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ON SOME PROPERTIES OF MULTIVALENT ALPHA-STARLIKE FUNCTIONS

1. Among functions holomorphic and univalent in the unit disc the families of convex and starlike functions are widely known. The first of them was introduced by E. Study [10] the second by J.W. Alexander [1]. Basic properties of functions of this kind were also investigated by K. Löwner [4] and R. Nevalinna [8]. In the year 1969 P.T. Mocanu [7] defined the family of univalent alpha-convex functions. The idea of P.T. Mocanu enabled a joint investigation of structures of classes of convex and starlike functions. (cf. e.g. [5], [6]). On the other hand also (certain) families of p-valent convex and starlike functions are known (cf. e.g. [2], [3]).

The aim of this paper is to introduce and investigate a certain class of holomorphic p-valent and α -starlike functions in a unit disc.

D e f i n i t i o n 1. Let p be an arbitrary fixed positive integer and let α be a fixed non-negative number. Denote by $M_{\alpha}^{(p)}$ the family of all functions

$$(1) \quad f(z) = z^p + a_{p+1} z^{p+1} + a_{p+2} z^{p+2} + \dots$$

holomorphic in the disc $K = \{z : |z| < 1\}$, $f(z) \cdot f'(z) \neq 0$ in $0 < |z| < 1$, and satisfying the condition

$$(2) \quad \operatorname{Re} J(f; \alpha) > 0, \quad z \in K,$$

where

$$(3) \quad J(f; \alpha) = (1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right).$$

If $f \in M_\alpha^{(p)}$, then f is said to be a p -valent α -starlike function.

Notice that $M_\alpha^{(1)} = M_\alpha$, where M_α is a known class of α -convex functions [7], [5] i.e. the class of functions

$$(4) \quad g(z) = z + b_2 z^2 + b_3 z^3 + \dots, \quad z \in K$$

satisfying the condition $g(z) \cdot g'(z) \neq 0$ in $0 < |z| < 1$ and $\operatorname{Re} J(g; \alpha) > 0$ for $z \in K$. Moreover, $M_0^{(p)}$ and $M_1^{(p)}$ are respectively subclasses of p -valent starlike and p -valent convex functions in the sense of [2].

2. We shall show now a relationship between the classes $M_\alpha^{(p)}$ and M_α , and applications of this relationship.

Theorem 1. For an arbitrary real number $\alpha \geq 0$ and any arbitrary positive integer $p \geq 1$ the equality

$$(5) \quad M_\alpha^{(p)} = \left(M_{\frac{\alpha}{p}} \right)^p$$

holds (where $\left(M_{\frac{\alpha}{p}} \right)^p$ is the set of p -th powers of α -convex functions).

Proof. For an arbitrarily fixed function $f \in M_\alpha^{(p)}$ we shall consider the function

$$(6) \quad g(z) = \sqrt[p]{f(z)} = z \sqrt[p]{\frac{f(z)}{z^p}}, \quad \sqrt[p]{1} = 1.$$

A simple calculation leads to the following identity

$$(7) \quad (1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) = p \left[\left(1 - \frac{\alpha}{p} \right) \frac{zg'(z)}{g(z)} + \frac{\alpha}{p} \left(1 + \frac{zg''(z)}{g'(z)} \right) \right]$$

in the disc K . If $f \in M_\alpha^{(p)}$, then according to Definition 1 inequality (2) holds. It follows from conditions (2) and (7) that $\operatorname{Re} J(g; \frac{\alpha}{p}) > 0$ for $z \in K$. Therefore $g \in M_{\frac{\alpha}{p}}$ and

$f = \xi^p \in \left(M_{\frac{\alpha}{p}}\right)^p$. Hence we have demonstrated that $M_{\alpha}^{(p)} \subset \left(M_{\frac{\alpha}{p}}\right)^p$.

The opposite inclusion can be demonstrated in a similar way.

R e m a r k. Let p be a fixed positive integer and m an arbitrary integer $1 \leq m \leq p$. Denote by $M_{\alpha, m}^{(p)}$ the family of all functions

$$h_m(z) = z^m + a_{m+1} z^{m+1} + a_{m+2} z^{m+2} + \dots,$$

where $h_m \in M_{\alpha}^{(m)}$. From the above quoted definitions and reasoning follows a generalized theorem:

$$(8) \quad M_{\alpha, m}^{(p)} = M_{\alpha} \cup \left(M_{\frac{\alpha}{2}}\right)^2 \cup \dots \cup \left(M_{\frac{\alpha}{p}}\right)^p.$$

It is evident that if $m = p$, then the decomposition (8) is reduced to (5).

T h e o r e m 2. If $0 \leq \beta < \alpha$, then

$$(9) \quad M_{\alpha}^{(p)} \subset M_{\beta}^{(p)}$$

holds.

P r o o f. It was noticed already in [5] that $M_{\alpha} \subset M_{\frac{\beta}{p}}$ for $0 \leq \beta < \alpha$. Hence follows the inclusion $\left(M_{\frac{\alpha}{p}}\right)^p \subset \left(M_{\frac{\beta}{p}}\right)^p$ which, by (5), is equivalent to (9).

As an immediate consequence of Definition 1 and Theorems 1,2 we obtain the following corollary.

C o r o l l a r y 1. $(M_{\alpha})^p \subset M_{\alpha}^{(p)}$, $\alpha \geq 0$, $p = 1, 2, \dots$.

T h e o r e m 3. The family $M_{\alpha}^{(p)}$ is a subclass of the class of p -valent starlike functions.

P r o o f. We obtain from Theorem 2 that $M_{\alpha}^{(p)} \subset M_0^{(p)}$, it follows herefrom and from Theorem 1 that $M_{\alpha}^{(p)}$ consists of p -th powers of univalent starlike function (more precisely: of α -convex functions). Therefore functions belonging to the class $M_{\alpha}^{(p)}$ are p -valent and starlike (cf. e.g. [9]). It should be emphasized here that any function $f \in M_{\alpha}^{(p)}$ has all its zeros in the origin and satisfies the well known differen-

tial inequality $\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0$ in the entire disc K . Hence it is obvious that $M_\alpha^{(p)} \subset M_0^{(p)}$ is a proper subclass of the class of p -valent starlike functions defined in [2] (c.f. also [3]).

3. We come now to some properties of initial coefficients of functions belonging to $M_\alpha^{(p)}$.

Theorem 4. If a function f of the form (1) belongs to the class $M_\alpha^{(p)}$, $0 < \alpha < p$, $p = 1, 2, \dots$, then for any arbitrary complex number η the sharp estimation

$$(10) \quad \left| a_{p+2} - \eta a_{p+1}^2 \right| \leq \frac{p^2}{(p+2\alpha)} \max \left[1, \frac{|4\eta(p+2\alpha)p^2 - (\alpha^2 + 4\alpha p + 4\alpha p^2 + p^2 + 2p^3)|}{(p+\alpha)^2} \right]$$

holds.

Proof. From (6) we obtain for the functions f and g of the form (1) and (4) respectively the following formulas for coefficients:

$$(11) \quad a_{p+1} = pb_2$$

and

$$(12) \quad a_{p+2} = pb_3 + \frac{p(p-1)}{2} b_2^2, \quad p = 1, 2, \dots$$

Therefore

$$(13) \quad \left| a_{p+2} - \eta a_{p+1}^2 \right| = p \left| b_3 - \mu b_2^2 \right|,$$

where

$$(14) \quad \mu = \frac{1}{2} (2p\eta + 1 - p).$$

In [11] the following estimation in M_α , $0 < \alpha \leq 1$, was obtained

$$(15) \quad |a_3 - \mu a_2^2| \leq \frac{1}{(1+2\alpha)} \max \left[1, \frac{|4\mu(1+2\alpha) + 4 - (1+\alpha)(7+\alpha)|}{(1+\alpha)^2} \right]$$

(μ stands for an arbitrary complex number). Therefore, it follows from Theorem 1 and from conditions (13), (14), (15) that the estimation (10) holds. As the estimation (15) is sharp therefore we get from conditions (6) and (5) the sharp estimation of (10).

C o r o l l a r y 2. If $f_\alpha \in M_\alpha^{(p)}$, $\alpha > 0$, then the sharp estimation

$$(16)) \quad |a_{p+1}| \leq \frac{2p^2}{p+\alpha}, \quad p = 1, 2, \dots$$

holds.

P r o o f. Inequality (16) follows from (11) and from the well known estimation $|b_2| \leq \frac{2p}{p+\alpha}$, $\alpha > 0$, in the class $M_{\frac{\alpha}{p}} [5]$. The extremal function $f^* \in M_\alpha^{(p)}$ is given by the formula

$$(17) \quad f^*(z; \alpha) = \left[g^*(z; \frac{\alpha}{p}) \right]^p,$$

where

$$(18) \quad g^*(z; \alpha) = \left[\frac{1}{\alpha} \int_0^z \zeta^{\frac{1}{\alpha}-1} (1-\zeta)^{-\frac{2}{\alpha}} d\zeta \right]^\alpha.$$

It is worth mentioning that the extremal function (17) is the solution of the differential equation

$$z \left\{ \frac{f(z)}{z^p} \left[\frac{zf'(z)}{pf(z)} \right]^\alpha \right\}^{\frac{1}{p}} = \frac{z}{(1-z)^2}$$

with the boundary condition $f(0) = 0$. It is therefore closely connected with the well known Koebe function.

From Corollary 2 we obtain in the classes $M_0^{(p)}$ and $M_1^{(p)}$ the following estimations: $|a_{p+1}| \leq 2p$, $|a_{p+1}| \leq \frac{2p^2}{1+p^2}$.

Putting in Theorem 4 $\eta = 0$ we obtain

C o r o l l a r y 3. If $f \in M_\alpha^{(p)}$, $0 \leq \alpha \leq 1$, then the sharp estimation

$$|a_{p+2}| \leq \frac{p^2}{(p+2\alpha)} \left[1 + \frac{2p(p^2 + 2\alpha p + \alpha)}{(p+\alpha)^2} \right], \quad p = 1, 2, \dots$$

holds.

In particular in the classes $M_0^{(p)}$ and $M_1^{(p)}$ the following inequalities hold: $|a_{p+2}| \leq p(1+2p)$, $|a_{p+2}| \leq \frac{p^2(1+2p)}{p+2}$.

4. In the sequel of the paper we shall apply the following notations

$$K(\alpha; r) = r \left[G\left(\frac{1}{\alpha}, \frac{2}{\alpha}, \frac{1}{\alpha} + 1; r\right) \right]^\alpha = \left[\frac{1}{\alpha} \int_0^r \varrho^{\frac{1}{\alpha}-1} (1-\varrho)^{-\frac{2}{\alpha}} d\varrho \right]^\alpha,$$

where

$$\begin{aligned} (19) \quad G(a, b, c; z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!} = \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1} (1-u)^{c-a-1} (1-zu)^{-b} du, \end{aligned}$$

$\operatorname{Re}(a) > 0$, $\operatorname{Re}(c-a) > 0$. It is known that the hypergeometric function (19) is holomorphic in the disc K .

T h e o r e m 5. If $f \in M_\alpha^{(p)}$, $\alpha > 0$, then for $|z| = r$, $0 < r < 1$ the following sharp estimations hold

$$(20) \quad \left[-K\left(\frac{\alpha}{p}; -r\right) \right]^p \leq |f(z)| \leq \left[K\left(\frac{\alpha}{p}; r\right) \right]^p.$$

P r o o f. We know by Theorem 1 that $f \in M_{\alpha}^{(p)}$ if and only if $g = \sqrt[p]{f} \in M_{\frac{\alpha}{p}}$. It has been demonstrated in [5] that if $g \in M_{\frac{\alpha}{p}}$, $\alpha > 0$, then

$$(21) \quad -K\left(\frac{\alpha}{p}; -r\right) \leq |g(z)| \leq K\left(\frac{\alpha}{p}; r\right), \quad |z| = r, \quad 0 < r < 1,$$

where the equality in (21) is realized by the function $g^*(z; \frac{\alpha}{p})$ defined by formula (18). Hence from estimation (21) we obtain (20). The extremal function $f^*(z; \alpha)$ is given by formula (17).

Note that inequality (21) implies the distortion theorem in class $M_1^{(p)}$

$$\frac{r^p}{(1+r)^p} \leq |f(z)| \leq \frac{r^p}{(1-r)^p}, \quad |z| = r, \quad 0 < r < 1.$$

Applying a similar method we can obtain e.g. the following properties.

T h e o r e m 6. If $f \in M_{\alpha}^{(p)}$, $\alpha > 0$, then the image of disc K under the mapping $w = f(z)$ embraces always the disc $|w| < d(\alpha, p)$ where

$$d(\alpha, p) = \left[\frac{p}{2\alpha} \frac{\Gamma(\frac{p}{\alpha})^2}{\Gamma(\frac{2p}{\alpha})} \right]^{\alpha}.$$

The result is sharp with an equality for the function $f^*(z; \alpha)$ given by formula (17).

As it follows from the Stirling formula that $\lim_{\alpha \rightarrow 0+} d(\alpha, p) = \left(\frac{1}{4}\right)^p$ therefore applying the limit process we obtain the covering theorem for the class $M_0^{(p)}$.

T h e o r e m 7. The radius of α -convexity of the family $M_0^{(p)}$ is given by the formula

$$R_{\alpha}(M_0^{(p)}) = \sup \left[r : \operatorname{Re} J(f; \alpha) > 0, |z| < r, f \in M_0^{(p)} \right] = \frac{1}{p} \left[\alpha + p - \sqrt{(\alpha + p)^2 - p^2} \right].$$

(In the proof of Theorem 7 the formula (5) as well as the relevant result of [6] can be applied).

5. Let us finally notice that similarly as in the family M_α also in the class $M_\alpha^{(p)}$ the following structural properties hold.

If $f \in M_\alpha^{(p)}$, then for $0 < \gamma \leq \alpha$

$$(22) \quad g(z) = z \left\{ \frac{f(z)}{z^p} \left[\frac{zf'(z)}{pf(z)} \right]^\gamma \right\}^{\frac{1}{p}}$$

(with appropriately chosen branches) is a univalent starlike function.

Reversely, if g is a univalent starlike function, then the function

$$f(z) = \left[\frac{p}{\alpha} \int_0^z \frac{g(\zeta)^{\frac{p}{\alpha}}}{\zeta} d\zeta \right]^\alpha$$

which is a solution of the differential equation (22) with boundary condition $f(0) = 0$ belongs to class $M_\alpha^{(p)}$.

These properties can be obtained from the relations existing between the families $M_\alpha^{(p)}$ and M_α or immediately from (2) defining the family $M_\alpha^{(p)}$ and suitable properties of the family S^* of univalent starlike functions. In the second case the basic results of this paper can be obtained without applying Theorem 1.

The obtained theorems are natural generalizations of results obtained earlier by P.T. Mocanu, S.S. Miller, M.O. Reade [5], [6] and J. Szynal [11] for the case $p = 1$.

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