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A CERTAIN CLASSIFICATION OF THE HEYTING-BROUWER INTERMEDIATE PROPOSITIONAL LOGICS

1. We shall deal with the Heyting-Brouwer propositional calculus (briefly L_{H-B}), defined in [6].

The language of L_{H-B} is an extension of the language of intuitionistic propositional calculus, obtained by adding an unary propositional connective $\neg$ and a binary propositional connective $\dot{-}$. For the technical reasons we assume that the constants $\bar{1}$, $\bar{0}$ are the symbols of this language. The class of all formulas F of L_{H-B} is defined in the usual way. Propositional variables will be denoted by $p, q, r, \dots$ with indices if necessary and formulas by $\alpha, \beta, \gamma, \dots$ with indices if necessary.

The axioms of L_{H-B} are the following formulas:

$$\begin{aligned}
 &(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r)), \\
 &p \Rightarrow (p \vee q), \\
 &q \Rightarrow (p \vee q), \\
 &(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow ((p \vee q) \Rightarrow r)), \\
 &(p \wedge q) \Rightarrow p, \\
 &(p \wedge q) \Rightarrow q, \\
 &(r \Rightarrow p) \Rightarrow ((r \Rightarrow q) \Rightarrow (r \Rightarrow (p \wedge q))), \\
 &(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \wedge q) \Rightarrow r), \\
 &((p \wedge q) \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r)), \\
 &(p \Rightarrow q) \Rightarrow (\neg q \Rightarrow \neg p), \\
 &p \Rightarrow (q \vee (p \dot{-} q)), \\
 &(p \dot{-} q) \Rightarrow \neg(p \Rightarrow q),
 \end{aligned}$$

$$\begin{aligned}
&((p \dot{-} q) \dot{-} r) \implies (p \dot{-} (q \vee r)), \\
&\neg(p \dot{-} q) \implies (p \implies q), \\
&(p \implies \bar{0}) \implies \neg p, \\
&\neg p \implies (p \implies \bar{0}), \\
&(\bar{1} \dot{-} p) \implies \neg p, \\
&\neg p \implies (\bar{1} \dot{-} p), \\
&p \implies \bar{1}, \\
&\bar{0} \implies p.
\end{aligned}$$

The consequence operation (of L_{H-B}) is the mapping C of 2^F to 2^F such that, for any $F' \subseteq F$, $C(F')$ is the smallest set containing F' , the set of all axioms, and it is closed under the rules of inference (of L_{H-B}): modus ponens, substitution and the (r) -rule $(r) \frac{\alpha}{\neg r \alpha}$, where α is a formula.

By L_{H-B} we understand $C(\emptyset)$. Generally, we shall identify any calculus with the set of all formulas provable in it.

For every $L \subseteq F$ we say that L is the Heyting-Brouwer intermediate propositional calculus if $L = C(L) \subseteq C(\{p \vee \neg p\})$. The set of all Heyting-Brouwer intermediate propositional calculi will be denoted by LP_{H-B} .

For every $\alpha \in F$ and $L \in LP_{H-B}$, if $\alpha \in L$, then we say that α is the theorem of L and write $\vdash_L \alpha$.

For formulas $\alpha_1, \alpha_2, \dots, \alpha_n$, by $L_{H-B} + \alpha_1 + \alpha_2 + \dots + \alpha_n$ we understand $C(\{\alpha_1, \alpha_2, \dots, \alpha_n\})$.

As algebraical models for $L \in LP_{H-B}$ serve semi-Boolean algebras (briefly s-Ba) defined in [6]. Namely, an s-BA is any abstract algebra $\mathcal{B} = (B, \vee, \wedge, \implies, \dot{-}, \neg, \neg, \neg, V, \Lambda)$ such that $(B, \vee, \wedge, \implies, \neg, V, \Lambda)$ is a pseudo-Boolean algebra with the greatest element V and the least element Λ , the binary operation $\dot{-}$ satisfies the condition:

$a \dot{-} b \leq c$ if and only if $a \leq b \vee c$, where a, b, c are any elements of B , and the unary operation $\neg$ is defined by $\neg a = V \dot{-} a$.

As usual, by a $\mathcal{B}$ -valuation we mean every mapping $\bar{v}$ of the set of all propositional variables to B . As it is well-known, every mapping $\bar{v}$ can be extended to a homomorphism $\bar{v}$ of the s-BA of formulas to $\mathcal{B}$.

H-B theory of $\mathcal{B}$ (briefly $\text{Th}_{\text{H-B}}(\mathcal{B})$) is the set of all formulas α such that $v(\alpha) = \check{V}$ for any $\mathcal{B}$ -valuation v . A formula α is an H-B propositional tautology if for every s-BA $\mathcal{B}$ $\alpha \in \text{Th}_{\text{H-B}}(\mathcal{B})$.

The purpose of this paper is to present a classification of $\text{LP}_{\text{H-B}}$ similar to the classification of intermediate propositional calculi given by Hosoi. In the analogous way as in [5] our classification is obtained from the properties of linear s-BA $\mathcal{A}$. The methods used here are similar to those of [1], [2], [4].

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2. Definition 1. Let $(A, \leq)$ be a linearly ordered infinite set with the greatest element V and the least element Λ . $(A, \leq)$ determines a linear infinite s-BA (briefly l.i.s-BA) $\mathcal{A} = (A, \vee, \wedge, \Rightarrow, \div, \neg, \neg\neg, V, \Lambda)$, where for every $a, b \in A$

$$\begin{aligned} a \vee b &= \max \{a, b\}, \\ a \Rightarrow b &= \begin{cases} V & \text{if } a \leq b, \\ b & \text{otherwise,} \end{cases} \\ \neg a &= \begin{cases} V & \text{if } a = \Lambda, \\ \Lambda & \text{otherwise,} \end{cases} \\ a \wedge b &= \min \{a, b\}, \\ a \div b &= \begin{cases} \Lambda & \text{if } a \leq b, \\ a & \text{otherwise,} \end{cases} \\ \neg\neg a &= \begin{cases} \Lambda & \text{if } a = V, \\ V & \text{otherwise.} \end{cases} \end{aligned}$$

In [1] Dummett showed that the intuitionistic theory of any linear infinite pseudo-Boolean algebra is axiomatized by adding $(p \Rightarrow q) \vee (q \Rightarrow p)$ to the axioms of the intuitionistic propositional calculus.

Let $\mathcal{A}$ be a l.i.s-BA. The purpose of this section is a proof of

Theorem 1. $Th_{H-B}(\mathcal{A}) = L_{H-B} + \neg(p \Rightarrow q) \vee \neg(q \Rightarrow p)$.

For the sake of brevity we define new propositional connectives $\rightarrow, \leftrightarrow, \Rightarrow$:

$$\alpha \rightarrow \beta = \neg(\alpha \Rightarrow \beta),$$

$$\alpha \leftrightarrow \beta = (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha),$$

$$\alpha \Rightarrow \beta = (\alpha \rightarrow \beta) \wedge ((\beta \Rightarrow \alpha) \rightarrow \alpha) \wedge (\beta \rightarrow (\beta \div \alpha))$$

and we denote by $LIN_{\omega} L_{H-B} + (p \rightarrow q) \vee (q \rightarrow p)$.

It is easy to observe that in every l.i.s-BA $\mathcal{A}$, for any $a, b \in A$, we have

$$(1) \quad a \rightarrow b = V \quad \text{if and only if } a \leq b,$$

$$(2) \quad a \leftrightarrow b = V \quad \text{if and only if } a = b,$$

$$(3) \quad a \Rightarrow b = V \quad \text{if and only if } a < b,$$

$$(4) \quad \neg a = V \quad \text{if and only if } a = \Lambda,$$

$$(5) \quad \neg \neg a = V \quad \text{if and only if } a \neq \Lambda,$$

$$(6) \quad \neg a = V \quad \text{if and only if } a \neq V,$$

$$(7) \quad \neg \neg a = V \quad \text{if and only if } a = V.$$

Let us notice that we are able to express the linear ordering in $\mathcal{A}$ using $\rightarrow, \leftrightarrow, \Rightarrow$. Moreover, we are able to express the least (the greatest) element of $\mathcal{A}$ using only the connective $\neg(r)$.

Definition 2. Let (P, φ) be a linearly ordered set of n distinct propositional variables ($n \geq 1$). Let us arrange the elements of P in the sequence $p_1, p_2, \dots, p_n$ determined by φ . Then we say that

(i) A formula R is an order formula on (P, φ) provided that if $n = 1$ then $R = p_1 \leftrightarrow p_1$, and if $n > 1$ then $R = \bigwedge_{i=1}^{n-1} R_i$, where R_i is one of the form: $p_i \leftrightarrow p_{i+1}, p_i \Rightarrow p_{i+1}$.

(ii) A formula E is an evaluation formula on (P, φ) provided that E is one of the form: $\neg p_1 \wedge R \wedge \neg p_n, \neg p_1 \wedge R \wedge \neg p_n, \neg p_1 \wedge R \wedge \neg p_n, \neg p_1 \wedge R \wedge \neg p_n$, where R and $R_{\supset}$ are any order formulas on (P, φ) and there exists $i = 1, 2, \dots, n-1$ such that $p_i \Rightarrow p_{i+1}$ is the subformula of $R_{\supset}$.

Lemma 1. (Rauszer [6]). Let $\mathcal{B} = (B, \vee, \wedge, \Rightarrow, \div, \neg, \neg \neg, V, \Lambda)$, be a semi-Boolean algebra. For every $a, b, c \in B$, we have

- (1) $a \vee \neg a = V$,
- (2) $a \div \neg a = a$,
- (3) $\neg a \leq a$,
- (4) $\neg a \leq \neg a, \neg \neg a \leq \neg a$,
- (5) if $a \leq b$ then $\neg b \leq \neg a$,
- (6) if $a \leq b$ then $a \div c \leq b \div c$,
- (7) $(a \div c) \div (a \div b) \leq b \div c$, $\neg a \div \neg b \leq b \div a$,
- (8) $a \div b \leq \neg(a \Rightarrow b)$,
- (9) $\neg(a \div b) \leq a \Rightarrow b$,
- (10) $\neg a \vee \neg b = \neg(a \wedge b)$,
- (11) $\neg \neg a \wedge \neg \neg b = \neg \neg(a \wedge b)$,
- (12) $\neg \neg(a \Rightarrow b) \leq \neg \neg a \Rightarrow \neg \neg b$.

L e m m a 2. In every s-BA $\mathcal{B} = (B, \vee, \wedge, \Rightarrow, \div, \neg, \neg, \neg, \vee, \wedge)$, for any $a, b, c, d \in B$, we have

- (1) $\neg a \Rightarrow a \leq \neg a$,
- (2) $a \Rightarrow \neg a \leq \neg \neg a \leq \neg a$,
- (3) $\neg a \Rightarrow a \leq \neg \neg a \leq \neg \neg a$,
- (4) $a \Rightarrow \neg a \leq \neg \neg \neg a \leq \neg a$,
- (5) $(a \Rightarrow b) \wedge (b \Rightarrow c) \leq a \Rightarrow c$,
- (6) $((b \Rightarrow a) \Rightarrow a) \wedge (b \Rightarrow c) \leq (c \Rightarrow a) \Rightarrow a$,
- (7) $(a \Rightarrow b) \wedge ((c \Rightarrow b) \Rightarrow b) \leq ((c \Rightarrow a) \Rightarrow a)$,
- (8) $\neg(a \Rightarrow b) \wedge (c \Rightarrow (c \div b)) \leq c \Rightarrow (c \div a)$,
- (9) $(b \Rightarrow (b \div a)) \wedge (c \Rightarrow b) \wedge \neg(b \Rightarrow c) \leq c \Rightarrow (c \div a)$,
- (10) $(a \Rightarrow b) \wedge \neg a \leq \neg b$,
- (11) $(a \Rightarrow b) \wedge \neg \neg a \leq \neg \neg b$,
- (12) $(a \Rightarrow b) \wedge \neg b \leq \neg a$,
- (13) $(a \Rightarrow b) \wedge \neg \neg b \leq \neg \neg a$,
- (14) $a \Rightarrow b \leq \neg a \wedge \neg b$,
- (15) $a \Rightarrow (a \Rightarrow b) \leq a \Rightarrow b$,
- (16) $(a \Rightarrow b) \Rightarrow a \leq (a \Rightarrow b) \Rightarrow b$,
- (17) $(a \div b) \Rightarrow b \leq a \Rightarrow b$,
- (18) $b \Rightarrow (a \div b) \leq a \Rightarrow (a \div b)$,
- (19) $a \Rightarrow b \leq \neg b \Rightarrow \neg a$,
- (20) $(a \Rightarrow c) \wedge (b \Rightarrow d) \leq (a \div d) \Rightarrow (c \div b)$.

P r o o f . We check only cases (1), (14), (20). The remaining cases are left to the reader.

P r o o f of (1). By Lemma 1 (2), (7) we have that $\neg a \div (\neg a \div a) = (\neg a \div \Lambda) \div (\neg a \div a) \leq a \div \Lambda = a$. Hence and from (8) in Lemma 1 $\neg a = ((\neg a \div a) \vee a) \wedge \neg a \leq \neg a \div a \leq \neg(\neg a \Rightarrow a)$. Thus $\neg a \rightarrow a = \neg\neg(\neg a \Rightarrow a) \leq \neg\neg a$.

P r o o f of (14). By the definition of $\supset$ and (1), (3) and (6) in Lemma 1 $a \supset b \leq (a \rightarrow b) \wedge (b \rightarrow (b \div a)) \leq (a \Rightarrow b) \wedge (b \Rightarrow (b \div a)) \leq a \Rightarrow (b \div a) \leq (a \Rightarrow \neg a) = (a \Rightarrow \neg a) \wedge (a \vee \neg a) \leq \neg a$. On the other hand $(a \supset b) \wedge \neg b \leq (a \rightarrow b) \wedge ((b \Rightarrow a) \Rightarrow a) \wedge \neg b \leq (\neg b \Rightarrow \neg a) \wedge \neg b \wedge ((b \Rightarrow a) \Rightarrow a) \wedge (b \Rightarrow a) \leq \neg a \wedge a = \Lambda$. Thus $a \supset b \leq \neg b \Rightarrow \Lambda = \neg\neg b$, which together with the above inequality gives (14).

P r o o f of (20). Note that the definition of $\div$ and (7), (8) and (10) in Lemma 1 imply $(a \div d) \div (c \div b) \leq (a \div c) \vee (b \div d) \leq \neg(a \Rightarrow c) \vee \neg(b \Rightarrow d) = \neg((a \Rightarrow c) \wedge (b \Rightarrow d))$. Hence and by (9), (11) in Lemma 1 $(a \rightarrow c) \wedge (b \rightarrow d) = \neg\neg((a \Rightarrow c) \wedge (b \Rightarrow d)) \leq \neg((a \div d) \div (c \div b)) \leq (a \div d) \Rightarrow \Rightarrow (c \div b)$.

The following theorem is proved in [6].

T h e o r e m 2. For every formula $\alpha \vdash_{\text{H-B}} \alpha$ if and only if α is an H-B propositional tautology.

In virtue of this theorem and Lemma 2 (1) (18) we obtain

L e m m a 3. For every formulas α, β, γ

- (0) $\frac{}{\text{LIN}_\omega} \neg\neg\alpha \Rightarrow \neg\neg\neg\alpha,$
- (1) $\frac{}{\text{L}_{\text{H-B}}} ((\alpha \leftrightarrow \beta) \wedge (\beta \leftrightarrow \gamma)) \Rightarrow (\alpha \leftrightarrow \gamma),$
 $\frac{}{\text{LIN}_\omega} ((\alpha \supset \beta) \wedge (\beta \leftrightarrow \gamma)) \Rightarrow (\alpha \supset \gamma),$
 $\frac{}{\text{LIN}_\omega} ((\alpha \leftrightarrow \beta) \wedge (\beta \supset \gamma)) \Rightarrow (\alpha \supset \gamma),$
 $\frac{}{\text{LIN}_\omega} ((\alpha \supset \beta) \wedge (\beta \supset \gamma)) \Rightarrow (\alpha \supset \gamma),$
- (2) $\frac{}{\text{L}_{\text{H-B}}} ((\alpha \leftrightarrow \beta) \wedge \neg\alpha) \Rightarrow \neg\beta,$
 $\frac{}{\text{L}_{\text{H-B}}} ((\alpha \leftrightarrow \beta) \wedge \neg\neg\alpha) \Rightarrow \neg\neg\beta,$
 $\frac{}{\text{L}_{\text{H-B}}} ((\alpha \leftrightarrow \beta) \wedge \neg\gamma) \Rightarrow \neg\alpha,$
 $\frac{}{\text{L}_{\text{H-B}}} ((\alpha \leftrightarrow \beta) \wedge \neg\neg\gamma) \Rightarrow \neg\neg\alpha,$
 $\frac{}{\text{L}_{\text{H-B}}} (\alpha \supset \beta) \Rightarrow \neg\alpha \wedge \neg\neg\beta,$

$$(3) \quad \frac{}{\text{LIN}_\omega} (\alpha \supset \beta) \vee (\alpha \multimap \beta) \vee (\beta \supset \alpha).$$

$$(4) \quad \frac{}{\text{LIN}_\omega} ((\neg \neg \alpha \wedge \neg \beta) \vee (\neg \neg \alpha \wedge \neg \neg \beta) \vee (\neg \alpha \wedge \neg \beta) \vee (\neg \alpha \wedge \neg \neg \beta)).$$

P r o o f . We verify only (3). On account of the fact that $\frac{}{\text{LIN}_\omega} (((\alpha \multimap (\alpha \Rightarrow \beta)) \vee ((\alpha \Rightarrow \beta) \multimap \alpha)) \wedge ((\beta \multimap (\alpha \div \beta)) \vee ((\alpha \div \beta) \multimap \beta)))$, and from (15)-(18) in Lemma 2 it follows that $\frac{}{\text{LIN}_\omega} (\alpha \multimap \beta) \vee (((\alpha \Rightarrow \beta) \multimap \beta) \wedge (\alpha \multimap (\alpha \div \beta)))$.

Obviously, $\frac{}{\text{LIN}_\omega} (\beta \multimap \alpha) \vee (((\beta \Rightarrow \alpha) \multimap \alpha) \wedge (\beta \multimap (\beta \div \alpha)))$.

Therefore $\frac{}{\text{LIN}_\omega} (\alpha \supset \beta) \vee (\alpha \multimap \beta) \vee (\beta \supset \alpha) \vee \varphi$, where

$\varphi = ((\beta \Rightarrow \alpha) \multimap \alpha) \wedge (\beta \multimap (\beta \div \alpha)) \wedge ((\alpha \Rightarrow \beta) \multimap \beta) \wedge (\alpha \multimap (\alpha \div \beta))$. Thus, because of $\frac{}{\text{LIN}_\varphi} \varphi \Rightarrow ((\varphi \wedge (\alpha \multimap \beta)) \vee (\varphi \wedge (\beta \multimap \alpha)))$, $\frac{}{\text{L}_{H-B}} (\varphi \wedge (\alpha \multimap \beta)) \Rightarrow (\alpha \supset \beta)$ and $\frac{}{\text{L}_{H-B}} (\varphi \wedge (\beta \multimap \alpha)) \Rightarrow (\beta \supset \alpha)$, we obtain (3).

L e m m a 4. Let (P, φ) be as in Definition 2. Then

(i) If $n > 1$ and R is an order formula on (P, φ) then for every $1 \leq i < j \leq n$ either $R \Rightarrow (p_i \supset p_j)$ or $R \Rightarrow (p_i \multimap p_j)$ is the theorem of LIN_ω .

(ii) If E is an evaluation formula on (P, φ) and $1 \leq i, j \leq n$, then among formulas $E \Rightarrow (\neg \neg p_i \wedge \neg p_j)$, $E \Rightarrow (\neg \neg p_i \wedge \neg \neg p_j)$, $E \Rightarrow (\neg p_i \wedge \neg p_j)$, $E \Rightarrow (\neg p_i \wedge \neg \neg p_j)$ is exactly one theorem of LIN_ω .

P r o o f . The existence of such theorems of LIN_ω follows from (1), (2) in Lemma 3. In order to prove that any two distinct formulas from (i) or (ii) are not simultaneously theorems of LIN_ω , it is sufficient to show that for any l.i.s-BA $\mathcal{A}$ there exists an $\mathcal{A}$ -valuation v such that $v(R) = V$ or $v(E) = V$, respectively. Observe that this property for R has every $\mathcal{A}$ -valuation v such that, for $k = 1, 2, \dots, n-1$,

$v(p_{k+1}) = v(p_k)$ if $p_k \multimap p_{k+1}$ is the subformula of R and $v(p_k) < v(p_{k+1})$, otherwise.

For E , the required property has every $\mathcal{A}$ -valuation v satisfying the analogous condition and, additionally, conditions:

$v(p_1) \neq \Lambda$ if and only if $\neg p_1$ is the subformula of E ,

$v(p_n) = V$ if and only if $\neg p_n$ is the subformula of E .

L e m m a 5. Let (P, φ) be as usual, let R be an order formula on (P, φ) and let p be a propositional variable not in P . Then there exist order formulas $Q_1, Q_2, \dots, Q_m$ on $(P \cup \{p\}, \varphi_1), (P \cup \{p\}, \varphi_2), \dots, (P \cup \{p\}, \varphi_m)$ respectively, where every φ_i is an extension of φ , such that $\overline{\text{LIN}}_{\omega} R \Rightarrow \Rightarrow \bigvee_{i=1}^m Q_i$.

P r o o f . Proof is by induction on n . For $n = 1$ it follows from (3) in Lemma 3. If $R = R' \wedge R_{n-1}$ is an order formula on (P, φ) , then, by the induction hypothesis,

$\overline{\text{LIN}}_{\omega} R' \Rightarrow \bigvee_{j=1}^m Q'_j$, where Q'_j is an order formula on $((P - \{p_n\}) \cup \{p\}, \varphi'_j)$ and $\varphi'_j|_{P - \{p_n\}} = \varphi|_{P - \{p_n\}}$. Now let us note that if T_j is

$$\begin{cases} Q'_j \wedge R_{n-1} & \text{if } p \varphi'_j p_{n-1}, \\ (R \wedge (p_n \supset p) \vee (R \wedge (p_n \leftarrow p)) \vee (Q'_j \wedge (p \supset p_n))) & \text{otherwise,} \end{cases}$$

then it is order formula on $(P \cup \{p\}, \varphi_j)$, where φ_j is the extension of φ , or a disjunction of such formulas. As

$\overline{\text{LIN}}_{\omega} R \wedge Q'_j \Rightarrow T_j$, the proof is completed.

Lemma 5 implies by induction on the number of propositional variables, the following

L e m m a 6. For every non-empty finite set of propositional variables P there exist linear orderings $\varphi_1, \varphi_2, \dots, \varphi_k$ and order formulas $Q_1, Q_2, \dots, Q_k$ on $(P, \varphi_1), (P, \varphi_2), \dots, (P, \varphi_k)$, respectively, such that $\overline{\text{LIN}}_{\omega} \bigvee_{i=1}^k Q_i$.

Note that from the above lemma and (4) in Lemma 3 we get

$$(1) \overline{\text{LIN}}_{\omega} \bigvee_{i=1}^k ((\neg p_1 \wedge Q_i \wedge \neg q_i) \vee (\neg p_1 \wedge Q_i \wedge \neg \neg q_i) \vee$$

$(\neg p_1 \wedge Q_i \wedge \neg \neg \neg q_i) \vee (\neg p_1 \wedge Q_i \wedge \neg q_i))$, where the propositional variable $p_i(q_i)$ occurs in Q_i as the first variable to the left (to the right). Observe that if $\supset$ does not occur in Q_i then $\overline{\text{LIN}}_{\omega} Q_i \Rightarrow (p_i \leftarrow q_i)$. Thus, by (4) in Lemma 1, Theorem 2 and (2) in Lemma 3 we obtain

$$(2) \frac{}{L_{H-B}} (\neg p_1 \wedge q_1 \wedge \neg \neg q_1) \Rightarrow (\neg q_1 \wedge \neg \neg q_1).$$

By (1) and (2) we obtain

Lemma 7. For every non-empty finite set P of propositional variables there exist linear orderings $\varphi_1, \varphi_2, \dots, \varphi_l$ on P and evaluation formulas $E_1, E_2, \dots, E_l$ on $(P, \varphi_1), (P, \varphi_2), \dots, (P, \varphi_l)$, respectively, such that $\frac{}{LIN_\omega} \bigvee_{i=1}^l E_i$.

Lemma 8. If $\beta_1, \beta_2, \dots, \beta_n$ can be simultaneously replaced in any formula, then for arbitrary formulas $\alpha, \delta_1, \delta_2, \dots, \delta_n$ we have $\frac{}{LIN_\omega} (\bigwedge_{i=1}^n (\beta_i \leftrightarrow \delta_i)) \Rightarrow \Rightarrow (\alpha \leftrightarrow \alpha(\beta_1/\delta_1, \beta_2/\delta_2, \dots, \beta_n/\delta_n))$.

Proof. The lemma follows from the fact that, for arbitrary formulas $\alpha, \alpha', \beta, \beta'$, we have

$$(1) \frac{}{L_{H-B}} (\alpha \leftrightarrow \beta) \Rightarrow (\neg \alpha \leftrightarrow \neg \beta)$$

$$\frac{}{L_{H-B}} ((\alpha \leftrightarrow \alpha') \wedge (\beta \leftrightarrow \beta')) \Rightarrow ((\alpha \circ \beta) \leftrightarrow (\alpha' \circ \beta')),$$

where $\circ$ is one of the $\vee, \wedge, \Rightarrow$.

$$(2) \frac{}{LIN_\omega} (\alpha \leftrightarrow \beta) \Rightarrow (\neg \alpha \leftrightarrow \neg \beta)$$

$$\frac{}{LIN_\omega} ((\alpha \leftrightarrow \alpha') \wedge (\beta \leftrightarrow \beta')) \Rightarrow ((\alpha \div \beta) \leftrightarrow (\alpha' \div \beta')).$$

Let us note that (2) is obtained from (19)-(20) in Lemma 2, Theorem 2 and (0) in Lemma 3.

For every formula α let V_α denote the set of all propositional variables appearing in α . Now, for every formula α , every linearly ordered non-empty finite set (P, φ) of propositional variables such that $V_\alpha \subseteq P$ and every evaluation formula E on (P, φ) we want define a formula α_E such that $(*) \alpha_E \in V_\alpha \cup \{\bar{0}, \bar{1}\}$ and $(**) \frac{}{LIN_\omega} E \Rightarrow (\alpha \leftrightarrow \alpha_E)$.

For this purpose let (P, φ) be as usual, E be an evaluation formula on (P, φ) and for a formula β let $V_\beta \subseteq P$. We distinguish some formula β' formed from β by the following rules 1 and 2:

Rule 1. From a formula β we obtain a new formula β' by the following replacements:

$$\beta_1 = \beta(p_1 \Rightarrow \bar{0}/\neg p_1, \dots, p_n \Rightarrow \bar{0}/\neg p_n, \bar{1} \div p_1/\neg p_1, \dots, \bar{1} \div p_n/\neg p_n),$$

$$\beta_2 = \beta_1(\varphi_1/\bar{0}, \dots, \varphi_k/\bar{0}, \psi_1/\bar{1}, \dots, \psi_l/\bar{1}),$$

where

$$\{\varphi_1, \dots, \varphi_k\} = \{\varphi \in V_\beta : \overline{\text{LIN}_\omega} E \Rightarrow \neg \varphi\},$$

$$\{\psi_1, \dots, \psi_l\} = \{\psi \in V_\beta : \overline{\text{LIN}_\omega} E \Rightarrow \neg \neg \psi\},$$

$$\beta_3 = \beta_2(\neg \varphi_1/\bar{0}, \dots, \neg \varphi_k/\bar{0}, \neg \psi_1/\bar{1}, \dots, \neg \psi_l/\bar{1}),$$

where

$$\{\varphi_1, \dots, \varphi_k\} = \{\varphi \in V_\beta : \overline{\text{LIN}_\omega} E \Rightarrow \neg \neg \varphi\},$$

$$\{\psi_1, \dots, \psi_l\} = \{\psi \in V_\beta : \overline{\text{LIN}_\omega} E \Rightarrow \neg \psi\},$$

$$\beta' = \beta_3(\neg \bar{0}/\bar{1}, \neg \bar{1}/\bar{0}, \neg \bar{1}/\bar{0}, \neg \bar{1}/\bar{0}).$$

Rule 2. For a formula β , we obtain a new formula β' by the following simultaneous replacements in β :
for each $\varphi \in V_\beta \cup \{\bar{0}, \bar{1}\}$, we replace:

$$\varphi \wedge \bar{1}, \bar{1} \wedge \varphi, \bar{0} \vee \varphi, \varphi \vee \bar{0}, \bar{1} \Rightarrow \varphi, \varphi \div \bar{0} \text{ by } \varphi,$$

$$\varphi \vee \bar{1}, \bar{1} \vee \varphi, \varphi \Rightarrow \bar{1}, \bar{0} \Rightarrow \varphi \text{ by } \bar{1},$$

$$\varphi \wedge \bar{0}, \bar{0} \wedge \varphi, \bar{0} \div \varphi, \varphi \div \bar{1}, \text{ by } \bar{0},$$

for each pair $p_i, p_j \in V_\beta$ such that $1 \leq i \leq j \leq n$, we replace

$p_i \wedge p_j$ by p_i and $p_i \vee p_j$ by p_j ,

for each pair $p, q \in V_\beta$ such that $\overline{\text{LIN}_\omega} E \Rightarrow (p \leftrightarrow q)$, we replace $p \Rightarrow q$ by $\bar{1}$ and $p \div q$ by $\bar{0}$,

for each pair $p_i, p_j \in V_\beta$ such that $1 \leq i < j \leq n$ and $\overline{\text{LIN}_\omega} E \Rightarrow (p_i \supset p_j)$, we replace:

$$p_i \Rightarrow p_j \text{ by } \bar{1},$$

$$p_j \Rightarrow p_i \text{ by } p_i,$$

$$p_i \div p_j \text{ by } \bar{0},$$

$$p_j \div p_i \text{ by } p_j.$$

L e m m a 9. Let β' be obtained from β with the rules 1 and 2. Then

$$(i) \quad V_{\beta} \subseteq V_{\beta'},$$

$$(ii) \quad \overline{\text{LIN}_{\omega}} E \Rightarrow (\beta \leftrightarrow \beta').$$

P r o o f. Observe that, by (5), (8), (11), (12) in Lemma 1, (1)-(4) in Lemma 2, (0) in Lemma 3, Theorem 2 and the (r)-rule,

$$\overline{\text{LIN}_{\omega}} \vdash \alpha \Rightarrow (\alpha \leftrightarrow \bar{0}),$$

$$\overline{\text{LIN}_{\omega}} \vdash \alpha \Rightarrow (\neg \alpha \leftrightarrow \bar{1}),$$

$$\overline{\text{LIN}_{\omega}} \vdash \neg \alpha \Rightarrow (\alpha \leftrightarrow \bar{1}),$$

$$\overline{\text{L}_{H-B}} (\alpha \rightarrow \beta) \Rightarrow (((\alpha \wedge \beta) \leftrightarrow \alpha) \wedge ((\alpha \vee \beta) \leftrightarrow \beta) \wedge ((\alpha \Rightarrow \beta) \leftrightarrow \bar{1})),$$

$$\overline{\text{L}_{H-B}} (\alpha \supset \beta) \Rightarrow (((\beta \Rightarrow \alpha) \leftrightarrow \alpha) \wedge ((\beta \div \alpha) \leftrightarrow \beta)),$$

$$\overline{\text{LIN}_{\omega}} (\alpha \rightarrow \beta) \Rightarrow ((\alpha \div \beta) \leftrightarrow \bar{0}).$$

The proof of (ii) follows from these facts and Lemma 8 by the induction on the number of logical connectives in β .

Now we can give the definition of α_E . For this purpose we construct a sequence of formulas $\alpha_0, \alpha_1, \dots$, in the following inductive manner:

$$\alpha_0 = \alpha,$$

α_{2n} is obtained from α_{2n} by using Rule 2,

α_{2n+1} is obtained from α_{2n} by using Rule 1.

It is easily seen that, if $\alpha_{2n} \notin V_{\alpha} \cup \{\bar{0}, \bar{1}\}$, then there is less logical connectives in α_{2n+2} than in α_{2n} . Thus, there exists the smallest $k \geq 0$ such that $\alpha_{2k} \in V_{\alpha} \cup \{\bar{0}, \bar{1}\}$. Let us note that $\alpha_n = \alpha_{2k+1}$ for every $n \geq 2k+1$. We define: $\alpha_E = \alpha_{2k+1}$.

Then, in virtue of Lemma 9, α_E satisfies the required conditions (*) and (**).

Let E be an evaluation formula on (F, φ) , $\alpha \in \text{Th}_{H-B}(\mathcal{A})$ be such that $V_{\alpha} \subseteq F$ and let v be the $\mathcal{A}$ -valuation from the proof of Lemma 4. On account of (*), (**) and $\text{LIN}_{\omega} \subseteq \text{Th}_{H-B}(\mathcal{A})$ we have

$$v(E \Rightarrow (\alpha \leftrightarrow \alpha_E)) = V,$$

$$v(\alpha \leftrightarrow \alpha_E) = V,$$

$$v(\alpha_E) = V,$$

$$\alpha_E \in V_\alpha \cup \{\bar{1}\}.$$

It follows from the definition of v that if $\alpha_E \in V_\alpha$ then $\vdash_{LIN} E \Rightarrow \neg \neg \alpha$, a contradiction to the definition of α_E .

Thus $\alpha_E = \bar{1}$, and we proved

L e m m a 10. For every evaluation formula E on (P, φ) , every l.i.s-BA $\mathcal{A}$ and every $\alpha \in Th_{H-B}(\mathcal{A})$ such that $V_\alpha \subseteq P$, we have $\alpha_E = \bar{1}$. This along with Lemma 7 gives $Th_{H-B}(\mathcal{A}) \subseteq LIN_\omega$, which yields us the proof of Theorem 1.

Let us note that LIN_ω may be axiomatized alternatively by adding the formulas $\neg(p \vee q) \Rightarrow \neg p \vee \neg q$ and $(p \Rightarrow q) \vee (q \Rightarrow p)$ to L_{H-B} . The first formula will be denoted by H .

3. For $n = 1, 2, \dots$ let $A_n = \{1, 2, \dots, n-1\} \cup \{\wedge, \vee\}$ and let $\leq$ be a linear ordering on A_n such that $\wedge \leq 1 \leq 2 \leq \dots \leq n-1 \leq \vee$. Then $(A_n, \leq)$ determines the linear s-BA

$$\mathcal{A}_n = (A_n, \vee, \wedge, \Rightarrow, \div, \neg, \neg, \neg, \neg, \neg, \neg).$$

Hosoi proved in [2] that the intuitionistic theory of the $(n+1)$ elemented linear pseudo-Boolean algebra can be axiomatized by adding $(p \Rightarrow q) \vee (q \Rightarrow p)$, X_{n+1} , Y_n to the axioms of the intuitionistic propositional calculus, where

$$X_n = \bigvee_{1 \leq i < j \leq n+1} (p_i \leftrightarrow p_j), \quad T_n = \bigvee_{i=1}^{n+1} p_i, \quad F_n = \bigvee_{i=1}^{n+1} \neg p_i,$$

$$Y_n = X_n \vee (T_n \wedge F_n).$$

Let us note that the formula $X_{n+1}(p_{n+2}/\bar{0}) \wedge X_{n+1}(p_{n+2}/\bar{1}) \Rightarrow Y_n$ is the theorem of the intuitionistic propositional calculus, hence Y_n may be omitted in the above axiomatization.

Now we prove

Theorem 3. For $n = 1, 2, \dots$ $\text{Th}_{H-B}(\mathcal{A}_n) = \text{LIN}_\omega + X_{n+1}$.

Proof. Let n be fixed. To prove the theorem it is sufficient to show that $\text{Th}_{H-B}(\mathcal{A}_n) \subseteq \text{LIN} + X'_{n+1}$, where

$$X'_{n+1} = \bigvee_{1 \leq i < j \leq n+2} (p_i \rightarrow p_j).$$

Proof. For propositional variables $q_1, q_2, \dots, q_k$, we let $X'_{n+1}\{q_1, q_2, \dots, q_k\} = \bigwedge_{\theta \in \Delta} X'_{n+1}(p_1/\theta(p_1), p_2/\theta(p_2), \dots, p_{n+2}/\theta(p_{n+2}))$ where Δ is the set of all mappings from $\{p_1, p_2, \dots, p_{n+2}\}$ to $\{q_1, q_2, \dots, q_k\} \cup \{\bar{0}, \bar{1}\}$. It is sufficient to prove that, if a formula $\alpha \in \text{Th}_{H-B}(\mathcal{A}_n)$ and $V_\alpha \subseteq \{q_1, q_2, \dots, q_k\}$, then for every l.i.s-BA $\mathcal{A}$ and every $\mathcal{A}$ -valuation v we have

$$v(X'_{n+1}\{q_1, q_2, \dots, q_k\} \Rightarrow \alpha) = V.$$

Let us note that it easily follows from the following implications (1)-(4):

(1) If v satisfies one of the conditions:

- a) $\text{Card}\{v(q_1), v(q_2), \dots, v(q_k)\} \leq n-1$,
- b) $\text{Card}\{v(q_1), v(q_2), \dots, v(q_k)\} = n$ and $\{\Lambda, V\} \cap \{v(q_1), v(q_2), \dots, v(q_k)\} \neq \emptyset$,
- c) $\text{Card}\{v(q_1), v(q_2), \dots, v(q_k)\} = n+1$ and $\{\Lambda, V\} \subseteq \{v(q_1), v(q_2), \dots, v(q_k)\}$, then $v(\alpha) = V$.

(2) If $\text{card}\{v(q_1), v(q_2), \dots, v(q_k)\} = \text{card}\{v(q_{i_1}), v(q_{i_2}), \dots, v(q_{i_n})\} = n$ ($1 \leq i_1, \dots, i_n \leq k$) and $\{\Lambda, V\} \cap \{v(q_1), v(q_2), \dots, v(q_k)\} = \emptyset$, then $v(X'_{n+1}(p_1/q_{i_1}, p_2/q_{i_2}, \dots, p_n/q_{i_n}, p_{n+1}/\bar{1}, p_{n+2}/\bar{0})) = \Lambda$.

(3) If $\text{card}\{v(q_1), v(q_2), \dots, v(q_k)\} = \text{card}\{v(q_{i_1}), v(q_{i_2}), \dots, v(q_{i_{n+1}})\} = n+1$ ($1 \leq i_1, \dots, i_{n+1} \leq k$) and $\{\Lambda, V\} \cap \{v(q_1), v(q_2), \dots, v(q_k)\} \neq \{\Lambda, V\}$, then $v(X'_{n+1}(p_1/q_{i_1}, p_2/q_{i_2}, \dots, p_{n+1}/q_{i_{n+1}}, p_{n+2}/\bar{0})) \wedge v(X'_{n+1}(p_1/q_{i_1}, p_2/q_{i_2}, \dots, p_{n+1}/q_{i_{n+1}}, p_{n+2}/\bar{1})) = \Lambda$.

(4) If $\text{card} \{v(q_1), v(q_2), \dots, v(q_k)\} \geq \text{card} \{v(q_{i_1}), v(q_{i_2}), \dots, v(q_{i_{n+2}})\} = n+2$ ($1 \leq i_1, \dots, i_{n+2} \leq k$), then $v(X'_{n+1}(p_1/q_{i_1}, p_2/q_{i_2}, \dots, p_{n+2}/q_{i_{n+2}})) = \Lambda$.

In [2] and [3] it was proved that, for $n = 1, 2, \dots$, the theory of pseudo-Boolean algebra $\mathcal{A}_n = (\mathcal{A}_n, \vee, \wedge, \Rightarrow, \neg, \vee, \wedge)$ can be alternatively axiomatised by adding $S_n = p_1 \vee (p_1 \Rightarrow p_2) \vee (p_2 \Rightarrow p_3) \vee \dots \vee (p_{n-1} \Rightarrow p_n) \vee \neg p_n$ or $(p \Rightarrow q) \vee (q \Rightarrow p)$ and P_n to the axioms of the intuitionistic propositional calculus, where P_n is defined inductively as follow:

$$P_1 = ((p_1 \Rightarrow p) \Rightarrow p_1) \Rightarrow p_1,$$

$$P_{n+1} = ((p_{n+1} \Rightarrow P_n) \Rightarrow p_{n+1}) \Rightarrow p_{n+1}.$$

Thus, as we can easily observe, $\text{LIN}_n = L_{H-B} + S'_n = \text{LIN}_\omega + P_n$, where $S'_n = \neg p_1 \vee (p_1 \Rightarrow p_2) \vee (p_2 \Rightarrow p_3) \vee \dots \vee (p_{n-1} \Rightarrow p_n) \vee \neg p_n$ and $\text{LIN}_n = \text{LIN}_\omega + X_{n+1}$.

Finally, let us note that $L_{H-B} + H \neq \text{LIN}_\omega \neq L_{H-B} + (p \Rightarrow q) \vee (q \Rightarrow p)$ and, for $n = 2, 3, \dots$, $L_{H-B} + S_n \neq \text{LIN}_n$. Indeed, $H \in \text{Th}_{H-B}(\mathcal{A})$, $(p \Rightarrow q) \vee (q \Rightarrow p) \notin \text{Th}_{H-B}(\mathcal{A})$ and, for $n = 2, 3, \dots$, S_n , we have $(p \Rightarrow q) \vee (q \Rightarrow p) \in \text{Th}_{H-B}(\mathcal{A}_n)$, $H \notin \text{Th}_{H-B}(\mathcal{A}_n)$, where s-BA's $\mathcal{A}$, $\mathcal{A}_n$ ($n \geq 2$) are given by the Figures 1 and 2, respectively.

Fig.1

Fig.2

4. We can prove as in [4]

Theorem 4. For $n = 1, 2, \dots$ and every formula α , $\text{LIN}_\omega + \alpha = \text{LIN}_n$ if and only if $\vdash_{\text{LIN}_n} \alpha$ and non $\vdash_{\text{LIN}_{n+1}} \alpha$.

Proof. Let us suppose that $\vdash_{\text{LIN}_n} \alpha$ and v is $\mathcal{A}_{n+1}$ -valuation such that $v(\alpha) \neq V$. Now, if $q_1, q_2, \dots, q_k$ are all propositional variables in α then let us take for α' the formula $\alpha(q_1/\varphi_1, q_2/\varphi_2, \dots, q_k/\varphi_k)$ where for $i=1, 2, \dots, k$

$$\varphi_i = \begin{cases} p_{n-j+1} & \text{if } v(q_i) = j \in \{1, 2, \dots, n\}, \\ \bar{0} & \text{if } v(q_i) = \Lambda, \\ \bar{1} & \text{if } v(q_i) = V. \end{cases}$$

Note that for every $\mathcal{A}_{n+1}$ -valuation v' such that $v'(S'_n) \neq V$ we have $v'(\neg \alpha') = \Lambda$. Thus $\vdash_{\text{LIN}_{n+1}} \neg \alpha' \Rightarrow S'_n$. But if there occur at most n , propositional variables in a formula β then $\vdash_{\text{LIN}_{n+1}} \beta$ is equivalent to $\vdash_{\text{LIN}_\omega} \beta$. Thus $\vdash_{\text{LIN}_\omega} \neg \alpha' \Rightarrow S'_n$. Hence, from the (r)-rule and the axiomatization of LIN_n it follows that $\text{LIN}_n \subseteq \text{LIN}_\omega + \alpha$.

Theorem 4 implies the classification of LP_{H-B} . Namely

Corollary. If $L \in \text{LP}_{H-B}$ then there exists exactly one natural number n such that $\text{LIN}_n = L + (p \rightarrow q) \vee (q \rightarrow p)$.

REFERENCES

- [1] M. Dummett: A propositional calculus with denumerable matrix, J. Symb. Logic 24 (1959) 97-106.
- [2] T. Hosoi: The axiomatization of the intermediate propositional systems S_n of Godel, J. Fac. Sci., Univ. Tokyo, Sec.I, 13 (1966) 183-187.
- [3] T. Hosoi: The separable axiomatization of the intermediate propositional systems S_n of Godel, Proc. Japan. Acad., 42 (1966) 1001-1006.

- [4] T. H o s o i : A criterion for the separable axiomatization of Godel's S_n , Proc. Japan. Acad., 43 (1967) 365-368.
- [5] T. H o s o i : On intermediate logics I, J. Fac. Sci., Univ. Tokyo, Sec.I, 14 (1967) 293-312.
- [6] C. R a u s z e r : Semi-Boolean algebras and their applications to intuitionistic logic with dual operations, Fund. Math., 83 (1974) 219-249.

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