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ON THE RADIUS OF CONVEXITY FOR CERTAIN FAMILY
OF FUNCTIONS MEROMORPHIC IN UNIT DISC

1. Let $\mathcal{P}$ denote the family of functions of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots$$

holomorphic in the disc $K = \{z: |z| < 1\}$ and for every $z \in K$ satisfying the condition $\operatorname{re} p(z) > 0$.

Let the symbol $\mathcal{P}(m)$ denote the subclass of m -symetric functions of the family $\mathcal{P}$, where $m \geq 1$ is a fixed natural number.

Denote by Σ the family of functions F meromorphic and univalent in the disc K and which in the neighbourhood $z=0$ have the expansion of the form

$$w = F(z) = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + \dots$$

In the family Σ we distinguish two following subclasses; subclass Σ^* of the functions $w = F(z)$ mapping the disc K onto regions whose complements to the plane are starlike regions in relation to the point $w = 0$, and subclass Σ_α^* of the functions $w = F(z)$, such that, for an arbitrary fixed α , $0 \leq \alpha < 1$, and for every $z \in K$, the condition is satisfied

$$(1.1) \quad \operatorname{re} [-zF'(z)/F(z)] > \alpha.$$

It follows that $F \in \Sigma_\alpha^*$ if and only if there is a function $p \in \mathcal{P}$ such that for every $z \in K$, the condition:

$$-zF'(z)/F(z) = (1-\alpha)p(z) + \alpha \quad (\text{see}[4], \text{p.518})$$

is satisfied.

Finally let $\Sigma_{\alpha}^*(\beta)$ denote the class of functions $F \in \Sigma$ for which the condition

$$(1.2) \quad -\frac{zF'(z)}{F(z)} = [(1-\alpha)p(z) + \alpha]^{\beta}, \quad z \in K$$

is satisfied, where $p \in \mathcal{P}$ and $0 \leq \alpha < 1$, $0 < \beta \leq 1$ are arbitrary fixed real numbers.

2. Denote

$$r(F) = \sup \left\{ r : \operatorname{re} \left[-\left(1 + \frac{zF''(z)}{F'(z)} \right) \right] > 0, |z| < r \right\}.$$

The number $r^c[\Sigma_{\alpha}^*(\beta)] = \inf \{ r(F) : F \in \Sigma_{\alpha}^*(\beta) \}$ will be called the radius of convexity of the family $\Sigma_{\alpha}^*(\beta)$. It is easy to show that the family $\Sigma_{\alpha}^*(\beta)$ is compact, hence the radius of convexity of the family $\Sigma_{\alpha}^*(\beta)$ is equal to the greatest value of r , $0 < r \leq 1$ for which

$$\operatorname{re} \left[-\left(1 + \frac{zF''(z)}{F'(z)} \right) \right] \geq 0$$

for every $F \in \Sigma_{\alpha}^*(\beta)$ and z , where $|z| \leq r$.

Thus the radius of convexity $r^c[\Sigma_{\alpha}^*(\beta)]$ is equal to the smallest root r_0 , $0 < r_0 \leq 1$, of the equation $\omega(r) = 0$, where

$$(2.1) \quad \omega(r) = \min \left\{ \operatorname{re} \left[-\left(1 + \frac{zF''(z)}{F'(z)} \right) \right] : |z| = r < 1, F \in \Sigma_{\alpha}^*(\beta) \right\}.$$

From condition (1.2) it follows that for every $F \in \Sigma_{\alpha}^*(\beta)$ there exists a function $p \in \mathcal{P}$ such that, for every $z \in K$, the equality is satisfied

$$-\left[1 + \frac{zF''(z)}{F'(z)} \right] = [(1-\alpha)p(z) + \alpha]^{\beta} - \frac{\beta(1-\alpha)zp'(z)}{(1-\alpha)p(z) + \alpha}.$$

From the last two equalities (see[3]) we obtain

$$(2.2) \quad \omega(r) = \min \left\{ \operatorname{re} \left[((1-\alpha)p^*(z)+\alpha)^\beta - \frac{\beta(1-\alpha)zp^*(z)}{(1-\alpha)p^*(z)+\alpha} \right] : |z|=r < 1, p^* \in \mathcal{P}_2 \right\},$$

where $\mathcal{P}_2$ is the family of functions of the form

$$p^*(z) = \frac{1+\lambda}{2} \frac{1+\varepsilon_1 z}{1-\varepsilon_1 z} + \frac{1-\lambda}{2} \frac{1+\varepsilon_2 z}{1-\varepsilon_2 z},$$

$$-1 \leq \lambda \leq 1, |\varepsilon_1| = |\varepsilon_2| = 1.$$

Making use of Lemma 1 and 2 (see [1], p.70) and assuming $h = \frac{\alpha}{1-\alpha}$ from (2.2) we obtain

$$(2.3) \quad \omega(r) = \min \left\{ \operatorname{re} \left[\left(\frac{p^*(z)+h}{1+h} \right)^\beta - \beta \frac{(p^*(z))^2-1}{2(p^*(z)+h)} + \right. \right. \\ \left. \left. + \beta \frac{(\varrho^2 - |p^*(z)-c|^2)\eta}{2(p^*(z)+h)} \right] : |z|=r < 1, p^* \in \mathcal{P}_2 \right\},$$

where $z = re^{i\varphi}$, $0 \leq r < 1$, $0 \leq \varphi \leq 2\pi$,

$$(2.4) \quad c = \frac{1+r^2}{1-r}, \quad \varrho = \frac{2r}{1-r^2}$$

$\eta = \eta_1 \eta_2$ at the same time

$$\eta_k = \varepsilon_k e^{i\varphi} \frac{1 - \bar{\varepsilon}_k e^{-i\varphi} r}{1 - \varepsilon_k e^{i\varphi} r}; \quad k = 1, 2; |\varepsilon_1| = |\varepsilon_2| = 1.$$

Assuming $p^*(re^{i\varphi}) + h = se^{it}$, $\operatorname{im} t = 0$ and taking into account the inequality $\operatorname{re}(e^{-it}\eta) \geq -1$ and the equality

$$\varrho^2 - |se^{it} - h - c|^2 = 2hs \cos t + 2cs \cos t - s^2 - h^2 - 2hc - 1$$

from (2.3), we obtain $\omega(r) \geq \min_{\varphi, t} L(s, t)$ where the function

$$(2.5) \quad L(s, t) = \frac{s^\beta \cos \beta t}{(1+h)^\beta} + \beta \frac{1-h^2-s^2}{2s} \cos t + \beta h - \beta(h+c) \cos t + \\ + \beta \frac{s^2+h^2+2hc+1}{2s}$$

is defined in the disc D_r , where

$$D_r = \left\{ (s, t) : s^2 - 2s(c+h) \cos t + h^2 + 2hc + 1 \leq 0; \right. \\ \left. -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}, c+h-\varrho \leq s \leq c+h+\varrho \right\}.$$

Direct calculations (see [4], p. 524) show that the function $L(s, t)$ can reach the minimum only at those points $(s, t) \in D_r$ for which $t = 0$. The problem of determining the minimum of the function $\omega(r)$ is thus reduced to determining the minimum of the function $L_1(s) = L(s, 0)$ in the interval $s \in [c+h-\varrho, c+h+\varrho]$.

We have the following lemmas.

L e m m a 1. The function

$$L_1(s) = \frac{s^\beta}{(1+h)^\beta} + \beta \frac{1+hc}{s} - \beta c$$

reaches the absolute minimum on the interval $[c+h-\varrho, c+h+\varrho]$ in the point

$$(2.6) \quad s_1 = c+h+\varrho$$

if the inequality $s_1 \leq s_2$ is satisfied, where

$$(2.7) \quad s_2 = (1+h)^{\frac{\beta}{1+\beta}} (1+hc)^{\frac{1}{1+\beta}}.$$

If, however, $s_1 > s_2$, then the function $L_1(s)$ reaches this minimum in the point s_2 . The minimum respectively amounts to

$$(2.8) \quad M_1(r) = M_1(r; \beta, h) = \frac{(c+h+q)^{1+\beta} + \beta(1+h)^\beta(1+hc)}{(1+h)^\beta(c+h+q)} - \beta c$$

$$(2.9) \quad M_2(r) = M_2(r; \beta, h) = (1+\beta) \left(\frac{1+hc}{1+h} \right)^{\frac{\beta}{1+\beta}} - \beta c.$$

L e m m a 2. For every function $F \in \Sigma_\alpha^*(\beta)$ and for every fixed z , $|z| = r$, $0 < r < 1$, and at fixed α and β the acute estimate holds:

$$(2.10) \quad \operatorname{re} \left\{ - \left[1 + \frac{zF''(z)}{F'(z)} \right] \right\} \geq \begin{cases} M_1(r) & \text{for } s_1 \leq s_2 \\ M_2(r) & \text{for } s_1 > s_2 \end{cases},$$

where $s_1, s_2, M_1(r), M_2(r)$ are respectively defined by the formulas (2.6) - (2.9). The equalities in the estimations (2.10) are realised respectively by functions of the form

$$(2.11) \quad F_1(z) = \frac{1}{z} \exp \int_0^z \left[1 - \frac{(1 + (1-2\alpha)e^{-i\varphi}z)^\beta}{1 - e^{-i\varphi}z} \right] \frac{dz}{z}$$

$$(2.12) \quad F_2(z) = \frac{1}{z} \exp \int_0^z \left[1 - \frac{(1+2K(1-\alpha)e^{-i\varphi}z + (1-2\alpha)e^{-2i\varphi}z^2)^\beta}{1 - e^{-2i\varphi}z^2} \right] \frac{dz}{z},$$

where $0 \leq \varphi \leq 2\pi$, however

$$K = \frac{(s_2 - h)(1 - r^2) - (1 + r^2)}{2r}.$$

3. We have the following theorem.

T h e o r e m. Denote by r_0 the only root of the equation

$$(3.1) \quad \left(\frac{1+\beta}{\beta} \right)^{\frac{1+\beta}{\beta}} \beta(1+r)(1-r^2)^{\frac{1}{\beta}} + (1+r^2) [r^2 - (1+\beta)r - \beta] = 0$$

in the interval $(0,1)$ and let α_0 denote the expression

$$(3.2) \quad \alpha_0 = \frac{(1+r_0)[r_0^2 - (2\beta+2)r_0+1]}{2r_0[r_0^2 - (1+\beta)r_0 - \beta]},$$

where $\beta \in (0,1]$, is an arbitrary fixed real number.

The radius of convexity of the family $\Sigma_\alpha^*(\beta)$ is given by the formula

$$(3.3) \quad r^c[\Sigma_\alpha^*(\beta)] = \begin{cases} r_2 & \text{for } 0 \leq \alpha \leq \alpha_0 \\ r_1 & \text{for } \alpha_0 \leq \alpha < 1, \end{cases}$$

where r_2 is equal to the only root in the interval $(0,1)$ of the equation

$$(3.4) \quad \left(\frac{1+\beta}{\beta}\right)^{1+\beta} (1-r^2+h+hr^2)(1-r^2) - (1+r^2)^{1+\beta} (1+h)^\beta = 0$$

and r_1 is equal to the only root in the interval $(0,1)$ of the equation

$$(3.5) \quad (1+r+h-hr)^{1+\beta} (1-r)^{1-\beta} - 2\beta(1+h)^\beta r = 0.$$

The forms of the extremal functions are respectively given by (2.12) and (2.11).

P r o o f. From Lemma 2 and from the formula (2.1) it follows that the radius of convexity of the family $\Sigma_\alpha^*(\beta)$ is equal to the root of the equation $M_1(r) = 0$ if $s_1 \leq s_2$, and it is equal to the root of the equation $M_2(r) = 0$ if $s_1 > s_2$.

It can be proved that the value α_0 given by formulas (3.1) and (3.2) satisfies the inequality $\frac{1}{2} < \alpha_0 < 1$ for every $\beta \in (0,1]$ and that for arbitrary fixed $\beta \in (0,1]$ and $\alpha \in [0, \alpha_0)$ the inequality $s_1 > s_2$ holds; for arbitrary fixed $\beta \in (0,1]$ and $\alpha \in (\alpha_0, 1)$ the inequality $s_1 < s_2$ holds, however for $\alpha = \alpha_0$ we have the equality $s_1 = s_2$. Solving the equations $M_2(r) = 0$ and $M_1(r) = 0$ respectively, we obtain (3.3).

C o r o l l a r y. Assuming $\beta = 1$ we obtain, from the last theorem, the radius of convexity of the family Σ_{α}^* marked by V.A. Zmorovič (see [4], p.524); assuming $\alpha = 0$ we obtain the radius of convexity of the family $\Sigma^*(\beta)$ for $m=1$, $r^c[\Sigma^*(\beta)] = \frac{1}{\sqrt{1+2\beta}}$ (see [1], p.81 - 82); however in the case when $\alpha = 0$ and $\beta = 1$ we obtain M. Robertson's result for the family Σ^* , $r = \frac{1}{\sqrt{3}}$ (see [3], p.239).

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