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GENERALIZED EPIMORPHISM, MONOMORPHISM AND BIMORPHISM PROPERTIES OF CATEGORIES

0. Introduction

In this paper we generalize the notion of epimorphisms, monomorphisms and bimorphism of categories in terms of ideals and study their properties. We finally examine our theory looking at semigroups.

Let $\mathcal{C}$ be a category. For two morphisms $\alpha: A \rightarrow B$ and $\beta: B \rightarrow C$ in $\mathcal{C}$, we denote their composition by $\alpha\beta: A \rightarrow C$.

Following Hilton-Ledermann [1], for any morphism μ in $\mathcal{C}$, we shall denote by $\mathcal{L}(\mu)$ the set $\mathcal{L}(\mu) = \{\alpha\mu \mid \alpha \in \mathcal{C}\}$ whenever $\alpha\mu$ is defined and call $\mathcal{L}(\mu)$ the left ideal generated by μ .

Similarly, $\mathcal{R}(\mu) = \{\mu\alpha \mid \alpha \in \mathcal{C}\}$ will denote the right ideal generated by μ and $\mathcal{M}(\mu) = \{\beta\mu\alpha \mid \alpha, \beta \in \mathcal{C}\}$ the two-sided ideal generated by μ .

We shall say that the category $\mathcal{C}$ has the generalized epimorphism property, generalized monomorphism property and generalized bimorphism property if, respectively:

$$(GE) \quad \mathcal{L}(\mu) = \mathcal{L}(\nu) \Rightarrow \mu = \nu$$

$$(GM) \quad \mathcal{R}(\mu) = \mathcal{R}(\nu) \Rightarrow \mu = \nu$$

and

$$(GBM) \quad \mathcal{M}(\mu) = \mathcal{M}(\nu) \Rightarrow \mu = \nu$$

holds, for every $\mu, \nu \in \mathcal{C}$, lifting the usual notions of epimorphisms, monomorphisms and bimorphisms from the pointwise case.

1. Main results

We proceed to examine when a category $\mathcal{C}$ has these properties. We shall usually look for one property, leaving the formulation of the mirror image to our reader.

Proposition 1.1. A category $\mathcal{C}$ satisfies (GE) if and only if the following implication holds

(*) whenever $\alpha\beta$ and $\beta\mu$ are defined and $\alpha\beta\mu = \mu$, then $\mu = \beta\mu$.

Proof. Suppose $\mathcal{C}$ has (GE) and $\mu = \alpha\beta\mu$, then $\mu \in \mathcal{C}(\beta\mu)$ that is $\mathcal{C}(\mu) \subseteq \mathcal{C}(\beta\mu)$. Again, $\beta\mu \in \mathcal{C}(\mu)$. Therefore $\mathcal{C}(\beta\mu) \subseteq \mathcal{C}(\mu)$, showing $\mathcal{C}(\mu) = \mathcal{C}(\beta\mu)$ which, by (GE) gives $\mu = \beta\mu$.

Next suppose $\mathcal{C}$ does not have property (GE), though it satisfies (*). Then there exist μ, ν with $\mathcal{C}(\mu) = \mathcal{C}(\nu)$ and $\mu \neq \nu$. Evidently, $\mu \in \mathcal{C}(\nu)$ and $\nu \in \mathcal{C}(\mu)$ and so $\mu = \gamma\nu$ and $\nu = \delta\mu$. Thus, $\mu = \gamma\delta\mu$ which by (*) implies $\mu = \delta\mu = \nu$, a contradiction to our assumption.

Corollary 1.2. If the category $\mathcal{C}$ has property (GE), the only invertible morphisms of $\mathcal{C}$, are the identity morphisms.

Example 1.3. A category having (GE).

Let S be a nilpotent semigroup, then for $a, b \in S$, there exist a positive integer n , such that $(H) \dots (ab)^n = b(ab)^n$ holds (take for example the least positive integer n , for which $(ab)^n = 0$).

Adjoin an identity 1 to S to get $\mathcal{C} = S \cup \{1\}$. Here $x.1 = x$ for $x \in S$ and $1.1 = 1$. Now $\mathcal{C}$ still has property (H).

In a semigroup (H) implies (*); for, if $x = abx$, then $x = (ab)^2x = \dots = (ab)^nx = b(ab)^nx = bx$.

Thus our category $\mathcal{C}$ has property (GE).

An application of our Proposition 1.1 will show that if $\mathcal{C}$ has property (GE) and the monoid $\text{Hom}(B, B)$ for an object B is periodic, then $\text{Hom}(B, B)$ is left weakly commutative.

This means, for $\alpha, \beta \in \text{Hom}(B, B)$, that there exists a $\gamma \in \text{Hom}(B, B)$ and a positive integer k , such that

$$(\alpha\beta)^k = \beta\gamma \quad [\text{cf. [2]}].$$

Specializing our results to categories with single objects, we have

Theorem 1.4. If S is a monoid with (GE), then the following conditions are equivalent:

1. S is left regular,
2. S is regular,
3. S is idempotent.

Proof. $1 \Rightarrow 3$. If S is left regular, then for $x \in S$, we have an $a \in S$, with $x = axx$ i.e. $x = ax^2$ which by (GE) gives $x = x^2$.

$2 \Rightarrow 3$. If S is regular, then for $x \in S$, there exist an $a \in S$ with $x = xax$; by (GE) we have $x = ax$ and therefore $x = x^2$.

$3 \Rightarrow 1$ and $3 \Rightarrow 2$ follow easily, using the identity element of the monoid to play the required role.

Next we investigate when (GBM) implies (GE) and (GM).

Theorem 1.5. A category $\mathcal{C}$ has property (GBM) if and only if it has property (GE), property (GM) and the following implication holds

(**) $\mu = \beta\mu\alpha$ implies $\mu = \beta\mu$ and $\mu = \mu\alpha$ for any $\mu \in \mathcal{C}$, $\beta \in \text{Hom}(D, D)$ for D , the domain of μ , and $\alpha \in \text{Hom}(C, C)$ for C , the codomain of μ .

Proof. Suppose $\mathcal{C}$ has property (GBM). If $\mathcal{L}(\mu) = \mathcal{L}(\nu)$ for $\mu, \nu \in \mathcal{C}$, then $\mu \in \mathcal{L}(\mu) = \mathcal{L}(\nu) \subseteq \mathcal{M}(\nu)$ and $\nu \in \mathcal{L}(\nu) = \mathcal{L}(\mu) \subseteq \mathcal{M}(\mu)$. Thus, $\mathcal{M}(\mu) = \mathcal{M}(\nu)$ and this implies $\mu = \nu$ by (GBM). Thus (GE) holds. Similarly (GM) and (**) hold.

Next suppose $\mathcal{C}$ has properties (GE), (GM) and let the implication (**) hold in $\mathcal{C}$.

If $\mathcal{C}$ does not have property (GBM), then there exist μ, ν such that $\mathcal{M}(\mu) = \mathcal{M}(\nu)$ with $\mu \neq \nu$. From this it follows that $\mathcal{L}(\mu) \neq \mathcal{L}(\nu)$ and $\mathcal{R}(\mu) \neq \mathcal{R}(\nu)$. Since $\mu \in \mathcal{M}(\nu)$, we have $\mu = \beta\nu\alpha$ for some properly chosen $\beta, \alpha \in \mathcal{C}$. Similarly, $\nu \in \mathcal{M}(\mu)$ implies $\nu = k\mu\theta$. Thus $\mu = \beta\nu\alpha = \beta k\mu\theta\alpha$. By (**) we have $\mu = \beta k\mu$ and $\mu = \mu\theta\alpha$.

It follows now from Proposition 1.1 and its dual, that $\mu = k\mu$ and $\mu = \mu\theta$ leading finally to $\mu = k\mu = k\mu\theta = \nu$ a contradiction. So $\mathcal{C}$ has property (GBM).

Now we wish to know whether the condition (GBM) is equivalent to (GE) and (GM). By Theorem 1.5 (GBM) always implies (GE) and (GM), though in general the reverse implication does not hold. However, trivially, (GBM) is equivalent to (GE) and (GM) for generalized posets, i.e., in categories whose objects are posets under $A \leq B$ if and only if $\text{Hom}(A, B) \neq \emptyset$ and which also satisfy $\text{Hom}(A, A) = \{\text{id}_A\}$ for every A .

The above observations enable us to deduce as a corollary some simple results of semigroup theory [2], [4] trivially from the

Theorem 1.6. If $\mathcal{C}$ is a category such that $\text{Hom}(A, A)$ is a periodic monoid for each object A , then $\mathcal{C}$ satisfies (GBM) precisely when (GE) and (GM) hold.

Proof. If $\text{Hom}(A, A)$ is periodic, for any $\beta \in \text{Hom}(A, A)$ $\beta^n = 1_A$ for some positive integer n ; this implies $\beta = 1_A$ by (GE), so the endomorphism hom sets will consist of identity alone and the category will then be a generalized poset.

We now look at an example to see that (GE) and (GM) do not always imply (GBM). One can check this by looking at the category $\mathcal{C}$, as constructed below.

Let $\mathcal{C}$ be the category of the edge-path semigroupoid of the indicated directed graph, subject to the relation $\beta\mu\alpha = \mu$, where $\alpha = 1_A$ and $\beta = 1_B$

The different morphisms here are

$$1_B, \beta, \beta^2, \dots, \beta^n, \dots$$

$$1_A, \alpha, \alpha^2, \dots, \alpha^n, \dots$$

$$\mu, \mu\alpha, \mu\alpha^2, \dots, \mu\alpha^n, \dots$$

$$\beta\mu, \beta^2\mu, \dots, \beta^n\mu, \dots$$

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