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ON THE OPERATORS RELATED TO STOCHASTIC INTEGRAL EQUATIONS

The purpose of this work is to consider the concept of a general stochastic equation and to investigate the problem of existence and unicity of its solution. Thus our aim is two-fold. Firstly, we intend to study various types of operators which map the class of adapted, right continuous and possessing left limits processes into itself. These examinations lead to the equation of the form

$$(1) \quad Z = \sum_{i=1}^N \phi_i Z + \sum_{j=1}^M \psi_j Z + H(\Gamma_1 Z, \dots, \Gamma_P Z),$$

where the mappings ϕ_i are piecewise $\mathfrak{s}^P$ -contractive, ψ_j denotes a strongly $\mathfrak{s}^P$ -contractive mapping and each $\Gamma_1, \dots, \Gamma_P$ stands for a piecewise defined operator (for the definitions see Section 2). The main theorem which we arrive at in this paper is closely related with the classical results concerning the existence and unicity of a global strong solution of a stochastic integral equation with driving semimartingale and Lipschitzian coefficients. Since the method of demonstration for the equation comprising only contractive mappings relies on an employment of the ideas due to C. Doléans-Dade [1] and P. Protter [9] (then extended among others by M. Emery [5] and J.-A. Yan [14]) the first part of the proof of the main Theorem 1 is merely sketched, while the second part, concerning the concept of a piecewise defined operator, is presented in detail. In this connection we investigate some limitations

of this method related to the existence of a maximal solution.

Secondly, in recent years, extensive attention has been shown to the studying of strong solutions of equations with continuous or jump reflection at the boundary (see e.g. N.El Karoui, M.Chaleyat-Maurel [4], P.Protter [11] and M.Rutkowski [13]). Therefore our goal would be to formulate the results which cover different types of reflecting boundary conditions.

The article itself is divided into four sections. Section 1 contains a brief survey of the notation. Section 2 starts out with a description of a concept of a piecewise contractive mapping. After that follows some lemmata and counter-examples leading to the notion of a piecewise defined operator. In Section 3 the main result and its proof are given. We investigate also briefly the problem of necessity of some nonstandard conditions imposed on the piecewise defined operators. Finally Section 4 is devoted to the studying of special classes of the considered equations. In particular we discuss in our general context some of the results contained in the above mentioned papers.

1. Notation

Assume that we are given a fixed probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ satisfying the usual conditions, i.e., $\mathcal{F}_0$ contains all P -null sets of $\mathcal{F}$ and $\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s$. All processes considered throughout this article are assumed to be real valued and adapted to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$. We say that the process Z is cadlag if it is right continuous with left limits. The class of all such processes will be denoted by $\mathcal{Z}$ (by a convention $Z_{0-} = 0$ for each process $Z \in \mathcal{Z}$). Let p be a fixed real number, $1 \leq p < \infty$. Following M.Emery [5] we denote $\|Z\|_{\mathcal{Z}^p} = \|Z^*\|_{L^p}$, where $Z^* = \sup_{t \in \mathbb{R}_+} |Z_t|$. We shall use the fact that $\mathcal{Z}^p = \{Z \in \mathcal{Z} : \|Z\|_p < \infty\}$ is a Banach space with the norm $\|\cdot\|_p$.

For the stopping times S, T (with respect to the filtration $(\mathcal{F}_t)$) we denote by $\llbracket S, T \rrbracket, [S, T]$ etc. the stochastic intervals, more precisely

$$\begin{aligned}\llbracket S, T \rrbracket &= \{(\omega, t) \in \Omega \times \mathbb{R}_+ : S(\omega) < t < T(\omega)\}, \\ [S, T] &= \{(\omega, t) \in \Omega \times \mathbb{R}_+ : S(\omega) \leq t < T(\omega)\} \quad \text{etc.}\end{aligned}$$

We use the standard notation for the stopped (weakly stopped resp.) processes, that is to say for a stopping time T

$$Z^T = Z \chi_{\llbracket 0, T \rrbracket} + Z_T \chi_{[T, \infty[},$$

$$(Z^{T-} = Z \chi_{\llbracket 0, T \rrbracket} + Z_{T-} \chi_{[T, \infty[} \text{ resp.})$$

where χ_A stands for the indicator function of the set A . We shall also adopt the following notation for processes

$$\Delta \llbracket S, T \rrbracket Z = (Z - Z^S)^{T-}, \quad \Delta [S, T] Z = (Z - Z^{S-})^{T-} \quad \text{etc.}$$

(so that in particular $\Delta \llbracket T \rrbracket Z = \Delta^T Z = Z^T - Z^{T-}$) and random variables

$$\Delta \llbracket S, T \rrbracket Z = (Z_{T-} - Z_S) \chi_{\{S < T\}}, \quad \Delta [S, T] Z = Z_{T-} - Z_{S-} \quad \text{etc.}$$

(so that $\Delta \llbracket T \rrbracket Z = \Delta^T Z = Z_T - Z_{T-}$).

Finally we say that a certain property is satisfied for a stopping time T arbitrarily large if there exists a sequence of stopping times $V_k \nearrow \infty$ ($k=1, 2, \dots$) such that for each k this property holds when T is replaced by V_k . Here and in what follows the equality (inequality, convergence) of random variables is implicitly supposed to hold almost surely, so that we shall systematically drop the sign "a.s.".

2. Contractive and piecewise defined mappings

Let ϕ denote an operator which maps the class $\mathcal{Z}$ into itself. The following condition $C1$ ($\tilde{C}1$ resp.) was introduced by M. Emery [5] (J.-A. Yan [14] resp.)

C1: for each $Z, Y \in \mathcal{Z}$ and arbitrary stopping time S

$$Z^{S-} = Y^{S-} \text{ implies } \phi Z^{S-} = \phi Y^{S-},$$

$\tilde{C}1$: for each $Z \in \mathcal{Z}$ and any stopping time S

$$\phi Z^{S-} = \phi(Z^{S-})^{S-}$$

(ϕZ^{S-} stands for $(\phi Z)^{S-}$). One easily checks that the above two conditions are equivalent. For the general mappings we shall also need a more restrictive condition

C2: for each $Z \in \mathcal{Z}$ and any stopping time S

$$\phi Z^S = \phi(Z^{S-})^S.$$

Evidently the condition C2 implies $\tilde{C}1$ (and thus also C1) but the converse is not true.

R e m a r k . Let $\emptyset$ denote the evanescent process, i.e., the process satisfying $\emptyset^* = 0$. Then for each mapping ϕ satisfying the condition C2 and arbitrary process $Z \in \mathcal{Z}$ the equalities $\Delta^0 \phi Z = \Delta^0 \phi \emptyset$, $\phi Z_0 = \phi \emptyset_0$ are valid.

2.1. Piecewise ϕ^p -contractive mappings. Fix p with $1 \leq p < \infty$. Assume to this end that the mapping ϕ satisfies the condition C2. Finally let $S \leq T$ be stopping times.

D e f i n i t i o n 1. A mapping ϕ is said to be ϕ^p -contractive on $\llbracket S, T \rrbracket$ if there exists a constant c , $0 \leq c < 1$, such that for each $Z, Y \in \mathcal{Z}$

$$(2) \quad Z^S = Y^S \text{ implies } \|(\phi Z - \phi Y)^T\|_{\phi^p} \leq c \|Z - Y\|_{\phi^p}.$$

R e m a r k . The condition (2) is equivalent to the following one

$$(3) \quad Z^S = Y^S \text{ implies } \|\Delta^{\llbracket S, T \rrbracket}(\phi Z - \phi Y)\|_{\phi^p} \leq c \|\Delta^{\llbracket S, T \rrbracket}(Z - Y)\|_{\phi^p}.$$

The following lemma is an easy consequence of the definition of the ϕ^p -contractive mapping, therefore its proof is omitted.

L e m m a 1. Assume that ϕ is $\mathfrak{L}^P$ -contractive on $\mathbb{IS}, \mathbb{T}\mathbb{I}$. Then

(i) ϕ is $\mathfrak{L}^P$ -contractive on $\mathbb{IS}, \tilde{\mathbb{T}}\mathbb{I}$ whenever the latter stochastic interval is a subset of $\mathbb{IS}, \mathbb{T}\mathbb{I}$,

(ii) the mapping ϕ^T , defined by the formula $\phi^T Z = \phi Z^T$ for $Z \in \mathcal{Z}$, is $\mathfrak{L}^P$ -contractive on $\mathbb{IS}, \infty\mathbb{I}$.

D e f i n i t i o n 2. A mapping ϕ is called piecewise $\mathfrak{L}^P$ -contractive if there exists a sequence of stopping times $T_n \uparrow \infty$, $T_0 = 0$, and a sequence of constants c_n , $0 \leq c_n < 1$ ($n = 0, 1, \dots$) such that for each n the mapping ϕ is $\mathfrak{L}^P$ -contractive on $\mathbb{I}T_n, T_{n+1}\mathbb{I}$ with a constant c_n .

We shall write briefly $\phi \in PC^P(T_n, c_n)$ in the sequel. Assume that the mapping ϕ is piecewise $\mathfrak{L}^P$ -contractive and let $\tilde{T}_1 \uparrow \infty$, $\tilde{T}_0 = 0$ ($l = 0, 1, \dots$) be a sequence of stopping times such that each stochastic interval $\mathbb{I}\tilde{T}_l, \tilde{T}_{l+1}\mathbb{I}$ is entirely contained in one of the intervals $\mathbb{I}T_n, T_{n+1}\mathbb{I}$ (that is to say, the sequence $\tilde{T}_l$ is a refinement of the sequence T_n). Then in view in Lemma 1(i) the mapping ϕ belongs to $PC^P(\tilde{T}_1, \tilde{c}_1)$ for a certain sequence $\tilde{c}_1$ of constants. Therefore we have an easily verified

C o r o l l a r y 1. Let $\phi_i \in PC^P(T_n^i, c_n^i)$, $i=1, \dots, N$, with the constants satisfying $\sum_{i=1}^N c_{j_i}^i < 1$ for arbitrary $j_1, \dots, j_N$.

Then the mapping $\phi = \sum_{i=1}^N \phi_i \in PC^P(T_1, c_1)$ for certain sequences T_1 and c_1 .

2.2. Strongly $\mathfrak{L}^P$ -contractive mappings. We shall now introduce a class of contractive mappings closely connected with the stochastic integral equations with driving semimartingales (cf. subsection 4.1). Namely we have

D e f i n i t i o n 3. A mapping ψ satisfying the condition C2 is called strongly $\mathfrak{L}^P$ -contractive if for each c , $0 < c < 1$, there exists a sequence $S_n(c)$ of stopping times such that $\psi \in PC^P(S_n(c), c)$.

We shall write briefly $\psi \in SC^P(S_n(c))$ in order to denote that the mapping ψ is strongly $\mathfrak{s}^P$ -contractive. The following counterpart of Corollary 1 is a straightforward conclusion of the above definition. We state it for easy later reference.

C o r o l l a r y 1. Suppose that $\psi_j \in SC^P(S_n^j(c))$, $j=1, \dots, M$. Then the mapping $\psi = \sum_{j=1}^M \psi_j \in SC^P(S_1(c))$ for a certain family $S_1(c)$, $0 < c < 1$, of sequences of stopping times.

R e m a r k. One easily checks that the sum of a piecewise $\mathfrak{s}^P$ -contractive mapping and a strongly $\mathfrak{s}^P$ -contractive mapping has the property of being piecewise $\mathfrak{s}^P$ -contractive. However it seems plausible to study those two classes of operators separately as we shall impose different additional hypotheses on them in the sequel.

2.3. Preliminary results. We shall now formulate the basic lemma which allows us to apply the Banach fixed point theorem. Suppose that $S \leq T$ are stopping times and Y is a fixed process from $\mathcal{Z}$ satisfying $Y = Y^S$. We introduce the following denotation for a subspace of $\mathfrak{s}^P$

$$\mathfrak{s}^P(S) = \{Z \in \mathfrak{s}^P : Z^S = 0\}$$

and a mapping

$$\phi_Y Z = (\phi(Z+Y) - \phi Y^S)^T, \quad Z \in \mathcal{Z}.$$

L e m m a 2. Let a mapping ϕ be $\mathfrak{s}^P$ -contractive on $\mathbb{I}S, \mathbb{I}T$. Assume that one of the following two conditions is satisfied:

- (i) the process $\Delta^{\mathbb{I}S, \mathbb{I}T} \phi Y$ belongs to $\mathfrak{s}^P$,
- (ii) the process Y belongs to $\mathfrak{s}^P$ and ϕ transforms $\mathfrak{s}^P$ into itself.

Then the following assertions hold:

- (a) the mapping ϕ_Y transforms $\mathfrak{s}^P(S)$ into itself,
- (b) ϕ_Y is contractive, namely for each $Z_1, Z_2 \in \mathfrak{s}^P(S)$

$$\|\phi_Y Z_1 - \phi_Y Z_2\|_{\mathfrak{s}^P} \leq c \|Z_1 - Z_2\|_{\mathfrak{s}^P}.$$

P r o o f . We shall first verify that the second assertion is valid. By the assumption we have $Z_1^S = Z_2^S$, hence also $(Z_1+Y)^S = (Z_2+Y)^S$. Therefore

$$(4) \quad \begin{aligned} \|(\phi_Y Z_1 - \phi_Y Z_2)\|_{\mathfrak{S}^P} &= \|(\phi(Z_1+Y) - \phi(Z_2+Y))^T -\|_{\mathfrak{S}^P} \leq \\ &\leq c \|Z_1 - Z_2\|_{\mathfrak{S}^P} \end{aligned}$$

as ϕ is assumed to be $\mathfrak{S}^P$ -contractive on $\mathbb{I}S, \mathbb{I}T$. This proves the second assertion. Observe now that by each of the conditions (i), (ii) the process $\phi_Y \Phi = \Delta^{\mathbb{I}S, \mathbb{I}T} \phi Y$ belongs to the space $\mathfrak{S}^P$. Thus by the inequality (4) we have

$$\|\phi_Y Z\|_{\mathfrak{S}^P} \leq \|\phi_Y \Phi\|_{\mathfrak{S}^P} + c \|Z\|_{\mathfrak{S}^P} < \infty$$

for arbitrary $Z \in \mathfrak{S}^P(S)$. Finally

$$\phi_Y Z^S = ((\phi(Z+Y) - \phi_Y^S)^T -)^S = (\phi(Z+Y)^S - \phi_Y^S)^T - = \Phi$$

since by C2 and the equality $Z^S = \Phi$

$$\phi(Z+Y)^S = \phi((Z+Y)^{S-})^S = \phi(Y^{S-})^S = \phi_Y^S.$$

Thus we are done.

Let us recall that for a fixed stopping time T we denote by ϕ^T the mapping defined by the formula $\phi^T Z = \phi Z^T -$ for $Z \in \mathcal{Z}$. The employment of the preceding lemma would be possible whenever satisfies one of the following definitions.

D e f i n i t i o n 4. We say that an operator ϕ maps locally $\mathfrak{S}^P$ into itself if there exists a stopping time arbitrarily large such that ϕ^T maps $\mathfrak{S}^P$ into itself.

D e f i n i t i o n 5. A mapping $\phi \in PC^P(T_n, c_n)$ is said to be locally of increments in $\mathfrak{S}^P$ if there exists a stopping time T arbitrarily large such that for each $n=0, 1, \dots$ the process

$$(5) \quad \Delta^{\mathbb{I}T_n, T_{n+1}} \phi^T(Z^{T_n})$$

belongs to $\mathfrak{S}^P$, where Z is an arbitrary process from $\mathcal{Z}$.

Note that for the class of piecewise $\mathfrak{s}^P$ -contractive operators neither the property of possessing locally increments in $\mathfrak{s}^P$ implies the property of mapping locally $\mathfrak{s}^P$ into itself nor the converse implication is true, as the following counter-example shows.

Counter-example 1. (i) Let ϕ be defined by the formula

$$\phi Z_t = Z_{[t-1]}^2 \chi_{[1, \infty[}(t), \quad Z \in \mathcal{Z}.$$

Then $\phi \in PC^P(T_n, c)$ for a sequence of stopping times $T_n = n$ ($n = 0, 1, \dots$) and arbitrary constant c , $0 \leq c < 1$. Moreover ϕ is of increments in $\mathfrak{s}^P$ (we drop the word "locally" as we may take $T = \infty$), however, as is easily seen, it does not map locally any space $\mathfrak{s}^P$ into itself.

(ii) We shall now verify that the inverse implication is false. To check this we put

$$\phi Z_t = Z_{t-1} \chi_{[1, \infty[}(t), \quad Z \in \mathcal{Z}.$$

Then clearly ϕ maps each space $\mathfrak{s}^P$ into itself and belongs to $PC^P(T_n, c)$ for a sequence of stopping times $T_n = n$ ($n = 0, 1, \dots$) and arbitrary constant c , $0 \leq c < 1$. To check that ϕ does not satisfy Definition 5 it suffices to consider the processes $Z_t = \varrho t$, $t \in \mathbb{R}_+$, where ϱ is an arbitrary $\mathcal{F}_0$ -measurable random variable.

Let us recall that the property of being piecewise $\mathfrak{s}^P$ -contractive is hereditary while considering the refinements of a given sequence T_n . Unfortunately it is not the case for the property of possessing increments in $\mathfrak{s}^P$.

Counter-example 2. Let the mapping ϕ be as follows

$$\phi Z_t = (Z_{t-1} - Z_1) \chi_{[2, 3[}(t), \quad Z \in \mathcal{Z}.$$

Clearly $\phi \in PC^P(T_n, c)$ for a sequence of stopping times $T_0 = 0$, $T_1 = 1$, $T_2 = 3$, $T_n = \infty$ for $n = 3, 4, \dots$ and arbitrary constant c , $0 \leq c < 1$. It is also of increments in $\mathfrak{s}^P$ as a mapping from the class $PC^P(T_n, c)$. However if we take a refine-

ment $\tilde{T}_1$, where $\tilde{T}_1 = 1$ for $l = 0, \dots, 3$ and $\tilde{T}_1 = \infty$ for $l = 4, 5, \dots$, then clearly $\phi \in PC^p(\tilde{T}_1, c)$ but is no longer locally of increments in $\mathcal{P}$. Actually it suffices to consider the processes $Z_t = \phi((t-1) \wedge 1) \chi_{[1, \infty)}(t)$, $t \in \mathbb{R}_+$, where as before ϕ is an arbitrary $\mathcal{F}_0$ -measurable random variable. More precisely we have the equality

$$\|\Delta_{\tilde{T}_2, \tilde{T}_3} \phi(Z^{\tilde{T}_2})\|_{\mathcal{P}} = \|\phi\|_{\mathcal{P}},$$

which contradicts Definition 5 for $T = \infty$. The analogous result for arbitrarily large stopping time T can be achieved by standard arguments and thus is omitted.

In fact, there is no necessity of considering the refinements if we restrict ourselves to the equation $Z = \phi Z$, where ϕ is a piecewise $\mathcal{P}$ -contractive mapping. Still, while searching for a solution of the equation (1) we are forced to apply the customary technique of carving the time axis into arbitrarily small random pieces. Thus the revealed lack of heredity forces us to introduce the next class of operators.

2.4. Piecewise defined mappings. Let I denote the set of all countable ordinal numbers. To each process $Z \in \mathcal{Z}$ we shall associate a family $Q_a = Q_a(Z)$, $a \in I$, of stopping times possessing the following properties:

P1: if $a < b$, then $Q_a(Z) < Q_b(Z)$ on the set $\{Q_a(Z) < \infty\}$,

$$Q_0(Z) = 0,$$

P2: for each $a \in I$ and arbitrary processes $Z, Y \in \mathcal{Z}$

$$(Z - Y)^{Q_a(Z)-} = 0 \text{ implies } Q_a(Z) = Q_a(Y),$$

P3: If $Q_a = \sup_{b < a} Q_b(Z)$, then $Q_a = Q_a(Z)$.

Observe that the condition P1 implies that $\bigcup_{a \in I} [Q_a, Q_{a+1}] = [0, \infty]$ up to an evanescent set (cf. [11], p.198). Moreover by virtue of P2 the equality $Q_a(Z) = Q_a(Z^{Q_a(Z)-})$ holds for each $a \in I$ and $Z \in \mathcal{Z}$. For each $a \in I$ we denote by $\Gamma_a = (\tilde{\Gamma}_a, \hat{\Gamma}_a)$

a mapping which is defined on Z and takes its values in the Cartesian product $Z \times Z$. Assume that an operator Γ maps the class Z into itself and satisfies the condition C2. Then we have

D e f i n i t i o n 6. An operator Γ is said to be piecewise defined if there exists a family of mappings Γ_a , $a \in I$, and for each process $Z \in Z$ a family of stopping times $Q_a = Q_a(Z)$, $a \in I$, satisfying the conditions P1-P3, such that for each $a \in I$ either

(i) $a-1$ exists and

$$\Gamma Z^{Q_{a-1}} = \Gamma(Z^{Q_{a-1}})^{Q_a} = \tilde{\Gamma}_a(Z^{Q_{a-1}})^{Q_a},$$

$$\Delta^{Q_a} \Gamma Z = \Delta^{Q_a} \Gamma(Z^{Q_{a-1}}) = \Delta^{Q_a} \hat{\Gamma}_a(Z^{Q_{a-1}})$$

or (ii) a is a limit ordinal number and

$$\Gamma Z^{Q_a} = \Gamma(Z^{Q_{a-1}})^{Q_a} = \hat{\Gamma}_a(Z^{Q_{a-1}})^{Q_a}$$

(here always $Q_a = Q_a(Z)$).

For convenience in writing we shall denote $\Gamma \in \text{PD}(Q_a, \Gamma_a)$ if it fulfills the above definition. Note that in fact for the limit ordinal number a the mapping $\tilde{\Gamma}_a$ plays no role.

R e m a r k . The notion of a piecewise defined mapping is implicitly contained in P. Protter's paper [11] concerning the stochastic integral equation with jump reflection at the boundary. For more details we refer to the subsection 4.4.

Let P be a given natural number. We denote by H an operator which maps the Cartesian product Z^P into Z . The natural counterpart of the condition C1 has the form

C3: for arbitrary stopping time S and processes $Z_1, \dots, Z_P$

$$H(Z_1, \dots, Z_P)^{S-} = H(Z_1^{S-}, \dots, Z_P^{S-})^{S-}.$$

Observe that in particular the condition C3 holds for the mappings $\underline{H}(Z_1, \dots, Z_P) = \sum_{l=1}^P Z_l$ and $\overline{H}(Z_1, \dots, Z_P) = \prod_{l=1}^P Z_l$.

L e m m a 3. Suppose that H maps Z^P into Z and satisfies C3. Moreover assume that the mapping $\Gamma_l \in PD(Q_a^1, \Gamma_a^1)$, $l = 1, \dots, P$.

Then the mapping $\Gamma Z = H(\Gamma_1 Z, \dots, \Gamma_P Z)$ is piecewise defined, i.e., $\Gamma \in PD(Q_a, \Gamma_a)$ for certain families Q_a and Γ_a .

P r o o f . The assertion follows by standard arguments, therefore we omit the proof.

Turning back to the piecewise contractive mappings of increments in $\mathfrak{z}^P$, we shall demonstrate the following result which links this class of mappings with the piecewise defined operators.

L e m m a 4. Let $\phi \in PC^P(T_n, c_n)$ be locally of increments in $\mathfrak{z}^P$.

Then there exists a mapping $\underline{\phi} \in PC^P(T_n, c_n)$ which maps locally $\mathfrak{z}^P$ into itself, and a piecewise defined operator Γ such that the decomposition $\phi = \underline{\phi} + \Gamma$ is valid.

P r o o f . Let us define the mappings $\underline{\phi}$ and Γ in a following way

$$\phi Z = \sum_{n=0}^{\infty} \Delta \mathbb{I}^{T_n, T_{n+1}} \underline{\phi} Z + \sum_{n=0}^{\infty} \Delta T_n \phi Z = \underline{\phi} Z + \Gamma Z.$$

Clearly the mapping ϕ belongs to $PC^P(T_n, c_n)$ and by the optional stopping we may assume, without loss of generality, that the sum in the formula defining $\underline{\phi}$ is finite. Moreover for arbitrary process $Z \in Z$ we have by (3)

$$\|\Delta \mathbb{I}^{T_n, T_{n+1}} (\phi Z - \phi(Z^{T_n}))\|_{\mathfrak{z}^P} \leq c_n \|\Delta \mathbb{I}^{T_n, T_{n+1}} Z\|_{\mathfrak{z}^P}.$$

Moreover by the assumption ϕ is locally of increments in $\mathfrak{z}^P$ so that, applying the optional stopping if necessary, we have

$$\|\Delta \mathbb{I}^{T_n, T_{n+1}} \phi(Z^{T_n})\|_{\mathfrak{z}^P} < \infty$$

for each $Z \in Z$ and natural n . Combining the above inequalities we conclude that (at least locally)

$$\|\Delta^{T_n, T_{n+1}} \phi Z\|_{\mathcal{D}^p} < \infty$$

whenever the process Z belongs to $\mathcal{D}^p$. Thus evidently ϕ maps locally $\mathcal{D}^p$ into itself.

On the other hand an operator Γ fulfills the definition of a piecewise defined mapping (with $Q_n(Z) = T_n$ and mappings $\hat{\Gamma}_n Z = Z$, $\tilde{\Gamma}_n Z = \sum_{k=0}^{n-1} \Delta^{T_k, T_{k+1}} \phi Z$, $n = 0, 1, \dots$). This ends the proof.

3. General stochastic equation

We shall search for a global strong solution of the equation (1) in the class $\mathcal{Z}$ so that its definition may be formulated as follows.

D e f i n i t i o n 7. The process $Z \in \mathcal{Z}$ is called a solution of the equation (1) if the processes appearing on the right and left hand side of the formula (1) are indistinguishable. Moreover we say a solution is unique if for any two solutions $\underline{Z}$ and $\underline{Z}$ they are indistinguishable, i.e., $(Z - \underline{Z})^* = 0$.

The following auxiliary result is easily derived.

L e m m a 5. Let S be a stopping time and ϕ an operator satisfying the condition C2. We denote by ϕ_S the mapping $\phi_S Z = \phi Z \chi_{[0, S]}$.

Then the following assertions hold:

(i) if the process $U \in \mathcal{Z}$ is a solution of the equation $U = \phi U^{S-}$, then the process $Y = U \chi_{[0, S]}$ solves the equation $Y = \phi_S Y$,

(ii) reciprocally, if Y is a solution of the latter equation, then $U = Y^{S-}$ satisfies the former.

We need to impose an additional condition on a piecewise defined mapping Γ in order to ensure the existence of a global solution. Namely suppose that for a stopping time Q there exists a limit ordinal number a and the family of processes $Z(b)$, $b < a$, such that:

- (i) $Z_{(b)}^{Q_b} = Z_{(b)}^{Q_b} = Z_{(b)}$, where $b < b'$ and $Q_b = Q_b(Z_{(b)})$ (i.e., the family of processes $Z_{(b)}$ is coherent),
 (ii) $Q = \sup_{b < a} Q_b$.

Then we may formulate the condition:

C4: the process of the form

$$\sum_{b < a} \Delta^{Q_b, Q_{b+1}} \Gamma^{Z_{(b+1)}} \chi_{[0, Q]}$$

belongs to the class $\mathcal{Z}$ (or simply possess the left hand limit at Q).

Let N , M and P be natural numbers. The main result of this paper is

Theorem 1. Assume that:

- (i) the mappings $\phi_i \in PC^p(T_n^1, c_n^1)$, $i = 1, \dots, N$, with the constants satisfying $\sum_{i=1}^N c_{j_i}^i < 1$ for arbitrary $j_1, \dots, j_N$.

Moreover each ϕ_i either maps locally $\mathcal{S}^P$ into itself or is locally of increments in $\mathcal{S}^P$.

- (ii) the mappings $\psi_j \in SC^p(S_n^j(c))$, $j = 1, \dots, M$, and transform locally the space $\mathcal{S}^P$ into itself,

- (iii) the mappings $\Gamma_l \in PD(Q_a^1, \Gamma_a^1)$, $l = 1, \dots, P$, and satisfy C4,

- (iv) an operator H maps $\mathcal{Z}^P$ into $\mathcal{Z}$ and fulfills C3.

Then the equation (1) has a unique solution.

Observe that in view of the preceding results it suffices to prove the following

Theorem 1. Suppose that $\phi \in PC^p(T_n, c_n)$ and maps locally $\mathcal{S}^P$ into itself. Moreover let a piecewise defined mapping satisfy the condition C4.

Then a solution of the equation $Z = \phi Z + \Gamma Z$ exists and is unique.

Proof. We separate the proof into two steps.

Step 1. To begin with let us assume that $\Gamma Z = 0$ for each Z , i.e., we deal with the equation $Z = \phi Z$. In this case the demonstration may be realized by a straightforward modification of that given by M. Emery [5] for the stochastic

integral equations with driving semimartingales. Therefore we shall only indicate how the mapping ϕ_Y , introduced in Lemma 2, appears. Namely, proceeding by induction, assume that $Z_{(n)}$ is a solution on an interval $[0, T_n]$, that is to say $Z_{(n)} = \phi Z_{(n)}^{T_n}$. Our task is to extend a solution over $[0, T_{n+1}]$ so that the equation under consideration has the form

$$U = \phi U^{T_{n+1}-}$$

or equivalently

$$U - Z_{(n)}^{T_{n+1}-} = (\phi U - Z_{(n)})^{T_{n+1}-}.$$

Denote the left-hand side of the last formula by $\tilde{U}$ we can rewrite it in the form

$$(6) \quad \tilde{U} = (\phi(\tilde{U} + Z_{(n)}) - Z_{(n)})^{T_n} = \phi_{Z_{(n)}} \tilde{U}.$$

Moreover we may and do assume that $\tilde{U}^{T_n} = 0$, so that we search for the process $\tilde{U}$ satisfying the equation (6) in the space $\mathcal{D}^P(T_n)$. Thus the usefulness of Lemma 2 is justified. The details as well as the problem of unicity of a solution is left to the reader.

Step 2. In order to construct a solution in the general case we shall apply transfinite induction based on a family of stopping times associated with a solution we search for. The existence of a unique solution $Z_{(0)}$ on $[0, Q_0(Z)]$ is trivial, since by P1 we have $Q_0(Z) = 0$ for each $Z \in \mathcal{Z}$ and therefore

$$Z_{(0)} = (\phi 0 + \Gamma 0)^{Q_0}$$

where $Q_0 = Q_0(Z_{(0)}) = 0$ (we make use here of the convention $Z_{0-} = 0$ for each $Z \in \mathcal{Z}$).

Now assume that for a certain ordinal number $\alpha-1$ there exists a solution of the equation

$$(7) \quad U = (\phi U + \Gamma U)^{Q_{a-1}},$$

more precisely, there exists a process $Z_{(a-1)} \in \mathcal{Z}$ satisfying

$$Z_{(a-1)} = (\phi Z_{(a-1)} + \Gamma Z_{(a-1)})^{Q_{a-1}}$$

and such that

$$Q_{a-1} = Q_{a-1}(Z_{(a-1)}).$$

Our aim is to extend this solution over an interval $[0, Q_a]$, where Q_a is a certain stopping time satisfying $Q_{a-1} < Q_a$ on the set $\{Q_{a-1} < \infty\}$. To begin with we shall search for a global solution of the equation

$$U = \phi U + \Gamma Z_{(a-1)}$$

under an additional restriction $U^{Q_{a-1}} = Z_{(a-1)}$. The existence of such solution may be verified as in Step 1, since $\Gamma Z_{(a-1)}$ is a fixed process from the class $\mathcal{Z}$ (or from $\mathcal{Z}^P$, as we may assume by optional stopping). Moreover, if we denote this solution by $\hat{Z}_{(a)}$, we state by the unicity that $\hat{Z}_{(a)}^{Q_{a-1}} = Z_{(a-1)}$. Moreover by the property P2 we have $Q_{a-1} = Q_{a-1}(\hat{Z}_{(a)})$ and consequently the inequality $Q_{a-1} < Q_a$ on $\{Q_{a-1} < \infty\}$ is valid for the stopping time $Q_a = Q_a(\hat{Z}_{(a)})$.

We wish to check that the process $\tilde{Z}_{(a)} = \hat{Z}_{(a)}^{Q_{a-1}}$ is a solution of the equation

$$U = (\phi U + \Gamma U)^{Q_{a-1}},$$

or in other words it solves the basic equation $Z = \phi Z + \Gamma Z$ on the stochastic interval $[0, Q_a]$.

Actually the equality $Q_a = Q_a(\tilde{Z}_{(a)})$ holds by virtue of P2. In addition by the part (i) of Definition 6 we have

$$\begin{aligned}
\tilde{Z}_{(a)} &= \hat{Z}_{(a)}^{Q_{a-}} = (\phi \hat{Z}_{(a)} + r Z_{(a-1)})^{Q_{a-}} = \\
&= (\phi (\hat{Z}_{(a)})^{Q_{a-}} + \tilde{r}_a (Z_{(a-1)})^{Q_{a-1}})^{Q_{a-}} = \\
&= (\phi (\tilde{Z}_{(a)}) + \tilde{r}_a (\tilde{Z}_{(a)})^{Q_{a-1}})^{Q_{a-}} = \\
&= (\phi \tilde{Z}_{(a)} + r \tilde{Z}_{(a)})^{Q_{a-}}
\end{aligned}$$

as $\tilde{Z}_{(a)}^{Q_{a-1}}$ equals to $Z_{(a-1)}^{Q_{a-1}}$.

We shall now introduce the process $Z_{(a)}$ which forms a solution on an interval $[0, Q_a]$, namely we put

$$Z_{(a)} = \tilde{Z}_{(a)} + \Delta^{Q_a} (\phi \tilde{Z}_{(a)} + r \tilde{Z}_{(a)}).$$

Then $Q_a = Q_a(Z_{(a)})$ and taking in view the condition C2 and again the part (i) of Definition 6 we get

$$\begin{aligned}
Z_{(a)} &= (\phi \tilde{Z}_{(a)} + r \tilde{Z}_{(a)})^{Q_{a-}} + \Delta^{Q_a} (\phi \tilde{Z}_{(a)} + r \tilde{Z}_{(a)}) = \\
&= \phi (\tilde{Z}_{(a)})^{Q_{a-}}^{Q_a} + \tilde{r}_a (\tilde{Z}_{(a)})^{Q_{a-1}}^{Q_{a-}} + \Delta^{Q_a} \tilde{r}_a (\tilde{Z}_{(a)})^{Q_{a-}} = \\
&= (\phi Z_{(a)} + r Z_{(a)})^{Q_a}
\end{aligned}$$

since $Z_{(a)}^{Q_{a-}} = \tilde{Z}_{(a)}$ and $Z_{(a)}^{Q_{a-1}} = \tilde{Z}_{(a)}^{Q_{a-1}}$. Concluding, we have extended the solution $Z_{(a-1)}$ over the closed stochastic interval $[0, Q_a]$.

Suppose that a is a limit ordinal number and there exists a family of processes $Z_{(b)}$, $b < a$, such that each process $Z_{(b)}$ is a solution of the basic equation on an interval $[0, Q_b]$, where $Q_b = Q_b(Z_{(b)})$. Moreover let the coherency condition be satisfied, i.e., $Z_{(b)}^{Q_b} = Z_{(b)}$ whenever $b < b' < a$. Thus the family of stopping time Q_b , $b < a$, is uniquely determined, more precisely

$$Q_c(Z_{(b')}) = Q_c(Z_{(b)}) \text{ if } c \leq b \leq b'$$

(it seems worth to note that the stopping times $Q_c(Z_{(b)})$, $c > b$, play no role in the sequel).

We introduce a stopping time $Q_a = \sup_{b < a} Q_b(Z_{(b)})$ and a process

$$(8) \quad \tilde{Z}_{(a)} = \sum_{b < a} \Delta^{[Q_b, Q_{b+1}[} Z_{(b+1)} \chi_{[0, Q_a[}$$

Observe that the equality $\tilde{Z}_{(a)} = Z_{(b)}$ holds on each stochastic interval $[0, Q_b]$, $b < a$. This implies in particular, that the stopped process $Z_{(a)}^{Q_b}$ is a solution on $[0, Q_b]$ for each $b < a$. Moreover in view of the property P3 we have $Q_a = Q_a(\tilde{Z}_{(a)})$. Using the assumptions on $Z_{(b)}$ we can rewrite (8) in the form

$$(9) \quad \begin{aligned} \tilde{Z}_{(a)} &= \sum_{b < a} \Delta^{[Q_b, Q_{b+1}[} (\phi Z_{(b+1)} + \Gamma Z_{(b+1)}) \chi_{[0, Q_a[} = \\ &= \sum_{b < a} \Delta^{[Q_b, Q_{b+1}[} (\phi Z_{(b+1)}) \chi_{[0, Q_a[} + \\ &+ \sum_{b < a} \Delta^{[Q_b, Q_{b+1}[} \Gamma(\tilde{Z}_{(a)})^{Q_{b+1}} \chi_{[0, Q_a[}. \end{aligned}$$

According to the condition C4 the second process in the last sum (denoted by $\bar{U}$ from now on) belongs to the class Z . Therefore by virtue of Step 1 the equation $U = \phi U + \bar{U}$ possess a unique solution, denoted by $Z_{\bar{U}}$ in the sequel. On the other hand it is easily seen that for each $b < a$ the process $\tilde{Z}_{(a)}^{Q_b}$ is a unique solution of the equation

$$U = (\phi U + \bar{U})^{Q_b}.$$

In order to verify that the process $\tilde{Z}_{(a)}^{Q_b}$ belongs to Z , it suffices to observe that $\tilde{Z}_{(a)}^{Q_b} = Z_{\bar{U}}^{Q_b}$ for each $b < a$, since the last equality implies $Z_{(a)} = Z_{\bar{U}} \chi_{[0, Q_a[}$. Consequently the

stopped process $Z_{(a)}^{Q_{a^-}}$ belongs to the class $\mathcal{Z}$ as well and satisfies the equation

$$U = (\phi U + rU)^{Q_{a^-}},$$

that is to say, is a solution of the basic equation on the stochastic interval $\llbracket 0, Q_a \rrbracket$. To verify the last statement it suffices to replace $Z_{(b+1)}$ by $\tilde{Z}_{(a)}^{Q_{b+1}}$ in the formula (9) and apply in particular Lemma 5(ii).

In order to extend this solution over $\llbracket 0, Q_a \rrbracket$ we put

$$Z_{(a)} = \tilde{Z}_{(a)}^{Q_{a^-}} + \Delta^Q_a(\phi \tilde{Z}_{(a)} + r \tilde{Z}_{(a)}).$$

Then again by the property P3 we have $Q_a = Q_a(Z_{(a)})$ and in virtue of the part (ii) of Definition 6

$$\begin{aligned} Z_{(a)} &= \tilde{Z}_{(a)}^{Q_{a^-}} + \Delta^Q_a(\phi(\tilde{Z}_{(a)}^{Q_{a^-}}) + \hat{r}_a(\tilde{Z}_{(a)}^{Q_{a^-}})) = \\ &= \phi(Z_{(a)}^{Q_{a^-}})^{Q_a} + r(Z_{(a)}^{Q_{a^-}})^{Q_{a^-}} + \Delta^Q_a \hat{r}_a(Z_{(a)}^{Q_{a^-}}) = \\ &= (\phi Z_{(a)} + r Z_{(a)})^{Q_a}. \end{aligned}$$

Thus we have extended a solution on the closed stochastic interval $\llbracket 0, Q_a \rrbracket$ for a limit ordinal number a . The existence of a global solution now follows by the transfinite induction, in view of the property P1.

The problem of unicity of a solution can be treated easily by transfinite induction in view of the results of the preceding step. Therefore the proof of the theorem is completed.

R e m a r k . Observe that actually we do not need the condition C4 to be satisfied for all coherent families of processes $Z_{(b)}$, $b < a$. We merely require C4 to be fulfilled for those families which arise while solving our equation on increasing sequence of stochastic intervals $\llbracket 0, Q_b \rrbracket$, $b < a$. Thus in special cases (cf. P. Protter [11] and subsection 4.4

of this paper) one may employ the specific properties of a solution in order to check that this essentially weaker form of the condition C4 holds. In general, however, we cannot dispense from a certain condition of this kind, as the counter-examples show.

Counter-example 3. We shall consider the equation

$$(9) \quad Z = \phi Z + \Gamma Z,$$

where ϕZ equals to a fixed process Y from $\mathcal{Z}$ which will be specified in the sequel. Moreover the mapping Γ has the form

$$\Gamma Z_t = (Z \cdot B(t))_t, \quad t \in R_+,$$

where $B(t) = B_u(t)$, $t \in R_+$, is a family of processes of finite variation, so that the dot $\cdot$ stands here for the Stieltjes stochastic integral. More precisely let us put

$$Y_t = 1/2 \chi_{[0, 2/3[}(t), \quad t \in R_+,$$

$$B_u(t) = \begin{cases} 0 & \text{if } u \in R_+, \quad t \in [0, 1/2 \cup [1, \infty[, \\ (-1)^{k+1} \chi_{[(k+1)/k, \infty[}(u) & \text{if } u \in R_+ \text{ and} \end{cases}$$

$$t \in [k/(k+1), (k+1)/(k+2)[, \quad k=1, 2, \dots$$

One easily checks, that Γ satisfies Definition 6 with $q_n = Q_n(Z) = n/(n+1)$, $n = 0, 1, \dots$, $Q_{\omega}(Z) = 1$, $Q_{\omega+1}(Z) = \infty$, where ω denotes the order type of the set of natural numbers ordered by the relation $\leq$. On the other hand by the direct calculations we obtain

$$Z_{(\omega)} = \begin{cases} 1 & \text{if } t \in [2k/(2k+1), (2k+1)/(2k+2)[, \\ -1 & \text{if } t \in [(2k+1)/(2k+2), (2k+2)/(2k+3)[, \end{cases}$$

$k = 1, 2, \dots$, where $Z_{(\omega)}$ is a unique solution of the equation (9) on the stochastic interval $[[0, Q_{\omega}[$. Thus adopting the stan-

dard notation we state that a pair $(Z_{(\omega)}, Q_{\omega})$ is a maximal solution. By a slight modification of the above example we deduce that in contrast with the situation studied in [3], [7] or [9] the maximal solution is defined up to a stopping time which is the moment of explosion and/or oscillation of a solution.

4. Stochastic integral equations

The purpose of this section is to study in our general context the various types of stochastic integral equations. We insist that some of the results described below are well known, at least under slightly more restrictive assumptions.

4.1. Stochastic integral equation with driving semimartingale. We shall consider this class of equations in a setting close to that of [5] and [14]. Suppose that F is an operator which maps the class $\mathcal{Z}$ into itself and satisfies the condition C1. We say that F is Lipschitzean (briefly $F \in \text{Lip}(K)$) if there exists a process K such that for each $Z, Y \in \mathcal{Z}$ the inequality $(FZ - FY)^* \leq (K(Z - Y))^*$ is valid. As usual we denote by $FZ^- \cdot X$ the Ito stochastic integral of the left continuous version of the process FZ with respect to the semimartingale X (for the basic properties of this integral we refer to [6] or [8]). We may and do assume that the process $FZ^- \cdot X$ belongs to the class $\mathcal{Z}$. The following result is implicitly proved in [5] for the case of a constant process K . The general case can be handled by the technique of optional stopping at S_k^- , where $S_k = \inf\{t \in R_+ : |K_t| \geq k\}$.

L e m m a 6. Suppose that $F \in \text{Lip}(K)$ and X is a semimartingale. Denote by ψ the mapping $\psi Z = FZ^- \cdot X$, $Z \in \mathcal{Z}$.

Then the mapping ψ^T belongs to $SC^p(S_1(c))$ and maps $\mathcal{Z}^p$ into itself for the stopping time T arbitrarily large.

By virtue of this lemma Theorem 1 covers a wide class of stochastic integral equations with driving semimartingales. Some of them we shall consider in detail.

4.2. Stochastic integral equations of a mixed Volterra-Stieltjes and Ito type. Denote by $G(t, \cdot)$, $t \in R_+$, a family of

mappings such that each mapping $G(t, \cdot)$ transforms the class Z and satisfies the condition C1. Moreover for each $Z \in Z$ the function $G(\cdot, Z)$ is doubly cadlag, more precisely, for each $u, t \in R_+$

$$\lim_{s \searrow t, v \searrow u} G(s, Z)_v = G(t, Z)_u$$

and the corresponding finite left hand limits exist. Assume that $G(t, \cdot)$ has the following properties:

(i) it is ultimately Lipschitzean, that is to say

$$\begin{aligned} \exists K \in Z \exists \delta > 0 \forall t \in R_+ \forall Z, Y \in Z \quad (\Delta^{[t-\delta, t]}(G(t, Z) - G(t, Y)))^* \leq \\ \leq (K(Z - Y))^* \end{aligned}$$

(ii) it is ultimately bounded, i.e., for some $p \in [1, \infty[$

$$\exists \delta > 0 \forall t \in R_+ \forall Z \in Z \quad \sup_{t-\delta \leq s \leq t} (\Delta^{[t-\delta, s]} G(s, Z^{t-\delta})) \in L^p.$$

Observe that the family $G(t, \cdot) = \underline{G}$, where $\underline{G}$ is a fixed mapping from $Lip(K)$, satisfies all preceding conditions.

Let $B(t) = B_u(t)$, $t \in R_+$, be a family of processes of finite variation such that:

(i) $B(t)$ is τ -cadlag in the sense of [12],

(ii) $B(t)$ is smooth, i.e.,

$$\forall \varepsilon > 0 \exists \delta > 0 \forall t \in R_+ \quad |\Delta^{[t-\delta, t]} B(t)|_\infty < \varepsilon,$$

where for a process A of finite variation we apply the notation $|A|_\infty = \int_{[0, \infty[} |dA_s|$. For simplicity we also assume that $|B(t)|_\infty \leq M$, $t \in R_+$, where M is a real positive constant.

We are now in position to introduce the mapping ϕ , namely we put

$$\phi Z_t = \int_{[0, t[} G(t, Z)_u dB_u(t), \quad t \in R_+.$$

Then ϕ transforms the class Z into itself. Moreover one may check that ϕ is piecewise $\mathfrak{L}^p$ -contractive and possess increments in $\mathfrak{L}^p$. Therefore an equation of the form

$$Z_t = U_t + (G(t, Z) \cdot B(t))_t + (FZ^- \cdot X)_t$$

may be considered within the framework of Theorem 1. In accordance with [12] we call it the stochastic integral equation of a mixed Volterra-Stieltjes and Ito type.

4.3. Stochastic integral equation with a continuous reflection at the boundary.

We shall consider the stochastic integral equation

$$Z = U + FZ^- \cdot X$$

with a reflecting barrier $(\underline{Y}, \bar{Y})$. More precisely, we assume that U is a fixed process from $\mathcal{Z}$, X is a semimartingale, $F \in \text{Lip}(K)$ and the processes $\underline{Y}, \bar{Y} \in \mathcal{Z}$ satisfy

$$\inf_{s \in [0, t]} (\bar{Y}_s - \underline{Y}_s) > 0$$

for each $t \in R_+$. For each process Z let the triplet of processes $(GZ, \underline{K}, \bar{K})$ be a unique solution of a stochastic problem of reflection $(Z, \underline{Y}, \bar{Y})$ provided by Lemma 3 of [13], so that in particular the equality $GZ = Z + \underline{K} - \bar{K}$ holds. Thus by virtue of Lemma 4, [13] we state that the mapping G satisfies the condition C1 and belongs to $\text{Lip}(K)$. By Theorem 1 we derive the existence of a unique solution of the equation

$$(10) \quad Z = G(U + FZ^- \cdot X).$$

In fact, it suffices to observe that the mapping $\tilde{\Psi}^T Z = G(\Psi Z)^T$ belongs to $SC^P(S_1(c))$ and transforms $\mathcal{S}^P$ into itself for the stopping time T arbitrarily large whenever Ψ has those properties.

Observe that the formula (10) is equivalent to

$$Z = U + FZ^- \cdot X + \underline{K} - \bar{K},$$

where $(Z, \underline{K}, \bar{K})$ is a solution of a stochastic problem of reflection $(U + FZ^- \cdot X, \underline{Y}, \bar{Y})$. Thus in particular the process Z satisfies

$$\underline{Y}_t \leq Z_t \leq \bar{Y}_t, \quad t \in R_+,$$

and the increasing process $\underline{K}$ ($\bar{K}$ resp.) is flat outside the set $\{Z = \underline{Y}\}$ ($\{Z = \bar{Y}\}$ resp.) as was expected.

4.4. Stochastic integral equation with jump reflection at the boundary.

Recently P. Protter [11] has studied the problem of existence and unicity of a solution of the equation

$$(11) \quad Z_t = U_t + (FZ^- \cdot X)_t + \sum_{s \leq t} (N_s - IZ_s)X_{]-\infty, 0]} (IZ)_s,$$

where $IZ_s = Z_s^- + \Delta_s U + FZ_s^- \Delta_s X$, U denotes a fixed process from $\mathcal{Z}$, X stands for an arbitrary semimartingale and F is an operator belonging to $\text{Lip}(K)$. Finally the kicking process N is assumed to be strongly adapted (that is to say for each stopping time S the function $N_S X_{[S, \infty)}$ is $\mathcal{F}_S$ -measurable random variable) and $0 < N < \infty$ up to evanescence. The main theorem in [11] states that there exists exactly one global strong solution of the equation (11).

Let us define the mappings

$$\Psi Z_t = U_t + (FZ^- \cdot X)_t, \quad t \in \mathbb{R}_+,$$

$$\Gamma Z_t = \sum_{s \leq t} (N_s - IZ_s)X_{]-\infty, 0]} (IZ)_s, \quad t \in \mathbb{R}_+.$$

Then the equation (11) may be rewritten as follows

$$(12) \quad Z = \Psi Z + \Gamma Z.$$

By virtue of Lemma 6 we claim that the mapping Ψ is strongly contractive. Moreover, as one easily verifies, the operator Γ is piecewise defined, in particular the role of the family of stopping times $Q_a(Z)$, $a \in I$, play the consecutive moments of leaving the halfplane $\{x \in \mathbb{R} : x > 0\}$ by a solution Z . Unfortunately Γ does not satisfy the condition C4, so that Protter's result cannot be derived directly from our theorem. More precisely, in [11] is shown that if α is a limit ordinal number and $Z_{(b)}$, $b < \alpha$, denotes a family of solutions of the equation

(11) on increasing family of stochastic intervals $[0, Q_b]$ such that $Q = \sup_{b < a} Q_b$, then the process

$$\sum_{b < a} (N_{Q_b} - IZ_{(b)Q_b})^X [Q_b, Q] = \sum_{b < a} [Q_b, Q_{b+1}] \cap Z_{(b+1)}^X [0, Q]$$

belongs to the class $\mathcal{Z}$.

On the other hand in the continuous case (i.e. when the processes U and X are continuous) in order to make Γ satisfy the condition C4 it suffices to assume that N belongs to the class $\mathcal{Z}$. In fact, we have in this case $IZ_s = Z_s^-$, so that the process

$$\sum_{b < a} (N_{Q_b} - IZ_{(b)Q_b})^X [Q_b, Q] = \sum_{b < a} N_{Q_b} [Q_b, Q_{b+1}]$$

belongs to $\mathcal{Z}$. We have used the fact that the stopping times Q_b are the sole moments of discontinuity of a solution when U and X are continuous processes.

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