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SOME PROPERTIES OF ALMOST EVERYWHERE NON-DIFFERENTIABLE FUNCTIONS

In the paper we shall apply the following notation:

$$\Delta_f = \{x: |f'(x)| < \infty\},$$

$$\Delta_f(\infty) = \{x: f'(x) = \pm\infty\},$$

D_f - the set of all points of discontinuity of a function f ,

C_f - the set of all points of continuity of the function f ,

M_f - the set of all those points at which there is no one-sided (finite or infinite) derivative,

I - an arbitrary closed interval with finite Lebesgue measure,

R - the set of all real numbers,

$|A|$ - the Lebesgue measure of a set A ,

$\underline{f}(x)$ - the lower derivative of the function f at a point x ,

$\bar{f}(x)$ - the upper derivative of the function f at the point x ,

$\bar{A}$ - the closure of the set A .

Theorem. If a set $B \in G_\delta$, $|B| = 0$, and there exists a function $f: I \rightarrow R$ satisfying the conditions:

$$a) \quad \Delta_f(\infty) = B$$

$$b) \quad |I \setminus M_f| = 0,$$

then the set B is nowhere dense.

P r o o f . It follows from the theorem of Denjoy-Young-Saks [3] that almost each point x of non-differentiability of a function, in particular, almost each point of the set M_f , is a point at which

$$(1) \quad \underline{f}(x) = -\infty \quad \text{and} \quad \bar{f}(x) = +\infty.$$

Hence and from assumption b) it follows that almost each point of the interval I has property (1). This means that the sets $\{x: \underline{f}(x) = -\infty\}$ and $\{x: \bar{f}(x) = +\infty\}$ are dense in I . Whence, in virtue of Lemma, [1] p.74, these sets are residual. Consequently, such is the set $E = \{x: \underline{f}(x) = -\infty\} \cap \{x: \bar{f}(x) = +\infty\}$.

Suppose that the set B is not nowhere dense. Then there exists an interval $(a,b) \subset I$ on which the set B is dense. Since $B \in G_\delta$, the set $B \cap (a,b)$ is residual on (a,b) . The set $E \cap B \cap (a,b)$ is therefore residual in (a,b) . This last sentence is false because the set $E \cap B \cap (a,b)$ is, of course, empty.

C o r o l l a r y 1. If a function f is continuous, $f: I \rightarrow R$, and a set $B = \Delta_f(\infty)$ and $I \setminus B = M_f$, then B is nowhere dense.

C o r o l l a r y 2. For an arbitrary set $B \in G_\delta$, $|B| = 0$, $\bar{B} = I$, there exists no function $f: I \rightarrow R$ such that

$$(2) \quad \Delta_f(\infty) = B,$$

$$(3) \quad C_f = \Delta_f(\infty) \cup M_f.$$

P r o o f . Suppose that there exists a function f satisfying the conditions given in Corollary 2. We shall show that then

$$(4) \quad |D_f| = 0.$$

Let A stand for the set of those points x of the interval I at which the function f possesses at least one of the one-sided derivatives, and if two - they are distinct finite or infinite. Note that

$$(5) \quad D_f \subset A.$$

It follows from the theorem of Denjoy-Young-Saks that $|A| = 0$, which, along with (5), gives (4).

From the assumptions of Corollary 2 and from (4) we get

$$0 = |D_f| = |I \setminus C_f| = |I \setminus (\Delta_f(\infty) \cup M_f)| = |I \setminus (B \cup M_f)| = |I \setminus M_f|,$$

which leads to a contradiction with our theorem.

Corollary 2 is a negative answer to the following question raised by Marcus in [2]:

Does there exist, for any dense set B of measure zero, $B \in G_\delta$, a function of type α for which

$$\Delta_f(\infty) = B \quad \text{and} \quad C_f = \Delta_f(\infty) \cup M_f?$$

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