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ON A PROBLEM CONNECTED WITH THE KOLMOGOROV INTEGRODIFFERENTIAL EQUATIONS FOR A COMPLETELY DISCONTINUOUS STOCHASTIC PROCESS

In the paper [1] are given the integrodifferenzial equations for a completely discontinuous stochastic process, which must not be a Markov process. This paper is a generalization of the paper [1]. The Kolmogorov Equations for a completely discontinuous Markov process may be found for instance in the works [2], [3].

Let us consider a stochastic process $X(t)$ with a phase space X in which a σ -field B and finite time interval T are fixed.

Let $P_{n+1}(t_0, x_0, t_1, x_1, \dots, t_n, x_n, t_{n+1}, A)$, where $t_0 < t_1 < t_2 < \dots < t_n < t_{n+1}$, $(t_0, t_1, \dots, t_{n+1}) \in T$, $(x_0, x_1, \dots, x_{n+1}) \in X$, $A \in B$, be the transition probabilities, i.e.

$$P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, A) = P(X(t_{n+1}) \in A | \overline{X(\bar{t}_n)} = \bar{x}_n) \quad (1)$$

where $\bar{t}_n = [t_0, \dots, t_n]$, $\bar{x}_n = [x_0, \dots, x_n]$, $[X(t_0), \dots, X(t_n)] = \overline{X(\bar{t}_n)}$.

We shall assume, that the transition probabilities (1) satisfy the continuity condition

$$\lim_{t_{n+1} \rightarrow t_n} P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, A) = \chi_A(x_n),$$

where χ_A is the characteristic function of the set A .

We assume further that:

I. There exist the limits

$$\lim_{t_{n+1} \rightarrow t_n} \frac{P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, A) - \chi_A(x_n)}{t_{n+1} - t_n} = q_n(\bar{t}_n, \bar{x}_n, A) \quad (2)$$

and the convergence is uniform with respect to (t_n, x_n, A) .

II. Function (2) is continuous with respect to t_n .

We say that the process $X(t)$ is a completely discontinuous stochastic process if the conditions I and II are satisfied.

Let us put

$$q_n(\bar{t}_n, \bar{x}_n) = -q_n(\bar{t}_n, \bar{x}_n, \{x_n\}) = q_n(\bar{t}_n, \bar{x}_n, x - \{x_n\})$$

$$\Pi_n(\bar{t}_n, \bar{x}_n, A) = \begin{cases} \frac{q_n(\bar{t}_n, \bar{x}_n, A - \{x_n\})}{q_n(\bar{t}_n, \bar{x}_n)} & \text{for } q_n(\bar{t}_n, \bar{x}_n) \neq 0 \\ \chi_A(x_n) & \text{for } q_n(\bar{t}_n, \bar{x}_n) = 0 \end{cases} \quad (3)$$

It is well known that for a completely discontinuous Markov process the following integrodifferential equations are satisfied

$$\frac{\partial P_1(t_0, x_0, t_1, A)}{\partial t_1} = - \int_A q_0(t_1, x_1) P_1(t_0, x_0, t_1, dx_1) +$$

$$+ \int_{\mathbb{R}} q_0(t_1, x_1) \Pi_0(t_1, x_1, A) P_1(t_0, x_0, t_1, dx_1)$$

$$\frac{\partial P_1(t_0, x_0, t_1, A)}{\partial t_0} = q_0(t_0, x_0) \left[P_1(t_0, x_0, t_1, A) + \right.$$

$$\left. - \int_{\mathbb{R}} P_1(t_0, z, t_1, A) \Pi_0(t_0, x_0, dz) \right]$$

In this paper we shall not assume that $X(t)$ is a Markov process and we shall find the integrodifferential equations for the functions P_{n+1} , Π_n, q_n .

It is interesting that for $n=0$ the form of the equations in the markovian and non-markovian case are very similar. Then the assumption that the process is a Markov process is not essential. The form of the equations is essentially different for $n > 0$. For the Markov process the case $n > 0$ is not interesting.

The following theorem¹ will be proved.

T h e o r e m. If

1°. $X(t)$ is a completely discontinuous stochastic process with the transition probabilities (1),

2°. Function (2) is continuous with respect to t_n and uniformly with respect to (x_n, A) ,

3°. There exists the limit

$$\lim_{t_{n+1} \rightarrow t_n} \int_{\bar{x}} P_{n+2}(\bar{t}_n, \bar{x}_n, t_{n+1}, z, t_{n+2}, A) \Pi_{n+1}(\bar{t}_n, \bar{x}_n, dz) \quad (5)$$

then the following integrodifferential equations hold

$$\begin{aligned} & \frac{\partial P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, A)}{\partial t_{n+1}} = \\ & = - \int_A q_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, z) P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, dz) + \\ & + \int_{\bar{x}} q_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, z) \Pi_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, z, A) P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, dz) \\ & \frac{\partial P_{n+2}(\bar{t}_n, \bar{x}_n, t_{n+1}, x_n, t_{n+2}, A)}{\partial t_{n+1}} \Big|_{t_{n+1}=t_n} = \\ & = q_n(\bar{t}_n, \bar{x}_n) \left[P_{n+2}(\bar{t}_n, \bar{x}_n, t_n, x_n, t_{n+2}, A) + \right. \end{aligned} \quad (6)$$

¹ In the paper [1] was considered the case $n=0$

$$= \lim_{t_{n+1} \rightarrow t_n} \int_{\mathbb{R}} P_{n+2}(\bar{t}_n, \bar{x}_n, t_{n+1}, z, t_{n+2}, A) \Pi_n(\bar{t}_n, \bar{x}_n, dz) \quad (7)$$

P r o o f. The proof is analogous to the proof in Markov case. The proof in the Markov case is based on the Chapman-Kolmogorov equation. If we not assume that the Markov property holds, then we have [4]

$$\begin{aligned} P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+2}, A) &= \\ &= \int_{\mathbb{R}} P_{n+2}(\bar{t}_n, \bar{x}_n, t_{n+1}, z, t_{n+2}, A) P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, dz) \end{aligned} \quad (8)$$

From (2) we have

$$\begin{aligned} P_{n+2}(\bar{t}_n, \bar{x}_n, t_{n+1}, z, t_{n+2}, A) &= \\ &= \chi_A(z) + (t_{n+2} - t_{n+1}) q_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, z, A) + o(t_{n+2} - t_{n+1}) \end{aligned}$$

From the last relation and (8) it follows that

$$\begin{aligned} P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+2}, A) &= \int_{\mathbb{R}} \chi_A(z) P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, dz) + \\ &+ (t_{n+2} - t_{n+1}) \int_{\mathbb{R}} q_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, z, A) P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, dz) + \\ &+ o(t_{n+2} - t_{n+1}) \end{aligned} \quad (9)$$

By using the properties of the functions $\chi_A(z)$, dividing both side (9) by $(t_{n+2} - t_{n+1})$ and passing to the limit, as $t_{n+2} \rightarrow t_{n+1}$, we obtain

$$\begin{aligned} \frac{\partial P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, A)}{\partial t_{n+1}} &= \\ &= \int_{\mathbb{R}} q_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, z, A) P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, dz) \end{aligned} \quad (10)$$

From (3) we have

$$q_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, z, A) = q_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, z) \left[\Pi_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, z, A) - \chi_A(z) \right] \quad (11)$$

and (10) can be written in the following form

$$\begin{aligned} \frac{\partial P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, A)}{\partial t_{n+1}} = \\ - \int_{\mathbb{R}} \chi_A(z) q_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, z) P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, dz) + \\ + \int_{\mathbb{R}} q_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, z) \Pi_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, z, A) P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, dz) \end{aligned}$$

Taking into account the properties of the function $\chi_A(z)$ from the last relation we obtain equation (6).

We shall now derive equation (7).

Equation (8) we can write in the form

$$\begin{aligned} P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+2}, A) + \\ + \int_{\mathbb{R}} P_{n+2}(\bar{t}_n, \bar{x}_n, t_{n+1}, z, t_{n+2}, A) P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, dz) \end{aligned} \quad (12)$$

From (2) we have

$$\begin{aligned} P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+1}, A) = \chi_A(x_n) + (t_{n+1} - t_n) q_n(\bar{t}_n, \bar{x}_n, A) + \\ + o(t_{n+1} - t_n) \end{aligned}$$

Therefore, from the last relation and (11) it follows that

$$\begin{aligned} P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+2}, A) - P_{n+2}(\bar{t}_n, \bar{x}_n, t_{n+1}, x_n, t_{n+2}, A) = \\ = (t_{n+1} - t_n) \int_{\mathbb{R}} P_{n+2}(\bar{t}_n, \bar{x}_n, t_{n+1}, z, t_{n+2}, A) q_n(\bar{t}_n, \bar{x}_n, dz) + o(t_{n+1} - t_n) \end{aligned} \quad (13)$$

Since $P_{n+1}(\bar{t}_n, \bar{x}_n, t_{n+2}, A) = P_{n+2}(\bar{t}_n, \bar{x}_n, t_n, x_n, t_{n+2}, A)$, Therefore, dividing both side (13) by $(t_{n+1} - t_n)$, and passing to the limit as $(t_{n+1} - t_n) \rightarrow 0$ we obtain

$$\left. \frac{\partial P_{n+2}(\bar{t}_n, \bar{x}_n, t_{n+1}, x_n, t_{n+2}, A)}{\partial t_{n+1}} \right|_{t_{n+1}=t_n} =$$

$$\lim_{t_{n+1} \rightarrow t_n} \int_X P_{n+2}(\bar{t}_n, \bar{x}_n, t_{n+1}, z, t_{n+2}, A) q_n(t_n, x_n, dz)$$

Hence, by using (11) and the properties of the function $\chi_A(x)$ we obtain equation (7).

This completes the proof of the theorem.

We shall now consider the special case of formula (6). Namely. Let X be a numerabl space. Then (6) can be written in the form

$$\frac{\partial p_{i_0, i_1, i_2, \dots, i_n, i_{n+2}}(t_0, t_1, t_2, \dots, t_n, t_{n+2})}{\partial t_{n+2}} =$$

$$= - p_{i_0, i_1, i_2, \dots, i_n, i_{n+2}}(t_0, t_1, t_2, \dots, t_n, t_{n+2}) \times$$

$$\times q_{i_0, i_1, i_2, \dots, i_n, i_{n+2}}(t_0, t_1, t_2, \dots, t_n, t_{n+2}) +$$

$$+ \sum_{i_{n+1} \neq i_n} p_{i_0, i_1, i_2, \dots, i_n, i_{n+1}}(t_0, t_1, t_2, \dots, t_n, t_{n+2}) \times$$

$$\times q_{i_0, i_1, i_2, \dots, i_n, i_{n+1}, i_{n+2}}(t_0, t_1, t_2, \dots, t_n, t_{n+2}) \quad (14)$$

where

$$p_{i_0, i_1, i_2, \dots, i_n, i_{n+2}}(t_0, t_1, t_2, \dots, t_n, t_{n+2}) =$$

$$= P_{n+1}(t_0, i_0, t_1, i_1, t_2, i_2, \dots, t_n, i_n, t_{n+2}, \{i_{n+2}\}) \times$$

$$\begin{aligned}
 & \times q_{i_0, i_1, i_2, \dots, i_n, i_{n+2}}(t_0, t_1, t_2, \dots, t_n, t_{n+2}) = \\
 & \lim_{\Delta t_{n+2} \rightarrow 0} \frac{1 - p_{i_0, i_1, i_2, \dots, i_n, i_{n+2}, i_{n+2}}(t_0, t_1, t_2, \dots, t_n, t_{n+2}, t_{n+2} + \Delta t_{n+2})}{\Delta t_{n+2}} \\
 & q_{i_0, i_1, i_2, \dots, i_n, i_{n+1}, i_{n+2}}(t_0, t_1, t_2, \dots, t_n, t_{n+2}) = \\
 & \lim_{\Delta t_{n+2} \rightarrow 0} \frac{p_{i_0, i_1, i_2, \dots, i_n, i_{n+1}, i_{n+2}}(t_0, t_1, t_2, \dots, t_n, t_{n+2}, t_{n+2} + \Delta t_{n+2})}{\Delta t_{n+2}}
 \end{aligned}$$

Let us assume that the process is a time-homogeneous process and the probability transition from the state j to a state different from $j+1$ in the interval $(t_{n+2}, t_{n+2} + \Delta t_{n+2})$ is $o(\Delta t_{n+2})$.

Denote $t_1 - t_0 = \tau_0, t_2 - t_1 = \tau_1, \dots, t_{n+2} - t_{n+1} = \tau_{n+1}$.

From the above it follows that equations (14) can be written in the form

for $i_{n+2} = i_n = j$

$$\begin{aligned}
 & \frac{\partial p_{i_0, \dots, i_{n-1}, jj}(\tau_0, \tau_1, \dots, \tau_n)}{\partial \tau_n} = \\
 & = -p_{i_0, \dots, i_{n-1}, jj}(\tau_0, \tau_1, \dots, \tau_n) q_{i_0, \dots, i_{n-1}, jj}(\tau_0, \tau_1, \dots, \tau_n)
 \end{aligned} \tag{15}$$

for $k = i_{n+2} > i_n = j$

$$\begin{aligned}
 & \frac{\partial p_{i_0, \dots, i_{n-1}, jk}(\tau_0, \tau_1, \dots, \tau_n)}{\partial \tau_n} = \\
 & = -p_{i_0, \dots, i_{n-1}, j, k}(\tau_0, \tau_1, \dots, \tau_n) q_{i_0, \dots, i_{n-1}, j, k}(\tau_0, \tau_1, \dots, \tau_n) + \\
 & + p_{i_0, \dots, i_{n-1}, j, k-1}(\tau_0, \tau_1, \dots, \tau_n) q_{i_0, \dots, i_{n-1}, j, k-1, k}(\tau_0, \tau_1, \dots, \tau_n)
 \end{aligned} \tag{16}$$

where

$$q_{i_0, \dots, i_{n-1}, j, k}(\tau_0, \tau_1, \dots, \tau_n) = \\ = q_{i_0, \dots, i_{n-1}, j, k-1, k}(\tau_0, \tau_1, \dots, \tau_n)$$

The solution of (15) which satisfies the initial condition $p_{i_0, \dots, i_{n-1}, j, j}(\tau_0, \tau_1, \dots, \tau_{n-1}, 0) = 1$ is

$$p_{i_0, \dots, i_{n-1}, j, j}(\tau_0, \tau_1, \dots, \tau_n) = \\ = e^{-\int q_{i_0, \dots, i_{n-1}, j, j}(\tau_0, \tau_1, \dots, \tau_n) d\tau_n}$$

The solution of (16) which satisfies the initial condition $p_{i_0, \dots, i_{n-1}, j, k}(\tau_0, \tau_1, \dots, \tau_{n-1}, 0) = 0$ is

$$p_{i_0, \dots, i_{n-1}, j, k}(\tau_0, \tau_1, \dots, \tau_n) = e^{-\int q_{i_0, \dots, i_{n-1}, j, k}(\tau_0, \tau_1, \dots, \tau_n) d\tau_n} \times$$

$$\times p_{i_0, \dots, i_{n-1}, j, k-1}(\tau_0, \tau_1, \dots, \tau_n) q_{i_0, \dots, i_{n-1}, j, k}(\tau_0, \tau_1, \dots, \tau_n) \times \\ \times e^{-\int q_{i_0, \dots, i_{n-1}, j, k}(\tau_0, \tau_1, \dots, \tau_n) d\tau_n}$$

By the same assumptions, i.e. if X is a numerable space and the process $X(t)$ is time homogenous process, than for $n = 0$ equation (7) can be written in the form

$$\frac{\partial p_{i_0 i_0 i_2}(\tau_0, \tau_1)}{\partial \tau_1} \bigg|_{\tau_0=0} = - q_{i_0} p_{i_0 i_0 i_2}(\tau_0, \tau_0 + \tau_1) + \\ + \lim_{\tau_0 \rightarrow 0} \sum_{i_1 \neq i_0} p_{i_0 i_1 i_2}(\tau_0, \tau_1) q_{i_0 i_1} \quad (17)$$

Taking into account that the series is uniformly convergent, and that

$$\lim_{\tau_0 \rightarrow 0} p_{i_0 i_1 i_2}(\tau_0, \tau_1) = p_{i_1 i_1 i_2}(0, \tau_1) = p_{i_1 i_2}(\tau_1),$$

$$p_{i_0 i_0 i_2}(0, \tau_1) = p_{i_0 i_2}(\tau_1)$$

from equation (17) we obtain

$$\frac{\partial p_{i_0 i_2}(\tau_1)}{\partial \tau_1} = -q_{i_0} p_{i_0 i_2}(\tau_1) + \sum_{i_1 \neq i_0} p_{i_1 i_2}(\tau_1) q_{i_0 i_1} \quad (18)$$

Equation (18) has the same form as the equation in Markov case. But in this paper equation (18) was received without assuming Markov property.

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