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A THEOREM ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS
FOR ABSTRACT, QUASI-LINEAR DIFFERENTIAL EQUATIONS1. Derivatives, integrals, limit conditions

Assume that we are given:

- a) real linear spaces: L^1 and $L^0, L^1 \subset L^0, *$
 b) m derivatives: i.e. linear operations

$$S_1, S_2, \dots, S_m : L^1 \longrightarrow L^0$$

from the space L^1 onto the space L^0 , so that

$$L^1 = \left\{ x \in L^0, \bigwedge_i S_i x \in L^0 \right\}.$$

The space L^2 is defined as a subspace L^1 such that

$$S_i : L^2 \longrightarrow L^1 \quad (\text{onto})$$

$$L^2 = \left\{ x \in L^0, \bigwedge_{i,j} S_i S_j x \in L^0 \right\}.$$

*) The symbols L^0, L^1 denote any real linear spaces such that $L^1 \subset L^0$ (the other notion of functional spaces $L^p = \left\{ x : \left(\int_0^1 |x(t)|^p dt \right)^{\frac{1}{p}} < +\infty \right\}$ is not used).

By induction we define the space L^n as

$$L^n = \left\{ x \in L^0 : \bigwedge_{(i_1, \dots, i_n)} S_{i_1} S_{i_2} \dots S_{i_n} x \in L^0 \right\}.$$

So we have

$$\dots \subset L^n \subset L^{n-1} \subset \dots \subset L^1 \subset L^0.$$

Let us assume that for derivatives S_i there exist integrals i.e. linear operations

$$\bigwedge_i T_i(q_i) : L^0 \longrightarrow L^1$$

(the q_i indices belongs to sets Q_i), such that we have

$$(1.1) \quad \bigwedge_i S_i T_i(q_i) = \text{id}_{L^0}$$

and the corresponding limit conditions hold

$$(1.2) \quad s_i(q_i)x = x - T_i(q_i) S_i x.$$

We have for $c \in \ker S$

$$(1.3) \quad s_i^2(q_i)c = s_i(q_i)c = c. \quad (\text{see [2]}).$$

We assume that various derivatives and integrals do commute

$$S_i S_j x = S_j S_i x, \quad \text{for } x \in L^2, \quad i, j = 1, 2, \dots, m.$$

$$T_i(q_i) T_j(q_j) f = T_j(q_j) T_i(q_i) f, \quad \text{for } f \in L^0, \quad i \neq j.$$

$$S_i T_j(q_j) x = T_j(q_j) S_i x, \quad \text{for } x \in L^1, \quad i \neq j.$$

It is then easy to prove, using the definition of limit condition, that we have

$$S_i s_j(q_j)x = s_j(q_j)S_i x, \quad \text{for } x \in L^2, \quad i \neq j,$$

$$T_i(q_i)s_j(q_j)x = s_j(q_j)T_i(q_i)x, \quad \text{for } x \in L^1, \quad i \neq j,$$

$$s_i(q_i)s_j(q_j)x = s_j(q_j)s_i(q_i)x, \quad \text{for } x \in L^2, \quad i \neq j.$$

(see [3]).

2. Differential equations. Singular integrals

We assume that L^0 is a Banach space and that integrals $T_i(q_i)$ are continuous in L^0 .

Let us consider the equation

$$(2.1) \quad \mathcal{F}(x, S_1 x, \dots, S_m x, S_{i_1} \dots i_\alpha x, \bar{f}) = 0,$$

where $\alpha < n$, $\mathcal{F}$ is a function of variables

$$x, S_1 x, \dots, S_m x, S_{i_1} \dots i_\alpha x, \bar{f}.$$

The symbol $\bar{f}_{D_f} = \left\{ S_{i_1}, \dots, i_n x \right\}$ denotes a vector $\bar{f}$ with coordinates $f_{i_1}, \dots, i_n = S_{i_1}, \dots, i_n x$.

Let us introduce the notations:

$$S_I = S_{i_1}, \dots, i_k = S_{i_1} \cdot S_{i_2} \dots S_{i_k}, \quad I = (i_1, \dots, i_k)$$

$$S_L = S_{i_1}, \dots, i_\alpha = S_{i_1} \cdot S_{i_2} \dots S_{i_\alpha}, \quad L = (i_1, \dots, i_\alpha)$$

$$T_I(q_I) = T_{i_1}, \dots, i_k(q) = T_{i_1}(q_{i_1}) \dots T_{i_k}(q_{i_k}),$$

$$T_L(q_L) = T_{i_1}, \dots, i_\alpha(q) = T_{i_1}(q_{i_1}) \dots T_{i_\alpha}(q_{i_\alpha}),$$

$$\bar{f} = \bar{S}^n x = \left\{ S_{i_1}, \dots, i_n x \right\} = \left\{ f_{i_1}, \dots, i_n \right\}, \quad \text{for } x \in L^{n+1},$$

$$S_L x \in S_L(L^{n+1}), \quad \bar{S}^n x \in \bigcap_J S_J(L^{n+1}),$$

where $J = (i_1, i_2, \dots, i_n)$.

Definition 1. If an element

$$W : \{S_J W\} = 0, \quad 1 \leq i_1 \leq \dots \leq i_n \leq m$$

is the solution of the equation

$$(2.2) \quad \mathcal{F}(W, S_1 W, \dots, S_m W, S_L W, \dots, \{0\}) = 0$$

it will be called a singular integral of the equation (2.1).

Definition 2. If an element x is a nonsingular integral: i.e.

$$\mathcal{F}(x, S_1 x, \dots, S_I x, \dots, \{S_J x\}) = 0$$

when $\{S_J x\} \neq 0$, then for

$$a_{Df} = \{x, \dots, S_L x, \dots\}, \quad G_a = \{a\} \subset \prod_L S_L(L^{n+1})$$

$$b_{Df} = \{S_J x\}, \quad G_b = \{b\} \subset Z,$$

where $Z = \prod_J L^1$, we have

$$\mathcal{F}(a, b) = 0, \quad (G_a \times G_b) \subset \left(\prod_L S_L(L^{n+1-k}) \times Z \right),$$

with norms

$$\|b\|_* = \sup_{Df} \|S_J x\|, \quad \|a\|_* = \sup_L \|x\|, \dots, \|S_L x\|, \dots,$$

in products of some Banach spaces.

Now we will give a theorem about the equivalence of the implicate equation (2.2) and an explicite equation for the n -th gradient $S^n x$ in certain neighbourhood of nonsingular integral.

We will use the theorem about implicate functions.

Let us consider the equation

$$(2.3) \quad \mathcal{F}(a, b) = 0$$

with an additional condition

$$(2.4) \quad g(a, b) = 0.$$

Let

$$\phi(a, b) = \underset{Df}{(\mathcal{F}(a, b), g(a, b))}.$$

We assume the function $\phi(a, b)$ has in a neighbourhood continuous Frechet's derivative $\mathcal{F}'_b(a_0, b_0)$, and that there exists $(\mathcal{F}'_b(a_0, b_0))^{-1}$. Then there are neighbourhoods $U_{a_0} \subset G_a$ and $W_{b_0} \subset G_b$, and a continuous operation

$$f = \underset{Df}{\left\{ f_j \right\}} : U_{a_0} \longrightarrow W_{b_0},$$

such that the equation

$$b = f(a)$$

is equivalent to the equations

$$\mathcal{F}(a, b) = 0, \quad b_0 = f(a_0)$$

$$g(a, b) = 0, \quad g(a_0, b_0) = 0.$$

By the properties of elements a and b we obtain an explicit differential equation of the n -th order

$$(2.5) \quad \{S_J x\} = f_J(\dots, S_L x, \dots) \quad \alpha < n$$

equivalent to the implicit differential equation (2.1) in the neighbourhood U_{a_0} .

D e f i n i t i o n 3. The points (a,b) such that

$$(2.6) \quad \mathcal{F}(a,b) = 0, \quad g(a,b) = 0$$

and that $(\mathcal{F}'_b(a,b))^{-1}$ does not exist or is not continuous, will be called degenerate points of the equation (2.1).

In the case of nondegenerate points (a_0, b_0) there is an equivalent explicite equation (2.6) with the continuous right side for $a \in U_{a_0}$ in a neighbourhood of (a_0, b_0) .

We shall further consider an explicite differential equation.

3. Necessary condition for the existence of solutions for a differential equation

Let us consider the equation

$$(3.1) \quad \bar{S}^n x = \bar{f}(a),$$

where $\bar{f}$ is a continuous function,

$$a = \left\{ x, \dots, S_L x, \dots \right\}_{Df}, \quad \alpha < n$$

$$(3.2) \quad s(q)a = \left\{ s(q)x, \dots, s(q)S_L x, \dots \right\}_{Df} = \left\{ a_0(x) \right\}.$$

By relation (3.2) we can develop x in Taylor's formula (see [3])^{*)}

$$(3.3) \quad x = W_{n-1}(q)x + d^n(q)S^n x,$$

^{*)} The Taylor formula for m derivatives $S_1, S_2, \dots, S_m$ can be obtained also by formulas from [1] and [4], [5].

where

$$W_{n-1}(q)x = \sum_L T_L(q)S(q)S_L x, \quad L = \{i_1, i_2, \dots, i_\alpha\}, \quad \alpha < n.$$

Let

$$(3.4) \quad \|d^n(q)\|_{Df} = \delta_H,$$

where $H = \{x, q, W_{n-1}(q)x, a\}$.

L e m m a . Let

$$(3.5) \quad \|x - W_{n-1}(q)x\| \leq \alpha,$$

where

$$\alpha = \delta_H \|S^n x\|.$$

P r o o f . By (3.3) we have

$$\|x - W_{n-1}(q)x\| = \|d^n(q)S^n x\| \leq \delta_H \|S^n x\| \leq \alpha.$$

L e m m a . If an element of the equation (3.1) with limit condition (3.2) in a set G is a nonsingular solution of the equation (3.5), then

$$(3.6) \quad \sup_{x \in K(W_{n-1}(q), x, \epsilon)} \frac{\|x - W_{n-1}(q)x\|}{\inf_Y \|\widehat{f}(a(y))\|} = \delta_H$$

$$Y = \left\{ y : \|y - W_{n-1}(q)x\| = \|x - W_{n-1}(q)x\|, \|y - W_{n-1}(q)x\| < \epsilon \right\}.$$

P r o o f . 1° Because f is a continuous function, we have

$$\|\widehat{S}^n x\| = \|\widehat{f}(x)\| \neq 0$$

in the neighbourhood

$$\|x - W_{n-1}(q)x\| \leq \alpha.$$

But $\|x - W_{n-1}(q)x\| \leq \delta_H \|\tilde{S}^n x\|$, so that

$$\frac{\|x - W_{n-1}(q)x\|}{\|\tilde{S}^n x\|} \leq \delta_H.$$

For $0 < \varepsilon < \delta_H$ there exists an x_ε such that

$$\delta_H - \varepsilon \leq \frac{\|x_\varepsilon - W_{n-1}(q)x_\varepsilon\|}{\|\tilde{S}^n x_\varepsilon\|} \leq \delta_H,$$

where $x \in K' \stackrel{\text{Df}}{=} K(W_{n-1}(q)x, \varepsilon)$. But

$$\delta_H - \varepsilon \leq \frac{\|x_\varepsilon - W_{n-1}(q)x_\varepsilon\|}{\inf_{K'} \|\tilde{f}(a(x_\varepsilon))\|} \leq \delta_H$$

and

$$\delta_H - \varepsilon \leq \sup_K \frac{\|x - W_{n-1}(q)x\|}{\inf_Y \|\tilde{f}(a(y))\|} \leq \delta_H.$$

If (3.6) is true then

$$\frac{\|x - W_{n-1}(q)x\|}{\inf_Y \|\tilde{f}(a(y))\|} \leq \delta_H$$

and

$$\frac{\|x - W_{n-1}(q)x\|}{\|\tilde{f}(a(x))\|} \leq \delta_H$$

so that $\|x - W_{n-1}(q)x\| \leq \delta_H \|S^n x\|$.

4. A theorem about the existence of a solution for a differential equation

T h e o r e m 1. (i) Consider two assumptions:

(A) there exists a solution x of the equation

$$(4.1) \quad \tilde{S}^n x = \tilde{f}(a(x))$$

with limit conditions

$$(4.2) \quad W_{n-1}(q)x = \xi_q^{n-1}$$

and

$$(4.3) \quad \|x - W_{n-1}(q)x\|_{L^0} \leq \delta \|\tilde{f}(a(x))\|_{L^0}.$$

(B) there exist solutions x_i of the equations

$$(4.4) \quad \tilde{S}_i^n x_i = \tilde{f}^i(a(x_i)), \quad 0 \leq i \leq N$$

with limit conditions

$$(4.5) \quad W_{n-1}(q_i)x_i = \xi_{q_i}^{n-1}, \quad 0 \leq i \leq N$$

and

$$(4.6) \quad \|x_i - W_{n-1}(q_i)x_i\|_{L_i^0} \leq \delta_i \|\tilde{f}^i(a(x_i))\|_{L_i^0},$$

where

$$\{(x, q, W_{n-1}(q)x, a(x)) \in H = \{(x_1, q_1, W_{n-1}(q_1)x_1, a(x_1)) \in H_1.$$

Moreover, let $0 < \delta_1 = \delta_H < +\infty$, $0 \leq i \leq N$ and

$$\sup_{H_1} \|\tilde{f}^1(a(x_1))\|_{L_1^0} = \mu_1.$$

We assume that (A) is equivalent with (B).

(ii) Let us denote

$$(4.7) \quad A_1 x_1 \stackrel{\text{Df}}{=} W_{n-1}(q_1)x_1 + d^n(q_1) \tilde{f}^1(a(x_1)),$$

where $d^n(q_1) : L_1^0 \longrightarrow L_1^0$

$$\{x_1, q_1, W_{n-1}(q_1)x_1, a(x_1)\} \in H_1.$$

We suppose that the set

$$\bar{W}_1 = A_1(V_1), \text{ where } x_1 \in K(W_{n-1}(q_1)x_1, \delta_1(\mu_1)),$$

$$V_1 \stackrel{\text{Df}}{=} W_{n-1}(q_1)x_1 = \xi_{q_1}^{n-1}$$

is compact for $i = 1, 2, \dots, N$.

It follows from (i) and (ii) that there exists a solution of the equation (4.1).

P r o o f . The set

$$V_1 = \left\{ K(W_{n-1}(q_1)x_1, \delta_1(\mu_1)) \right\} \cap \left\{ W_{n-1}(q_1)x_1 = \xi_{q_1}^{n-1} \right\}$$

is a convex set.

$$\text{For } \|x_1 - W_{n-1}(q_1)x_1\| \leq \delta_{G_1}\mu_1$$

we have

$$\begin{aligned} \|A_1 x_1 - W_{n-1}(q)x_1\| &= \|d^n(q_1) \bar{f}^1(a(x_1))\| \leq \\ &\leq \|d_1^n(q_1)\| \cdot \|\bar{f}^1(a(x_1))\| \leq \delta_{G_1} \mu_1 = \delta_1 \mu_1, \end{aligned}$$

or $A_1(V_1) \subset \bar{W}_1$ and set $\bar{W}_1$ is compact.

The assumptions of Schauder's theorem are fulfilled and there exists x_1 such that

$$A_1 x_1 = x_1$$

i.e. there exist solutions of the equation (4.7).

We have

$$\bar{S}_1^n x_1 = \bar{f}^1(a(x_1))$$

$$W_{n-1}(q_1)x_1 = \xi_{q_1}^{n-1}$$

$$\text{for } \|x_1 - W_{n-1}(q_1)x_1\|_{L_0} \leq \delta_1 \|\bar{f}^1(a(x_1))\|_{L_0}.$$

5. The uniqueness of a solution of the equation (4.1)

Theorem 2. Let A_1 be defined in (4.7). Suppose that

$$x_{p+1}^i = A_1(x_p^i), \quad x_0^i = W_{n-1}(q)x_1.$$

Elements x_p^i for the operations A_1 are obtained by the successive approximations method. We assume the conditions:

$$\sum_{n=1}^{\infty} (\alpha \theta)^{[n]} < +\infty,$$

where $(\alpha \theta)^{[n]}$ denotes the n -th iteration of the function $(\alpha \theta)$,

$$(\alpha\theta)^{[n+1]} \underset{\text{Df}}{=} (\alpha\theta)(\alpha\theta)^{[n]}$$

$$\|f(a(\bar{x}_1)) - f(a(x_1^*))\| \leq \theta \|\bar{x}_1 - x_1^*\|,$$

$$\frac{1}{N} \sum_I \|T_1(q)\| \|A_I^{-1}(x)s(q)\| \underset{\text{Df}}{=} \alpha < 1,$$

$$R \underset{\text{Df}}{=} \|x_1 - x_0\|.$$

Then there does exist for $R \in (0, +\infty)$ the unique solution of the equation (4.1).

P r o o f . We have, for $\bar{x}_1 = \lim_{p \rightarrow \infty} x_{p+1}^1$, $\bar{x}_1 = A_1(\bar{x}_1)$ and

$$\begin{aligned} \|\bar{x}_1 - x_1^*\| &\leq \alpha, \quad \|f(a(\bar{x}_1)) - f(a(x_1^*))\| \leq (\alpha\theta) \|\bar{x}_1 - x_1^*\| \\ &\leq \dots \leq (\alpha\theta)^{[n]}(R) \longrightarrow 0. \end{aligned}$$

But $\sum_{n=1}^{\infty} (\alpha\theta)^{[n]}(R) < +\infty$, so that

$$\|\bar{x}_1 - x_1^*\| \longrightarrow 0, \quad \text{or} \quad \|\bar{x}_1 - x_1^*\| = 0.$$

6. A quasi-linear equation of the (l, n) order

Let us consider the derivative

$$(6.1) \quad \sum_{(l, n)} u = \Big|_{z=x} \sum_I A_I(z)s(z)S_I^x u,$$

where $I = (i_1, \dots, i_k)$, $1 \leq k \leq n$, $x \in \bar{\Omega}$, operations $A_I(z)$ are continuous and have continuous inverses in a domain $\bar{\Omega} \subset \theta^m$.

There is $A_I(z) : \text{Ker } S \longrightarrow \text{Ker } S$ for $z \in \bar{\Omega}$. We understand the symbol $\left|_{z=x} \sum_I A_I(z) s(z) S_I^x u\right.$ in such a way that first of all we find the derivative $S_I^x u$ for the function $u = \{u(x)\}$ then we take limit condition $s(z) S_I^x u$, multiply $A(z) s(z) S_I^x u$ and repeat substitution $\left|_{z=x} \sum_I A_I(z) s(z) S_I^x u\right.$

Remark. For example substitute for $u = \{u(x)\}$, $x \in \bar{\Omega}$, $u = u(k)$, $k \in N^m$

$$A_I(x) \frac{\partial u}{\partial x} = \left|_{z=x} \frac{\partial u}{\partial x}\right.$$

$$a_I(k) \Delta_k u = \left|_{z=k} a_I(z) \right|_{k=z} \Delta_k u.$$

Definition. The equation

$$(6.2) \quad \sum_{(1,n)} u = \mu u + f(x, u, S_I^x u), \quad x \in \bar{\Omega},$$

$L = (i_1, i_2, \dots, i_\alpha)$ will be called a quasi-linear equation of the order $(1, n)$.

Let us consider for the equation (6.2) a limit condition

$$(6.3) \quad \lambda_{(1,n)} u = \left|_{z=p} \sum_L B_L(z) s(z) S_L^x u + \right|_{z=p} h(z) u = \\ = \varphi(p, u, S_J^x u),$$

where $\alpha < 1$, $p \in \bar{k}$, $J = (j_1, \dots, j_\beta)$, $\beta < \alpha$, B_L are continuous and have continuous inverses in a set $\bar{k} \subset \bar{\Omega} \subset \theta^m$.

Theorem 3. If

$$\sum_L F_L(p) = 1, \quad \sum_I G_I(x) = 1, \quad F_L \geq 0, \quad G_I \geq 0$$

then the element

$$(6.4) \quad u = \sum_L \left(T_L^x(z) \Big|_{z=p} F_L(z) B_L^{-1}(z) \right) \Big|_{p=z} \left(\phi(p, u, S_L^x u) - h(p)u \right) + \\ + \sum_I \left(T_I^x(z) \Big|_{z=x} G_I(z) A_I^{-1}(z) \right) \Big|_{x=z} \left(\mu u + f(x, u, S_L^x u) \right) \stackrel{\text{Df}}{=} \phi(u)$$

with limit condition (6.3) is a solution of the equation (6.2).

Proof. We shall prove that the element (6.4) is a solution of the equation (6.2).

We have

$$(a) \quad \left| \sum_I A_I(z) s(z) S_I^x \sum_L T_L^x(z) \right|_{z=p} F_L(z) B_L^{-1}(z) \Big|_{p=z} \cdot \\ \cdot \left(\phi(p, u, S_L^x u) - h(p)u \right) = 0,$$

because $I = (i_1, \dots, i_k)$, $L = (i_1, \dots, i_\alpha)$, $\alpha < l \leq k \leq n$, and in the expression S_I^x the derivatives are of higher order than the order of integrals in the expression in T_L .

$$(b) \quad \left| \sum_I A_I(z) s(z) S_I^x \sum_{I'} T_{I'}^x(z) \right|_{z=x} G_{I'}(z) A_{I'}^{-1}(z) \Big|_{x=z} \cdot \\ \cdot \left(\mu u + f(x, u, S_L^x u) \right) = \mu u + f(x, u, S_L^x u),$$

because

$$\left| s(z) S_I^x \sum_{I'} T_{I'}^x(z) \right|_{z=x} G_{I'}(z) A_{I'}^{-1}(z) \Big|_{x=z} =$$

$$= \begin{cases} 0, & \text{for } I \neq I' \\ G_I(x) A_I^{-1}(x), & \text{for } I = I' \end{cases}$$

$$\text{and } \sum_I G_I(x) = 1.$$

From (a) and (b) we have the equation (6.2). We will prove that the limit condition (6.3) is fulfilled. We have

$$(c) \quad \left| s(z) S_L^x \sum_L B_L(z) s(z) S_L^x \sum_{L'} T_{L'}^x(z) F_{L'}(z) B_{L'}^{-1}(z) \right|_{p=z} \cdot$$

$$\cdot \left(\varphi(p, u, S_L^x u) - h(p)u \right) + h(p)u = \varphi(p, u, S_L u),$$

because

$$\left| s(z) S_L^x \sum_{L'} T_{L'}^x(z) F_{L'}(z) B_{L'}^{-1}(z) \right|_{p=z} = \begin{cases} 0 & \text{for } L \neq L' \\ F_L(p) B_L^{-1}(p) & \text{for } L = L' \end{cases}$$

$$\text{and } \sum_L F_L(p) = 1.$$

$$(d) \quad \sum_L B_L(z) \Big|_{z=p} s(z) S_L^x \sum_I T_I^x(z) G_I(z) A_I^{-1}(z) \Big|_{x=z} \cdot$$

$$\cdot \left(\mu u + f(x, u, S_L^x u) \right) = 0$$

and the orders of derivatives in S_L^x are lower than the order of integrals in T_I^x . Then $s(z) S_L^x T_I^x(z) = 0$.

From (c) and (d) we obtain that the limit condition (6.3) for the equation (6.4) is fulfilled.

R e m a r k . It results from 1° and 2° that

$$(\alpha) \quad \sum_L T_L^x(z) F_L(z) B_L^{-1}(z) \Big|_{z=p} \left(\varphi(p, u, S_L^x u) - h(p)u \right)$$

is a constant for the derivative (6.1).

$$(\beta) \quad \text{The operation } \sum_I T_I^x(z) G_I(x) A_I^{-1}(z) \Big|_{x=z}$$

is the integral for the derivative (6.1).

(\gamma) By (\alpha) and (\beta) we have

$$\sum_{(1,n)} \phi \{f(x)\} = \{f(x)\}, \quad \lambda_{(1,n)} \phi \{f(x)\} = 0,$$

$$\sum_{(1,n)} \{\varphi(p)\} = 0, \quad \lambda_{(1,n)} \{\varphi(p)\} = \{\varphi(p)\}.$$

T h e o r e m 4. If the functions f and φ fulfill Lipschitz's condition

$$(6.5) \text{ (i) } \|f(x, u, S_L^x u) - f(x, v, S_L^x v)\| \leq M_O^f \|u - v\| + \sum_L M_L^f \|S_L^x u - S_L^x v\|,$$

$$M_O^f \geq 0, \quad M_L^f \geq 0, \quad M_O^f + \sum_L M_L^f > 0,$$

$$(6.6) \text{ (ii) } \|\varphi(p, u, S_L^x u) - \varphi(p, v, S_L^x v)\| \leq M_O \|\varphi(p, u, S_L^x u) - \varphi(p, v, S_L^x v)\| + \sum_L M_L \|S_L^x u - S_L^x v\|,$$

$$M_O^\varphi \geq 0, \quad M_L^\varphi \geq 0, \quad M_O^\varphi + \sum_L M_L^\varphi > 0,$$

(iii) then the solution $\phi(u)$ (see (6.4)) of the equation (6.2) in a sufficiently small neighbourhood $\bar{\omega}$ of $\bar{K}$, does exist. It can be obtained by the method of successive approximations in the norm

$$(6.7) \quad \|u\|_* = \sup_L (\|u\|, \|S_L^x u\|)$$

and is independent of the choice of the F_L and G_I coefficients. The neighbourhood $\bar{\omega}$ is defined by the inequalities

$$(6.8) \quad \sum_{L'} \|T_{L'}^x(z)\| \sup_{p,L} \|B_L^{-1}(p)\| \leq \frac{\alpha_1}{M_0^f + \sum_L M_L^f u + \sup_p \|h(p)\|},$$

$$(6.9) \quad \sum_I \|T_I^x(z)\| \sup_{x,I} \|A_I^{-1}(x)\| \leq \frac{\alpha_2}{M_0^f + \sum_L M_L^f + \|\mu\|},$$

where $\alpha_1 + \alpha_2 = 1$.

(iv) ϕ is continuous in the norm $\|u\|_*$.

P r o o f . The following estimation is true:

$$(6.10) \quad \begin{aligned} \|\phi(u) - \phi(v)\| &\leq \left\| \sum_L T_L^x(z) F_L(p) B_L^{-1}(p) \right\| \cdot \\ &\cdot \left\| \phi(p, u, S_L^x u) - h(p)u - \phi(p, v, S_L^x v) + h(p)v \right\| + \\ &+ \left\| \sum_I T_I^x(z) G_I(x) A_I^{-1}(x) \right\| \cdot \left\| \mu u + f(x, u, S_L^x u) - \mu v - f(x, v, S_L^x v) \right\| \leq \\ &\leq \sum_L \|T_L^x(z)\| \sup_{p,L} \|B_L^{-1}(p)\| \left(M_0^f \|u-v\| + \sum_L M_L^f \|S_L^x u - S_L^x v\| \right) + \\ &+ \sup_p \|h(p)\| \cdot \|u - v\| + \\ &+ \sum_I \|T_I^x(z)\| \sup_{x,I} \|A_I^{-1}(x)\| \left(M_0^f \|u-v\| + \sum_L M_L^f \|S_L^x u - S_L^x v\| \right) + \end{aligned}$$

$$\begin{aligned}
& + \|\mu\| \|u-v\| \leq \sum_L \|T_L^x(z)\| \sup_{p,L} \|B_L^{-1}(p)\| \left(M_0^p + \sum_L M_L + \sup \|h(p)\| \right) + \\
& + \sum_I \|T_I^x(z)\| \sup_{x,I} \|A_I^{-1}(x)\| \left(M_0^f + \sum_L M_L^f + \|\mu\| \|u-v\|_* \right).
\end{aligned}$$

Further we have

$$\begin{aligned}
(6.11) \quad & \|S_L^x \phi(u) - S_L^x \phi(v)\| \leq \left(\sum_{L'} \|S_L^x T_{L'}^x\| \sup \|B_L^{-1}(p)\| \right) \cdot \\
& \cdot \left(M_0^p + \sum_L M_L^p + \sup_p \|h(p)\| \right) + \\
& + \sum_I \|S_L^x T_I^x\| \sup_{x,I} \|A_I^{-1}(x)\| \left(M_0^f + \sum_L M_L^f + \|\mu\| \right) \|u-v\|_*.
\end{aligned}$$

The inequality (6.10) is a particular case of the inequality (6.11) for $L = (0, 0, \dots, 0)$.

The sums $\sum_L \|T_L^x(z)\|$, $\sum_I \|T_I^x(z)\|$, $\sum_L \|S_L^x T_L^x\|$, $\sum_I \|S_L^x T_I^x\|$, as well as the coefficients $B_L^{-1}(p)$ fulfill the conditions (6.8) and (6.9) when $\sup \|B_L^{-1}(p)\| < 1$ and the norms $\delta_1 = \|T_1\|$, ..., $\delta_m = \|T_m\|$, are sufficiently small.

It results from the inequality (6.10) and (6.11) that ϕ is the operation continuous in the norm $\|u\|_*$, so we use the method of successive approximations.

Theorem 5. Let u and v be two solutions of the equation (6.2) in the common space $\bar{\Omega}$ for coefficients $\bar{F}_L, \bar{F}_I, \bar{G}_I, \bar{G}_I$. Then $u = v$.

Proof.

$$\begin{aligned}
u &= \sum_L T_L^x(z) \bar{F}_L(p) B_L^{-1}(z) \Big|_{p=z} \left(\varphi(p, u, S_L^x u) - h(p) u \right) + \\
&+ \sum_I T_I^x(z) \bar{G}_I(x) A_I^{-1}(z) \Big|_{x=z} \left(\mu u + f(x, u, S_L^x u) \right) \stackrel{\text{Df}}{=} \bar{\phi}(u).
\end{aligned}$$

$$v = \sum_L T_L^x(z) \bar{\bar{F}}_L(p) B_L^{-1}(z) \Big|_{p=z} \left(\varphi(p, v, S_L^x v) - h(p)v \right) + \\ + \sum_I T_I^x(z) \bar{\bar{G}}_I(x) A_I^{-1}(z) \Big|_{x=z} \left(\mu(u + f(x, v, S_L^x v)) \right) \stackrel{\text{Def}}{=} \bar{\bar{\phi}}(v).$$

Then $\|u - v\|_* = \|\bar{\phi}(u) - \bar{\phi}(v)\|_* \leq \|\bar{\phi}(u) - \bar{\phi}(v)\|_* +$
 $+ \|\bar{\phi}(v) - \bar{\bar{\phi}}(v)\| \leq (\alpha_1 + \alpha_2) \|u - v\|_* + \sup_{p, L} \|\bar{F}_L(p) - \bar{\bar{F}}_L(p)\| +$
 $+ \sup_{x, I} \|\bar{G}_I(x) - \bar{\bar{G}}_I(x)\| \cdot \|v\|_*$. Moreover we can take coefficients $\bar{\bar{F}}_L(p)$ and $\bar{\bar{G}}_I(x)$ for coefficients $\bar{F}_L(p)$ and $\bar{G}_I(x)$ so that

$$\left(\sup_{p, L} \|\bar{F}_L(p) - \bar{\bar{F}}_L(p)\| + \sup_{x, I} \|\bar{G}_I(x) - \bar{\bar{G}}_I(x)\| \right) \cdot \|v\|_* \leq \\ \leq (1 - (\alpha_1 + \alpha_2)) \|u - v\|_* \leq \|u - v\|_*, \text{ because } \alpha_1 + \alpha_2 < 1.$$

Therefore we must have $u = v$.

7. About an extension of solutions

D e f i n i t i o n . The function $u' = \{R|_{\Omega'}(x)\}$ is called a restriction of the function $u = \{R(x)\}_{\Omega}$ from Ω to Ω' .

L e m m a . If for $u_{\alpha} = \{R(x)\}_{\Omega_{\alpha}}$, $\alpha = 1, 2, \dots, N$ the equation

$$(7.1) \quad \sum_{(1, n)} u_{\alpha} = \mu u_{\alpha} + f(x, u_{\alpha}, S_L^x u_{\alpha}), \quad x \in \Omega_{\alpha}$$

with the limit condition

$$(7.2) \quad \varphi(p_{\alpha}, u_{\alpha}, S_L^x u_{\alpha}) = \sum_L B_L(p_{\alpha}) \Big|_{z=p} x(z) S_L^x u_{\alpha} \Big|_{z=p} + h(p_{\alpha}) u_{\alpha}$$

has for each $\alpha = 1, 2, \dots, N$ a unique solution u_α in Ω_α , and $\Omega_\alpha \cap \Omega_\beta \neq \emptyset$ then the equation (6.2), condition (6.3) has the unique solution u in Ω , where $u = \{u_\alpha(x)\}$ for $x \in \Omega_\alpha$, $\Omega = \bigcup \Omega_\alpha$.

P r o o f . The equation (7.1) is considered when $x \in \Omega$, which means $x \in \bigcup \Omega_\alpha$. It has the unique solution in each of Ω_α . In this case these solutions must coincide one with another in $\Omega_\alpha \cap \Omega_\beta \neq \emptyset$, which leads to the common solution u in $\Omega = \bigcup \Omega_\alpha$. This solution is the only one in Ω , because if u and $\bar{u}$ were different in Ω , then we would have $u = \bar{u}$ in Ω_α for $\alpha = 1, 2, \dots, N$, which means $u = \bar{u}$.

C o r o l l a r y . If $\|(T^x)_1(q)\|_\alpha < \delta$, where δ fulfills the condition (6.6) and Ω_α is the set from the lemma, in this case there is exactly one solution in the sum $\bigcup \Omega_\alpha$ in the (7.1) equation. This solution can be obtained using the method of successive approximation.

E x a m p l e 1. The equation

$$(7.3) \quad x \frac{\partial^3 u}{\partial x^2 \partial y} + x^2 y^2 \frac{\partial^2 u}{\partial x^2} = f(x, y), \quad (x, y) \in D,$$

where $D = \{(x, y) : x^2 + y^2 < 1, x, y > \alpha > 0\}$, $S_1^x = \frac{\partial}{\partial x}$, $S_2^y = \frac{\partial}{\partial y}$ leads as in the formula (6.2) to the equation

$$(7.4) \quad \left| \begin{matrix} z \\ z=x \\ w=y \end{matrix} \right|_{\substack{x=z \\ y=w}} (S_1^x)^2 S_2^y u + \left| \begin{matrix} z^2 w^2 \\ z=x \\ w=y \end{matrix} \right|_{\substack{x=z \\ y=w}} (S_1^x)^2 u = f(x, y).$$

The limit condition

$$(7.5) \quad x_0 \frac{\partial u}{\partial x} \Big|_{(x_0, y_0)} = u(x_0, y_0)$$

for $x_0^2 + y_0^2 = 1$, $x_0, y_0 > \alpha > 0$ give the equation

$$(7.6) \quad \left| \begin{matrix} p \\ p=x_0 \\ r=y_0 \end{matrix} \right|_{\substack{x_0=p \\ y_0=r}} S_1 u = u(x_0, y_0)$$

on the boundary $\Gamma = \{(x_0, y_0) : x_0^2 + y_0^2 = 1, x_0, y_0 > \alpha > 0\}$ of domain D.

By Theorem 3, the solution u can be written in the formula (6.4), if

$$T_1^x(z) = \int_z^x, \quad T_2^y(w) = \int_w^y, \quad \{(x, y, z, w) : x^2 + y^2 < 1, x, y > \alpha > 0\}$$

$$(7.7) \quad u = T_1^x(x_0) \Big|_{\substack{z=x_0 \\ w=y_0}} F_{1,0}(z, w) \frac{1}{z} u(z, w) + T_1^x(z)^2 T_2^y(w) \frac{1}{z} G_{2,1}(z, w) u(z, w) +$$

$$+ (T_1^x(z))^2 \frac{1}{z^2 w^2} G_{2,0}(z, w) u(z, w) = \phi(u),$$

Df

where

$$F_{1,0}(x_0, y_0) = 1, \quad G_{2,1}(x, y) + G_{2,0}(x, y) = 1, \quad G_{2,1}, G_{2,0} \geq 0.$$

Further we use Theorems 2 and 3.

For solution (7.7) we have

$$\|\phi(u) - \phi(v)\| \leq (\sup \|x - x_0\| \frac{1}{\|z\|} + \sup \frac{\|x-z\|^2}{2} \|y-w\| +$$

$$+ \sup \|x-z\| \frac{1}{\|z^2 w^2\|} \|u-v\|).$$

Let the set D_0 be defined by the following formula

$$D_0 = \left\{ (x, y) : \sup_{z=x_0} \|x-x_0\| \frac{1}{\|z\|} < \frac{x}{3}, \sup \frac{\|x-z\|^2}{2} \|y-w\| < \frac{x}{3}, \right.$$

$$\left. \sup \|x-z\| \frac{1}{\|z^2 w^2\|} < \frac{x}{3} \right\}.$$

Then in the set D_0 , $\|\phi(u) - \phi(v)\| \leq x \|u - v\|$ and if $0 < x < 1$ we can use the method of successive approximations.

We obtain some neighbourhood $D_0 \subset D$ of the boundary Γ in which the equation (7.3) with the condition (7.5) has a so-

lution. Taking another equations we can have even $D = D_0$, if α is so large that the estimation $\|\varphi(u) - \varphi(v)\| \leq \alpha \|u - v\|$ for $0 < \alpha < 1$ is fulfilled at all points of D .

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