect to a kernel $E \in D^b_{\text{coh}}(X \times_k Y)$. Then $E_K$ gives rise to a functor $F_K : D^b_{\text{coh}}(X_K) \rightarrow D^b_{\text{coh}}(Y_K)$ which remains fully faithful by the calculus of kernels, see [12, Lemma 2.12].

We have $\text{CRdim}(D^b_{\text{coh}}(X_K)) = \text{dim}(X_K) = \text{dim}(X)$ by Corollary 10 and similarly for $Y$. Thus by Lemma 7 applied to the right adjoint of $F_K$ (which exists by [2, Theorem 1.1] and is essentially surjective since $F_K$ is fully faithful), we have $\text{dim}(X) = \text{CRdim}(D^b_{\text{coh}}(X_K)) \leq \text{CRdim}(D^b_{\text{coh}}(Y_K)) = \text{dim}(Y)$, as needed.

4. Strong generators for regular schemes

Recall that a strong generator of a triangulated category $\mathcal{T}$ is an object $G \in \mathcal{T}$ which generates $\mathcal{T}$ in a finite number of steps, i.e., $\mathcal{T} = \langle G \rangle_d$ for some integer $d$.

Example 13. The object $\bigoplus_{i=0}^N \mathcal{O}_{\mathbb{P}^N}(-i)$ is a strong generator of $D^b_{\text{coh}}(\mathbb{P}^N_Z)$. This can be seen for example from the existence of a semiorthogonal decomposition $D^b_{\text{coh}}(\mathbb{P}^N_Z) = \langle D^b_{\text{coh}}(Z) \otimes \mathcal{O}_{\mathbb{P}^N}(-N), D^b_{\text{coh}}(Z) \otimes \mathcal{O}_{\mathbb{P}^N}(-N+1), \ldots, D^b_{\text{coh}}(Z) \otimes \mathcal{O}_{\mathbb{P}^N} \rangle$ (see [10, Theorem 2.6]) and the fact that $Z$ is a strong generator of $D^b_{\text{coh}}(Z)$.

Theorem 14 ([7, Theorem 0.5]). Let $X$ be a Noetherian regular scheme of finite Krull dimension which possesses an ample line bundle. Then $D^b_{\text{coh}}(X)$ has a strong generator.

Proof. Let $\mathcal{L}$ be an ample line bundle on $X$. After possibly replacing $\mathcal{L}$ with a positive tensor power, we may assume $\mathcal{L}$ is generated by finitely many global sections so that $\mathcal{L} = f^* \mathcal{O}_{\mathbb{P}^N}(1)$ for some morphism $f : X \rightarrow \mathbb{P}^N_Z$. We claim there is an integer $M$ such that $
abla_{\mathcal{L}^\otimes n} \subseteq \left( \bigoplus_{i=0}^N \mathcal{L}^\otimes i \right)_M$. 


This suffices since together with Theorem 4 it implies that the object \( G = \bigoplus_{i=0}^{N} \mathcal{L} \otimes^{-i} \) is a strong generator of \( D^b_{\text{coh}}(X) \). To prove the claim, note that since \( Lf^* \mathcal{O}_P(n) = \mathcal{L} \otimes^n \) and the subcategories \( (-)_i \) are preserved by exact functors, it suffices to prove the claim with \( X \) replaced by \( P^N_Z \) and \( \mathcal{L} \) replaced by \( \mathcal{O}_P(1) \). But this follows from Example 13.

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References


[7] A. Neeman, Strong generators in \( D^b_{\text{perf}}(X) \) and \( D^b_{\text{coh}}(X) \), Ann. of Math. (2) 193 (2021), no. 3, 689–732.


