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ON ESTIMATION OF THE FUNCTIONAL $|a_3 - \alpha a_2^2|$, $\alpha \in \mathbb{C}$, IN THE CLASSES OF BOUNDED UNIVALENT FUNCTIONS

Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

holomorphic and univalent in the disc $K = \{z: |z| < 1\}$, whereas S_M , $M > 1$, the subclass of functions of S satisfying the condition

$$|f(z)| < M \quad \text{for } z \in K.$$

In 1933 M. Fekete and G. Szego [2] estimated in the class S the functional

$$(1) \quad J(f) = |a_3 - \alpha a_2^2|$$

for some α 's from the interval $(0, \frac{1}{2})$. The estimate of functional (1) in the class S for any real α was obtained by J. Jenkins [7], and that for any complex α - by A. Szwankowski ([9], [10]).

From the point of view of the present paper also the results obtained for α real in the class S_M by Z. J. Jakubowski ([4], [5], [6]) and R. Guzek [3] seem to be interesting.

In this paper, functional (1) defined in the family S_M is considered for any complex α .

In order to estimate this functional, the variational method is applied. In particular, use is made of the general equation for extremal functions, obtained by Z.Charzyński [1].

Fundamental theorem

1. To begin with, we shall make a few general observations essential to our further considerations.

Note that, for each $M > 1$, the family S_M is compact, whereas functional (1)-continuous. So, for every fixed complex number α , there exists a function of the class S_M for which this functional attains its maximum.

Moreover, together with every function $f(z)$, to the class S_M also belongs the function $e^{-i\theta} f(e^{i\theta} z)$, $\theta \in \mathbb{R}$. Hence it appears that the maximum of functional (1) coincides in this class with the maximum of the functional

$$(2) \quad \hat{J}(F) = \operatorname{re}(a_3 - \alpha a_2^2),$$

which allows us to confine the considerations to investigating the latter.

From the Charzyński theorem [1] it follows directly that each function $w = f(z)$ extremal with respect to functional (2) satisfies the following differential-functional equation

$$(3) \quad \left(\frac{zw'}{w} \right)^2 \kappa(w) = \kappa(z), \quad 0 < |z| < 1,$$

where

$$(4) \quad \kappa(w) = \frac{2(w^4 + \bar{\beta} M w^3 - \mathcal{P} M^5 w^2 + \beta M^3 w + M^4)}{M^5 w^2},$$

$$(5) \quad \kappa(z) = \frac{2[z^4 + \bar{\beta} M^{-1} z]^3 + 2(\mathcal{P} - \mathcal{P} M) z^2 + \beta M^{-1} z + 1}{M z^2},$$

$$(6) \quad \beta = 2(1-\alpha)M a_2,$$

$$\mathcal{P} = 2M^{-3} \min_{0 \leq x < 2\pi} \operatorname{re}(\beta e^{ix} + e^{2ix}),$$

$$(7) \quad \gamma = a_3 - \alpha a_2^2.$$

Functions (4) and (5) are non-negative on the circles $|w| = M$, $|z| = 1$, respectively, and each of the two functions has on the respective circle at least one double zero. Besides, from their forms it follows that, if $\Re(w_0) = 0$, then $\Re\left(\frac{M^2}{\bar{w}_0}\right) = 0$, and if $\Re(z_0) = 0$, then $\Re\left(\frac{1}{\bar{z}_0}\right) = 0$.

Taking account of the above properties, we may represent the functions $\Re(w)$ and $\Re(z)$ in the form

$$(8) \quad \Re(w) = \frac{2(w - M e^{i\phi})^2 (w - \tau e^{-i\phi}) \left(w - \frac{M^2}{\tau} e^{-i\phi}\right)}{M^5 w^2}$$

$$(9) \quad \Re(z) = \frac{2(z - e^{i\psi})^2 (z - \varphi e^{-i\psi}) \left(z - \frac{1}{\varphi} e^{-i\psi}\right)}{M z^2},$$

with that $\psi, \phi \in (-\pi, \pi)$ and $\varphi \in (0, 1)$, $\tau \in (0, M)$.

Comparing the respective coefficients of functions (4) and (8) as well as of (5) and (9), we get

$$(10) \quad \begin{cases} \operatorname{Re} \beta = -\frac{(\tau + M)^2}{\tau M} \cos \phi = -M \frac{(\varphi + 1)^2}{\varphi} \cos \psi, \\ \operatorname{Im} \beta = -\frac{(\tau - M)^2}{\tau M} \sin \phi = -M \frac{(\varphi - 1)^2}{\varphi} \sin \psi, \end{cases}$$

$$(11) \quad \gamma = \varphi + \frac{1}{\varphi} + \cos 2\psi - \frac{1}{M^2} \left(\frac{\tau}{M} + \frac{M}{\tau} + \cos 2\phi \right).$$

Thus it follows from (7) and (11) that, in order to determine the maximum of functional (2), it is necessary to find relations between $\psi, \phi, \varphi, \tau$ and α, M .

For the purpose, let us notice that if, for given α and $M > 1$, the function $w = f(z)$ is an extremal function with respect to functional (2), so is the function

$$(12) \quad g(z) = -f(-z),$$

whereas

$$(13) \quad h(z) = \overline{f(\bar{z})}$$

is then an extremal function with respect to the functional $\operatorname{re}(a_3 - \bar{\alpha} a_2^2)$. Hence it appears that we may confine the range of variability of the parameters ψ, ϕ to the interval $\langle 0, \frac{\pi}{2} \rangle$, and the remaining extremal functions will be obtained from relations (12) and (13). In the case when $\operatorname{im} \beta \neq 0$, this confinement causes simultaneously the limitation of our seeking for the maximum of functional (2) to α 's such that either $\operatorname{im} \alpha \leq 0$ or $\operatorname{im} \alpha \geq 0$. What is more, the respective extremal values for α and $\bar{\alpha}$ are equal.

In the further part of the paper, we shall investigate all possible forms of functions (8), (9) with respect to the parameters $\psi, \phi \in \langle 0, \frac{\pi}{2} \rangle$, $\varphi \in (0, 1)$, $\tau \in (0, M)$, bearing in mind that the fulfilment of equation (3) by the function f is only a necessary condition for it to be an extremal function.

2. Let us first consider the case when $\psi, \phi \in \langle 0, \frac{\pi}{2} \rangle$, $\tau = M$, $\varphi \in (0, 1)$. From (8) it follows that if $\tau = M$, the function $\pi(w)$ possesses its zeros only on the circle $|w| = M$. Besides, when $\tau = M$, there must be $\varphi = 1$. Indeed, if $\varphi \in (0, 1)$, then, by (9), the point $z_0 = \varphi e^{-i\psi}$ would be a zero of $\pi(z)$ and, in consequence, at the point $w_0 = f(z_0)$, $|w_0| < M$, we would have $\pi(w_0) = 0$.

From (7), (10) and (11) we thus have

$$\gamma = a_3 - \alpha a_2^2 = 1 - \frac{1}{M^2}.$$

In the case we examine, equation (3) has the form

$$(14) \quad \left(\frac{zw'}{w}\right)^2 \frac{(w - Me^{i\phi})^2 (w - Me^{-i\phi})^2}{M^4 w^2} = \frac{(z - e^{i\psi})^2 (z - e^{-i\psi})^2}{z^2}.$$

Since $f(0) = 0$, therefore

$$(15) \quad \frac{zw'}{w} \frac{(w-Me^{i\phi})(w-Me^{-i\phi})}{L^2 w} = \frac{(z-e^{i\psi})(z-e^{-i\psi})}{z}.$$

Integrating both sides of equation (15), we have by (10)

$$(16) \quad \frac{1}{L^2} w + \frac{\beta}{2L} \log \frac{w}{z} - \frac{1}{w} = z - \frac{1}{z} + C,$$

where the branch of the logarithm is so chosen that, for $z = 0$, it assumes the value zero, while C is a constant.

Expanding the left-hand side of equation (16) in a power series in a neighbourhood of the point $z = 0$ and comparing the absolute terms, we have

$$(17) \quad a_2 = C.$$

On the other hand, it is known [1] that there exists a point $z = e^{i\theta}$, $\theta \in (0, 2\pi)$, such that $f(e^{i\theta}) = Me^{i\lambda}$, $\lambda \in \mathbb{R}$. Consequently, from (16) we get

$$(18) \quad C = -2\cos\psi \log M + 2i \left[\frac{1}{M} \sin\lambda - \sin\theta - (\lambda - \theta)\cos\psi \right].$$

If $\beta \neq 0$, we may represent equation (16), in virtue of (17), (6), in the form

$$(19) \quad \frac{1}{M^2} w + \frac{\beta}{2M} \log \frac{w}{z} - \frac{1}{w} = z - \frac{1}{z} + \frac{\beta}{2(1-\alpha)M}.$$

Moreover, from (17) it follows that $\operatorname{re} \frac{a_2}{\beta} = \operatorname{re} \frac{C}{\beta}$, whence, in view of (6), (10) and (18),

$$\operatorname{re} \frac{1}{1-\alpha} = \log M.$$

If $\beta = 0$, then from (10) we have $\psi = \phi = \frac{\pi}{2}$. Hence and from (18) $\operatorname{re} C = 0$. So, in order that the function $w = f(z)$, given by equation (16), belong to the class S_L , there must be

$$-2 \left(1 - \frac{1}{M}\right) \leq \operatorname{im} C \leq 2 \left(1 - \frac{1}{M}\right).$$

To sum up, we have shown

L e m m a 1. If in the family S_M , $M > 1$, for any α such that

$$\operatorname{re} \frac{1}{1-\alpha} = \log M,$$

there exists an extremal function $w = f(z)$ satisfying equation (14), where $\psi, \phi \in (0, \frac{\pi}{2})$, then

$$\hat{J}(f) = 1 - \frac{1}{M^2}$$

and, besides, this function satisfies equation (19), where $\beta = 2M(1-\alpha)a_2 \in (-4M, 0)$.

For α such that $\operatorname{re} \frac{1}{1-\alpha} \neq \log M$, there exists no extremal function satisfying equation (14), where $\psi, \phi \in (0, \frac{\pi}{2})$.

L e m m a 2. If in the family S_M , $M > 1$, for any α , there exists an extremal function $w = f(z)$ satisfying equation (14), where $\psi = \phi = \frac{\pi}{2}$, then

$$\hat{J}(f) = 1 - \frac{1}{M^2}$$

and the function satisfies equation (16), where $\operatorname{re} C = 0$, $\operatorname{im} C = \operatorname{im} a_2$. What is more, if $\alpha \neq 1$, then $\operatorname{im} C = 0$; if $\alpha = 1$, then $-2\left(1 - \frac{1}{M}\right) \leq \operatorname{im} C \leq 2\left(1 - \frac{1}{M}\right)$.

3. Let us next consider the case when $\psi, \phi \in (0, \frac{\pi}{2})$, $\rho = 1$, $\tau \in (0, M)$. It follows from (10) and (11) that if $\rho = 1$, $\tau \in (0, M)$, then $\phi = 0$, $\psi \in (0, \frac{\pi}{2})$ and

$$\tau = 1 - \frac{1}{M^2} + 2\left(\frac{1}{M} - \cos \psi\right)^2.$$

Besides, from (10) and the assumption that $\psi \in (0, \frac{\pi}{2})$ it follows that

$$\frac{1}{M} < \cos \psi \leq 1.$$

In the case in question, equation (3) may be represented in the form

$$(20) \quad \left(\frac{zw}{w}\right)^2 \frac{(w-M)^2 \left(w - \frac{M^2}{\tau}\right)(w-\tau)}{M^4 w^2} = \frac{(z-e^{i\psi})^2 (z-e^{-i\psi})^2}{z^2}.$$

From (20) it follows that the point $w = \tau$ is not an interior point of the domain $f(K)$, for if it were, the right-hand side of (20) would have a root at some interior point of the disc K . Since $f(0) = 0$, therefore, proceeding analogously as in 2., after suitable transformations we obtain, by (20), the equation

$$(21) \quad 2\cos\psi \log \frac{1 - \sqrt{\frac{w-\tau}{w - \frac{M^2}{\tau}}}}{1 + \sqrt{\frac{w-\tau}{w - \frac{M^2}{\tau}}}} - 2\cos\psi \log \frac{\tau - M\sqrt{\frac{w-\tau}{w - \frac{M^2}{\tau}}}}{z \left(\tau + M\sqrt{\frac{w-\tau}{w - \frac{M^2}{\tau}}} \right)} +$$

$$+ \frac{M^2 - \tau^2}{M} \frac{\sqrt{\frac{w-\tau}{w - \frac{M^2}{\tau}}}}{M^2 \frac{w-\tau}{w - \frac{M^2}{\tau}} - \tau^2} + \frac{M^2 - \tau^2}{\tau M^2} \frac{\sqrt{\frac{w-\tau}{w - \frac{M^2}{\tau}}}}{\frac{w-\tau}{w - \frac{M^2}{\tau}} - 1} = z - \frac{1}{z} + C,$$

where $\sqrt{\frac{\tau^2}{M}} = \frac{\tau}{M}$, $\log 1 = 0$, and C stands for a constant.

Expanding the left-hand side of equation (21) in a power series in a neighbourhood of the point $z = 0$ and comparing the absolute terms, we shall obtain

$$(22) \quad C = a_2 - \frac{1}{M} + \frac{\tau^2 + M^2}{2\tau M^2} - 2\cos\psi \log \frac{M^2 - \tau^2}{4\tau M^2} +$$

$$+ 2\cos\psi \log \frac{M - \tau}{M + \tau} + 2i(\psi \cos\psi - \sin\psi).$$

On the other hand, since there exists a point $z_0 = e^{i\theta}$, $\theta \in (0, 2\pi)$, such that $f(z_0) = -M e^{i\lambda}$ [1], therefore from equation (21) (cf. e.g. [5], p.12) we have

$$(23) \quad \operatorname{re} C = 0.$$

Consequently, from (22) and (6) we get

$$(24) \quad C = i \left(i m \frac{\beta}{2M(1-\alpha)} + 2\psi \cos \psi - 2\sin \psi \right).$$

In view of (23), (10), (6), from (22) we have

$$(25) \quad \lambda \log \lambda + \lambda \operatorname{re} \frac{\alpha}{1-\alpha} + \frac{1}{M} = 0,$$

where $\lambda = \cos \psi$, with that in the case under consideration $\lambda \in \left(\frac{1}{M}, 1\right)$. In order for equation (25) to possess a root belonging to the interval $\left(\frac{1}{M}, 1\right)$, the conditions

$$\operatorname{re} \frac{1}{1-\alpha} < \log M, \quad \operatorname{re} \frac{\alpha}{1-\alpha} \geq -\frac{1}{M}$$

must be satisfied, with that then the larger of the two roots of the equation belongs to the interval considered.

We have thus shown

L e m m a 3. If in the family S_M , $M > 1$, for any α such that

$$(26) \quad \operatorname{re} \frac{1}{1-\alpha} < \log M, \quad \operatorname{re} \frac{\alpha}{1-\alpha} \geq -\frac{1}{M},$$

there exists an extremal function $w = f(z)$ satisfying equation (20). then

$$\hat{J}(f) = 1 - \frac{1}{M^2} + 2 \left(\frac{1}{M} - \lambda \right)^2,$$

where λ is the only root of equation (25) belonging to the interval $\left(\frac{1}{M}, 1\right)$. Besides, the function $w = f(z)$ satisfies equation (21), where C is expressed by formula (24),

$$\cos \psi = \frac{-\beta}{4M}, \quad \frac{(\tau+M)^2}{\tau} = -M\beta, \quad \beta = 2M(1-\alpha)a_2 \in (-4M, -4).$$

For α not satisfying conditions (26), there exists no extremal function satisfying equation (20).

Note that the equations $\operatorname{re} \frac{1}{1-\alpha} = \log M$, $\operatorname{re} \frac{\alpha}{1-\alpha} = -\frac{1}{M}$ ($M > 1$ fixed) are, respectively, equations of the circles

$$(27) \quad \begin{cases} K_1 : \left| \alpha - 1 + \frac{1}{2 \log M} \right| = \frac{1}{2 \log M} \\ K_2 : \left| \alpha - \frac{M-2}{2(M-1)} \right| = \frac{M}{2(M-1)} \end{cases}$$

lacking the point $\alpha = 1$.

4. Let us finally consider the case when $\psi, \phi \in (0, \frac{\pi}{2})$, $\varphi \in (0, 1)$, $\tau \in (0, M)$. In this case $\operatorname{im} \beta \neq 0$, whereas equation (3) takes the form

$$(28) \quad \begin{aligned} & \left(\frac{zw'}{w} \right)^2 \frac{(w - Me^{i\phi})^2 (w - \tau e^{-i\phi}) \left(w - \frac{M^2}{\tau} e^{-i\phi} \right)}{M^4 w^2} = \\ & = \frac{(z - e^{i\psi})^2 (z - \varphi e^{-i\psi}) \left(z - \frac{1}{\varphi} e^{-i\psi} \right)}{z^2}. \end{aligned}$$

Besides, it follows from (7) and (11) that, for any fixed α and for each function f of the family S_M satisfying equation (28),

$$\hat{J}(f) = \varphi + \frac{1}{\varphi} + \cos 2\psi - \frac{1}{M^2} \left(\frac{\tau}{M} + \frac{M}{\tau} + \cos 2\phi \right).$$

Proceeding analogously as in 2., after suitable transformations we obtain from (28) the equation

$$(29) \quad \begin{aligned} & \frac{1}{2} B \log \frac{\tau - MP(w)}{\varphi - p(z)} - \frac{1}{2} B \log \frac{\tau + MP(w)}{\varphi + p(z)} - \\ & - \frac{1}{2} \bar{B} \log \frac{1 - P(w)}{1 - p(z)} + \frac{1}{2} \bar{B} \log \frac{1 + P(w)}{1 + p(z)} + \\ & + e^{i\phi} \frac{M^2 - \tau^2}{M} \frac{P(w)}{M^2 P(w) - \tau^2} + e^{-i\phi} \frac{M^2 - \tau^2}{\tau M^2} \frac{P(w)}{p^2(w) - 1} = \\ & = e^{i\psi} (1 - \varphi^2) \frac{p(z)}{p^2(z) - \varphi^2} + e^{-i\psi} \frac{1 - \varphi^2}{\varphi} \frac{p(z)}{p^2(z) - 1} + C, \end{aligned}$$

where C is a constant,

$$(30) \quad B = -\frac{(\varphi+1)^2}{\varphi} \cos \psi - i \frac{(\varphi-1)^2}{\varphi} \sin \psi,$$

$$F(w) = \sqrt{\frac{w - \tau e^{-i\phi}}{w - \frac{M^2}{\tau} e^{-i\phi}}}, \quad p(z) = \sqrt{\frac{z - \varphi e^{-i\psi}}{z - \frac{1}{\varphi} e^{-i\psi}}},$$

with that $p(0) = \varphi$, $P(0) = \frac{\tau}{M}$. Equation (29) determines the holomorphic function in the domain $\hat{K} = K - \langle \varphi e^{-i\psi}, e^{-i\psi} \rangle$, for all singularities of the functions occurring in this equation are removable.

Choose the branches of logarithms so that $\log 1 = 0$. Since if $z \rightarrow \varphi e^{-i\psi}$, then $w \rightarrow \tau e^{-i\phi}$, therefore, passing in (29) to the limit with $z \rightarrow \varphi e^{-i\psi}$, with the branches thus chosen we get $C = 0$.

Next, expanding both sides of equation (29) in a power series in a neighbourhood of the point $z = 0$ and comparing the absolute terms, we obtain

$$(31) \quad \frac{e^{-i\phi}}{M} - a_2 - e^{i\phi} \frac{\tau^2 + M^2}{2\tau M^2} - \frac{1}{2} B \log \frac{M^2 - \tau^2}{4\tau M^2} e^{i\phi} + \frac{1}{2} \bar{B} \log \frac{M - \tau}{M + \tau} =$$

$$= e^{-i\psi} - e^{i\psi} \frac{1 + \varphi^2}{2\varphi} + \frac{1}{2} \bar{B} \log \frac{1 - \varphi}{1 + \varphi} - \frac{1}{2} B \log \frac{1 - \varphi^2}{4\varphi} e^{i\psi},$$

with that B is expressed by formula (30). In view of (6), (10) and (30), equation (31) is equivalent to the system of equations

$$(32) \quad \left\{ \begin{aligned} & \frac{1}{M} \left[(u+2) \cos \phi \log \frac{u+2}{M} - (u-2) \cos \phi - \phi \sin \phi (u-2) + \right. \\ & \quad \left. + \frac{(u+2) \cos \phi \operatorname{re}(1-\alpha) + (u-2) \sin \phi \operatorname{im}(1-\alpha)}{|1-\alpha|^2} \right] = \\ & = (v+2) \cos \psi \log(v+2) - (v-2) \cos \psi - \psi \sin \psi (v-2), \\ & \frac{1}{M} \left[(u-2) \sin \phi \log \frac{u-2}{M} - (u+2) \sin \phi + \phi \cos \phi (u+2) + \right. \\ & \quad \left. + \frac{(u-2) \sin \phi \operatorname{re}(1-\alpha) - (u+2) \cos \phi \operatorname{im}(1-\alpha)}{|1-\alpha|^2} \right] = \\ & = (v-2) \sin \psi \log(v-2) - (v+2) \sin \psi + \psi \cos \psi (v+2), \end{aligned} \right.$$

where

$$u = \frac{\tau}{M} + \frac{M}{\tau}, \quad v = \varrho + \frac{1}{\varrho}.$$

Moreover, it follows from (10) that

$$(33) \quad \begin{cases} (u+2) \cos \phi = M(v+2) \cos \psi, \\ (u-2) \sin \phi = M(v-2) \sin \psi. \end{cases}$$

So we have proved

L e m m a 4. For each extremal function $w = f(z)$ of the family S_M , $M > 1$, satisfying equation (28),

$$(34) \quad \hat{J}(f) = v + \cos 2\psi - \frac{1}{M^2} (u + \cos 2\phi),$$

where v, u, ψ, ϕ are roots of system of equations (32), (33).

We shall now determine the set of those α 's for which, with $M > 1$ fixed system (32), (33) possesses a solution and demonstrate that this is the only solution satisfying the conditions $\psi, \phi \in (0, \frac{\pi}{2})$, $v, u > 2$.

Assume first that $\psi \in (0, \frac{\pi}{2})$. It follows from (33) that also $\phi \in (0, \frac{\pi}{2})$. Besides, from (33), for $M > 1$, we have

$$\frac{u+2}{M(v+2)} = \frac{\cos \psi}{\cos \phi} < 1, \quad \frac{u-2}{M(v-2)} = \frac{\sin \psi}{\sin \phi} > 1;$$

consequently, $\psi > \phi$, $v < u$.

For $\psi, \phi \in (0, \frac{\pi}{2})$, system (32), (33) can be represented in the form

$$(35) \quad (v-2)(1 - \operatorname{tg} \psi \operatorname{ctg} \phi + A_1 \operatorname{tg} \psi) + A_3(v+2) = 0,$$

$$(36) \quad (v+2)(1 - \operatorname{ctg} \psi \operatorname{tg} \phi - A_1 \operatorname{ctg} \psi) + A_2(v-2) = 0,$$

$$(37) \quad (v+2) \frac{\cos \psi}{\cos \phi} - (v-2) \frac{\sin \psi}{\sin \phi} - \frac{4}{M} = 0,$$

$$(38) \quad u \sin \phi = M(v-2) \sin \psi + 2 \sin \phi,$$

where

$$(39) \quad \begin{cases} A_1 = \psi - \phi + \frac{\operatorname{Im}(1-\alpha)}{|1-\alpha|^2}, \\ A_2 = \log \frac{\sin \psi}{\sin \phi} + \frac{\operatorname{Re}(1-\alpha)}{|1-\alpha|^2}, \\ A_3 = \log \frac{\cos \psi}{\cos \phi} + \frac{\operatorname{Re}(1-\alpha)}{|1-\alpha|^2}. \end{cases}$$

So, in order to show that system (32), (33) possesses exactly one solution (v, u, ψ, ϕ) , where $\psi, \phi \in (0, \frac{\pi}{2})$, $v, u > 2$, it is enough to show, by (35) - (38), that system (35) - (37) possesses exactly one solution (v, ψ, ϕ) , where $\psi, \phi \in (0, \frac{\pi}{2})$, $v > 2$.

From (35) - (37) we obtain

$$(40) \quad \alpha = 1 + (2e^{-i\psi+ve^{i\psi}}) \left\{ e^{-i\phi(v-2e^{2i\phi})} \left(\frac{\cos \psi}{\cos \phi} - \frac{\sin \psi}{\sin \phi} \right) + \right. \\ \left. + (v+2)\cos \psi \left[\log \frac{\cos \psi}{\cos \phi} - i(\psi-\phi) \right] + i(v-2)\sin \psi \left[\log \frac{\sin \psi}{\sin \phi} - i(\psi-\phi) \right] \right\}^{-1},$$

$$(41) \quad M = \frac{2\sin 2\phi}{(v+2)\cos \psi \sin \phi - (v-2)\sin \psi \cos \phi}.$$

Relations (40), (41) define in the set

$$T = \left\{ (v, \psi, \phi) : \psi, \phi \in \left(0, \frac{\pi}{2}\right), v > 2, \psi > \phi \right\}$$

the locally homeomorphic mapping since

L e m m a 5. The jacobian of mapping (40), (41) is different from zero at each point of the set T .

In the proof of Lemma 5 we make use of the fact that the jacobian of system (35) - (37), calculated with respect to the unknown quantities v, ψ, ϕ , is different from zero at each point of the set T .

Fixing now $M = M_0$, $M_0 > 1$, observe that the set of points $(v, \psi, \phi) \in T$ satisfying the condition

$$M_0 = \frac{2 \sin 2\phi}{(v+2) \cos \psi \sin \phi - (v-2) \sin \psi \cos \phi}$$

is a two-dimensional regular surface; denote this surface by H_{M_0} . It is not difficult to check that the boundary of H_{M_0} consists of: a half-line $\psi = 0$, $\phi = 0$, $v \geq 2$, a half-line $\psi = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$, $v \geq 2$, a segment $\phi = 0$, $v = 2$, $0 \leq \psi \leq \arccos \frac{1}{M_0}$, a curve with equations $v = 2$, $\phi = \arccos(M_0 \cos \psi)$, $\arccos \frac{1}{M_0} \leq \psi \leq \frac{\pi}{2}$, and infinity.

Mapping (40), (41) is in $H_{M_0} \subset T$ locally homeomorphic, with that the values it then assumes are points of the plane π_{M_0} , where π_{M_0} is the set of points $(\operatorname{re} \alpha, \operatorname{im} \alpha, M_0)$. We shall demonstrate that this mappings is homeomorphic in H_{M_0} . To prove this, use will be made of the method applied in paper [10].

For the purpose, denote by D_{M_0} the set of values of mapping (40), (41) for $(v, \psi, \phi) \in H_{M_0}$. In the case under consideration $\beta \neq 0$, so, as was noticed in 1., the set D_{M_0} lies in one of the half-planes of the plane π_{M_0} : either $\operatorname{im} \alpha \leq 0$ or $\operatorname{im} \alpha \geq 0$. Since, for $\psi = \frac{\pi}{4}$, $\phi = \frac{\pi}{6}$, we have $\operatorname{im} \alpha < 0$, therefore D_{M_0} is contained in the half-plane $\operatorname{im} \alpha \leq 0$.

We shall now construct on the plane π_{M_0} some set G'_{M_0} whose boundary is obtained through mapping (40), (41) as (v, ψ, ϕ) tends to $\delta H_{M_0} = \bar{H}_{M_0} - H_{M_0}$, where $\bar{H}_{M_0}$ is the closure of the set H_{M_0} .

Assume first that $v \rightarrow v_0$, $v_0 > 2$, $\psi \rightarrow 0$, $\phi \rightarrow 0$. It follows from (41) that

$$\alpha \rightarrow 1 + \frac{M_0(2+v_0)}{4(1-M_0)} < \frac{1}{1-M_0}.$$

Of course, if $v \rightarrow 2$, $\psi \rightarrow 0$, $\phi \rightarrow 0$, then $\alpha \rightarrow \frac{1}{1-M_0}$.

Analogously, if $v \rightarrow v_0$, $v_0 > 2$, $\psi \rightarrow \frac{\pi}{2}$, $\phi \rightarrow \frac{\pi}{2}$, we have

$$\alpha \rightarrow 1 + \frac{M_0(2-v_0)}{4(1-M_0)} > 1;$$

if, with $\psi \rightarrow \frac{\pi}{2}$, $\phi \rightarrow \frac{\pi}{2}$, simultaneously $v \rightarrow 2$, then $\alpha \rightarrow 1$.

Let now $v \rightarrow 2$, $\phi \rightarrow \arccos(M_0 \cos \psi_0)$, $\psi \rightarrow \psi_0$, with that $\arccos \frac{1}{M_0} < \psi_0 < \frac{\pi}{2}$. From (35), (36) we get that $\alpha \rightarrow \alpha_0$, where α_0 is a point of an arc of the circle K_1 (see (27)), lying in the half-plane $\operatorname{Im} \alpha < 0$ and joining the points $(1, 0, M_0)$ and (a, b, M_0) , where

$$(42) \quad \begin{cases} a = 1 - \frac{\log M_0}{\log^2 M_0 + \left(\sqrt{M_0^2 - 1} - \arccos \frac{1}{M_0} \right)^2} \\ b = \frac{\arccos \frac{1}{M_0} - \sqrt{M_0^2 - 1}}{\log^2 M_0 + \left(\sqrt{M_0^2 - 1} - \arccos \frac{1}{M_0} \right)^2} \end{cases}$$

Next, let $v \rightarrow 2$, $\phi \rightarrow 0$, $\psi \rightarrow \psi_0$, where $0 < \psi_0 < \arccos \frac{1}{M_0}$. It follows from (35), (36) that $\alpha \rightarrow \alpha^*$, where α^* satisfies the equation

$$(43) \quad \frac{1}{1 - \bar{\alpha}^*} = 1 - \frac{1}{M_0 \cos \psi_0} - \log \cos \psi_0 + i(\operatorname{tg} \psi_0 - \psi_0),$$

$$0 < \psi_0 \leq \arccos \frac{1}{M_0}.$$

The point α^* describes in the plane π_{M_0} a regular curve Γ . The curve lies in the half-plane $\operatorname{Im} \alpha < 0$ and joins the points $\left(\frac{1}{1-M_0}, 0, M_0 \right)$ and (a, b, M_0) , where a, b are given by formulae (42). Moreover, the curve Γ is contained in the disc whose boundary is K_2 and has exactly one point in common with each straight line $\alpha = c$, where $\frac{1}{1-M_0} < c \leq a$.

Let finally $(v, \psi, \phi) \in H_{M_0}$, with that $v \rightarrow \infty$; from (40) we have that $|\alpha| \rightarrow \infty$.

It follows from the considerations we carried out that mapping (40), (41) can be continuously extended to the set δH_{M_0} .

Let us now denote by G'_{M_0} a closed set in the plane π_{M_0} , containing the half-plane $\text{im } \alpha \geq 0$, consisting of all half-lines parallel to the axis $\text{im } \alpha$ such that their terminal points lie on the axis $\text{im } \alpha = 0$ for $\text{re } \alpha \leq \frac{1}{1-M_0}$ or $\text{re } \alpha \geq 1$, on the curve Γ when $\frac{1}{1-M_0} < \text{re } \alpha \leq a$, and on the circle K_1 when $a < \text{re } \alpha < 1$.

Evidently, the set G'_{M_0} being a complement of the set G'_{M_0} to the plane π_{M_0} is a simply connected domain in this plane. It follows from the construction of this set and from Lemma 5 that $D_{M_0} \subset G'_{M_0}$ and $\delta G_{M_0} \subset \delta D_{M_0}$. Moreover, use being made of Lemma 5, it is not difficult to prove by reduction to absurdity that $\delta D_{M_0} \subset D G_{M_0}$. Consequently, $D_{M_0} = G_{M_0}$.

We shall now prove

L e m m a 6. For each point $(v_0, \psi_0, \phi_0) \in H_{M_0}$ and for any curve $\hat{L}$ with parametric description

$$x = x(s), \quad y = y(s), \quad M = M_0, \quad 1 \leq s \leq m,$$

where $x = \text{re } \alpha$, $y = \text{im } \alpha$, issuing from a point $(x_0, y_0, M_0) = F(v_0, \psi_0, \phi_0)$ and running in D_{M_0} , there exists exactly one curve L with parametric description

$$v = v(s), \quad \psi = \psi(s), \quad \phi = \phi(s), \quad 1 \leq s \leq m$$

issuing from the point (v_0, ψ_0, ϕ_0) , running in H_{M_0} and such that along it we have $F(v(s), \psi(s), \phi(s)) = (x(s), y(s), M_0)$, where F is the mapping defined by formulae (40), (41).

P r o o f . Let a point $(\hat{x}, \hat{y}, M_0) = (x(\hat{s}), y(\hat{s}), M_0)$ be a point of the curve $\tilde{L}$, corresponding to a parameter $\hat{s}$, whereas $(\hat{v}, \hat{\psi}, \hat{\phi}) = (v(\hat{s}), \psi(\hat{s}), \phi(\hat{s})) \in H_{M_0}$ some point such that $F(\hat{v}, \hat{\psi}, \hat{\phi}) = (\hat{x}, \hat{y}, M_0)$. Since the mapping F is locally homeomorphic in H_{M_0} , in some neighbourhood of $\hat{s}$ there is exactly one curve $\tilde{L}$ with description $v = v(s)$, $\psi = \psi(s)$, $\phi = \phi(s)$, running in H_{M_0} , along which $F(v(s), \psi(s), \phi(s)) = (x(s), y(s), M_0)$, and such that $v(\hat{s}) = \hat{v}$, $\psi(\hat{s}) = \hat{\psi}$, $\phi(\hat{s}) = \hat{\phi}$.

Note next that, if there are given two curves L_I, L_{II} with descriptions $v_i = v_i(s)$, $\psi_i = \psi_i(s)$, $\phi_i = \phi_i(s)$ for $c \leq s \leq d$, where $1 \leq c \leq d \leq m$, $i = I, II$, running in H_{M_0} and such that, for each s belonging to the interval $\langle c, d \rangle$, we have $F(v_i(s), \psi_i(s), \phi_i(s)) = (x(s), y(s), M_0)$, $i = I, II$, then the coincidence of the initial points of these curves causes their coincidence in the whole run.

Let now (v_0, ψ_0, ϕ_0) be any point of the surface H_{M_0} , and $\tilde{L}$ any curve with description $x = x(s)$, $y = y(s)$, $M = M_0$, $1 \leq s \leq m$, issuing from the point $(x_0, y_0, M_0) = F(v_0, \psi_0, \phi_0)$ and running in D_{M_0} . It follows from the considerations we carried out that there exist a non-empty interval $\langle 1, \tilde{s} \rangle$ and exactly one curve L with description $v = v(s)$, $\psi = \psi(s)$, $\phi = \phi(s)$, $1 \leq s < \tilde{s}$, running in H_{M_0} and such that along it $F(v(s), \psi(s), \phi(s)) = (x(s), y(s), M_0)$ and $F(v(1), \psi(1), \phi(1)) = (x_0, y_0, M_0)$. Let us take any sequence of parameters (s_n) , $n = 1, 2, \dots$, tending to $\tilde{s}$ from the left. Of course, for every n , $(v_n, \psi_n, \phi_n) = (v(s_n), \psi(s_n), \phi(s_n)) \in H_{M_0}$, $(x(s_n), y(s_n), M_0) \in D_{M_0}$ and $F(v_n, \psi_n, \phi_n) = (x_n, y_n, M_0)$. Besides, let

$$(44) \quad \lim_{n \rightarrow \infty} (x_n, y_n, M_0) = (x(\tilde{s}), y(\tilde{s}), M_0) \in D_{M_0}.$$

The sequence of points (v_n, ψ_n, ϕ_n) is then convergent to a point of the set H_{M_0} , or else it would follow from (40), (41) that

$$\lim_{n \rightarrow \infty} (x_n, y_n, M_0) = (x^0, y^0, M_0) \notin D_{M_0},$$

which contradicts (44). From the other remark we made in the proof it follows that the largest interval in which the required curve exists is the interval $\langle l, m \rangle$, as well as that this is the only curve satisfying the assumptions of the lemma.

Since the set D_{M_0} is a simply connected domain in π_{M_0} , and the surface H_{M_0} a connected set, therefore it follows from Lemma 6 and the theorem of R. Naresimhan ([8], p.20) that mapping (40), (41) is in H_{M_0} homeomorphic.

We have thus shown

L e m m a 7. For each point $(\operatorname{re} \alpha, \operatorname{im} \alpha, M) \in D_M$, where $M > 1$ is any fixed real number, there exist exactly one point $(v, \psi, \phi) \in H_M$ such that system (35) - (37) is satisfied.

Denote now by E'_M (M fixed) a set of points α of the complex plane C , consisting of:

1° closed segments parallel to the axis $\operatorname{re} \alpha = 0$, such that their terminal points, for $a \leq \operatorname{re} \alpha \leq 1$ (see (43)), lie on the circle K_1 , whereas for $\frac{1}{1-M} < \operatorname{re} \alpha < a$, on the curves Γ (see (43)) and $\bar{\Gamma}$, with that α is a point of the curve $\bar{\Gamma}$ if and only if the point $\bar{\alpha}$ is a point of the curve Γ ,

2° the half-line $\operatorname{im} \alpha = 0, \operatorname{re} \alpha \geq 1$,

3° the half-line $\operatorname{im} \alpha = 0, \operatorname{re} \alpha \leq \frac{1}{1-M}$.

Let next $E_M = C - E'_M$.

From the considerations carried out in 1 and 4 follows

L e m m a 8. If in the family S_M , $M > 1$, for $\alpha \in E_M$, there exists an extremal function $w = f(z)$ satisfying equation (28), then

$$\hat{J}(f) = v + \cos 2\psi - \frac{1}{M^2} (u + \cos 2\phi),$$

where v, u, ψ, ϕ is the only solution to system (35) - (38), such that $v, u > 2$, $\psi, \phi \in (-\frac{\pi}{2}, 0)$ or $\psi, \phi \in (0, \frac{\pi}{2})$. Besides, the function $w = f(z)$ satisfies equation (29) for $C = 0$.

For $\alpha \in E'_M - B_M$, where

$$B_M = \left\{ \alpha : \operatorname{im} \alpha = 0, \operatorname{re} \alpha > 1 \text{ or } \operatorname{re} \alpha < \frac{1}{1-M} \right\},$$

there exists no extremal function satisfying equation (28).

Finally, note that when $\psi = 0$, from (32), (33) we have $\phi = 0$ and

$$u = 4\alpha - 6 - 4M(\alpha - 1), \quad v = 2 - 4\alpha + \frac{4}{M}(\alpha - 1),$$

with that $u, v > 2$ if $\alpha = \bar{\alpha} < \frac{1}{1-M}$. So we obtain

L e m m a 9. If, for $\alpha < \frac{1}{1-M}$, in the family S_M , $M > 1$, there exists an extremal function $w = f(z)$ satisfying equation (28), then

$$\hat{J}(f) = \frac{1}{M^2} (5 - 4\alpha) - \frac{8}{M} (1 - \alpha) + 3 - 4\alpha.$$

Adopting in (32), (33): $\psi = \frac{\pi}{2}$, one can analogously show

L e m m a 10. If, for $\alpha > 1$, in the family S_M , $M > 1$, there exists an extremal function $w = f(z)$ satisfying equation (28), then

$$\hat{J}(f) = \frac{1}{M^2} (4\alpha - 5) - \frac{8}{M} (\alpha - 1) - 3 + 4\alpha.$$

5. In the examinations carried out so far, we have considered all possible forms of differential-functional equation (3). At present, we shall proceed to proving the basic theorem.

To that end, we shall introduce, with any fixed $M > 1$, the following notation:

$$(45) \quad D_M^1 = \left\{ \alpha; \alpha \in \mathbb{C}, \operatorname{im} \alpha = 0, \operatorname{re} \alpha < \frac{1}{1-M} \right\},$$

$$(46) \quad D_M^2 = \left\{ \alpha; \alpha \in \mathbb{C}, \operatorname{im} \alpha = 0, \operatorname{re} \alpha > 1 \right\},$$

$$(47) \quad D_M^3 = \left\{ \alpha; \alpha \in \mathbb{C}, \left| \alpha - 1 + \frac{1}{2 \log M} \right| \leq \frac{1}{2 \log M} \right\},$$

$$(48) \quad D_M^4 = \left\{ \alpha; \alpha \in \mathbb{C}, \left| \alpha - \frac{M-2}{2(M-1)} \right| > \frac{M}{2(M-1)} \right\} \setminus (D_M^1 \cup D_M^2),$$

$$(49) \quad D_M^5 = \left(\left\{ \alpha; \alpha \in \mathbb{C}, \frac{1}{1-M} < \operatorname{re} \alpha < a, \operatorname{im} \tilde{\alpha} \leq \operatorname{im} \alpha \leq -\operatorname{im} \tilde{\alpha} \right\} \cup \left\{ \frac{1}{1-M} \right\} \right) \setminus D_M^3,$$

with that

$$a = 1 - \frac{\log M}{\log^2 M + \left(\sqrt{M^2 - 1} - \arccos \frac{1}{M} \right)^2},$$

whereas $\tilde{\alpha}$ is the respective point ($\operatorname{re} \tilde{\alpha} = \operatorname{re} \alpha$) of a curve with equation

$$\frac{1}{1-\tilde{\alpha}} = 1 - \frac{1}{M \cos t} - \log \cos t + i(\operatorname{tg} t - t), \quad 0 < t \leq \arccos \frac{1}{M}.$$

We shall prove the following

Theorem. For any function $w = f(z)$ of the family S_M , $M > 1$, the following exact estimate

$$(50)-(54) \quad |a_3 - \alpha a_2^2| \leq \begin{cases} \frac{1}{M^2} (5-4\alpha) - \frac{8}{M} (1-\alpha) + 3-4\alpha & \text{if } \alpha \in D_M^1, \\ \frac{1}{M^2} (4\alpha-5) - \frac{8}{M} (\alpha-1) - 3+4\alpha & \text{if } \alpha \in D_M^2, \\ 1 - \frac{1}{M^2} & \text{if } \alpha \in D_M^3, \\ v + \cos 2\psi - \frac{1}{M^2} (u + \cos 2\phi) & \text{if } \alpha \in D_M^4, \\ 1 - \frac{1}{M^2} + 2 \left(\frac{1}{M} - \lambda \right)^2 & \text{if } \alpha \in D_M^5 \end{cases}$$

takes place, where in estimate (53) v, u, ψ, ϕ are the only roots of system (35) - (38), such that $v, u > 2$, $\psi, \phi \in (-\frac{\pi}{2}, 0)$ or $\psi, \phi \in (0, \frac{\pi}{2})$; in (54) λ is the only root of equation (25), belonging to the interval $(\frac{1}{M}, 1)$, while the sets D_M^j , $j = 1, 2, 3, 4, 5$, are defined by formulae (45) - (49).

P r o o f . In Lemmas 1-3, 8-10 we have obtained some estimates of functional (2), and thereby - of functional (1). Since the sets obtained are not disjoint, the proof of the theorem will consist in determining the exact estimates in the common parts of the sets in question.

Let us first notice that the set D_M^1 is disjoint from the sets (26), E_M and D_M^2 . Consequently, it follows from Lemmas 2, 9, that, for $\alpha \in D_M^1$, either (50) or (52') may hold true. Since, for $\alpha < \frac{1}{1-M}$

$$\frac{1}{M^2} (5-4\alpha) - \frac{8}{M} (1-\alpha) + 3 - 4\alpha > 1 - \frac{1}{M^2}$$

and, besides, the Pick function $w = f(z)$, $|f(z)| < M$, $f(0) = 0$, defined by the equation

$$(55) \quad \frac{M^2 w}{(M+w)^2} = \frac{z}{(1+z)^2}, \quad |z| < 1,$$

realizing equality in estimate (50), belongs to the family S_M , therefore, for each $\alpha \in D_M^1$, estimate (50) takes place.

In an analogous way one proves that, for each $\alpha \in D_M^2$, estimate (51) holds true, with that the extremal function is function (55), too.

It follows from the considerations carried out in 2, 3, 4 that, for $\alpha \in D_M^3$ and any function of the family S_M ,

$$|a_3 - \alpha a_2^2| \leq 1 - \frac{1}{M^2},$$

equality taking place for the function defined by equation (16).

Note next that, for $\alpha \in D_M^5$,

$$1 - \frac{1}{M^2} + 2 \left(\lambda - \frac{1}{M} \right)^2 > 1 - \frac{1}{M^2}.$$

It follows from (33) that also

$$(56) \quad v + \cos 2\psi - \frac{1}{M^2} (u + \cos 2\phi) > 1 - \frac{1}{M^2}$$

for $\alpha \in D_M^4$. Indeed, in view of (33),

$$v + \cos 2\psi - \frac{1}{M^2} (u + \cos 2\phi) - 1 + \frac{1}{M^2} = \frac{(M^2-1)G(v,u) + (u-v)^3}{4M^2(uv-4)},$$

where the function

$$G(v,u) = -v^3 + 3v^2u - 4uv - 4u - 4v + 16$$

assumes in the domain $v, u > 2, u > v$, positive values, which proves inequality (56).

Since the upper bound of functional (2) is a continuous function of the variable α , therefore from the above considerations, suitable passing to the limit as well as from Lemmas 3, 8 it follows that estimate (52) holds true only for $\alpha \in D_M^3$.

Let us further observe that the set $D_M^4 \subset E_M$ is disjoint from the sets (26) and D_M^1, D_M^2 , and so, it follows from Lemmas 1, 3, 8 that, for $\alpha \in D_M^4$, estimate (53) holds, with that v, u, ψ, ϕ are the only roots of system (35) - (38), and one of the extremal functions satisfies (29).

Since the set D_M^5 is contained in (26) and, besides, it is disjoint from the set E_M , therefore, for $\alpha \in D_M^3$, (54) holds, where λ is the only root of equation (25), belonging to the interval $\left(\frac{1}{M}, 1 \right)$. The theorem has thus been proved.

The above theorem includes no information on estimating the functional in question for $\alpha \in D_M^6$, where

$$D_M^6 = C \setminus \bigcup_{j=1}^5 D_M^j.$$

It follows from the considerations carried out that, for $\alpha \in D_M^6$, either (53) or (54) may hold true. The definitive decision which of the above-mentioned estimates takes place requires some extra consideration. It seems to us that the maximal value of the functional in question for $\alpha \in D_M^6$ will be expressed by formula (34), i.e. estimate (53) will hold in $D_M^4 \cup D_M^6$.

Of course, in the case $\alpha = \bar{\alpha}$, from the basic theorem we obtain the earlier results included in papers ([4], [5], [6], [7]). What is more, by suitable passing in the very theorem to the limit with $M \rightarrow \infty$, we obtain the well-known estimate of functional (1) in the family S ([9], [10]).

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