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CONTINUOUS SOLUTIONS OF THE FUNCTIONAL EQUATION

$$f(xf^k(y) + yf^l(x)) = f(x)f(y)$$

1. Introduction

We shall consider the functional equation

$$(1) \quad f(xf^k(y) + yf^l(x)) = f(x)f(y),$$

where k and l belong to the set of non-negative integers ($f^k(x) := (f(x))^k$) and f is the unknown function defined in the set R of all real numbers and assuming values from the set R_0 of all real numbers different from zero. In this paper we will find the general solution of (1) (for $k \neq l$) in the class of continuous functions.

The equation (1) is generalization of the equation

$$f(xf(y) + yf^k(x)) = f(x)f(y),$$

which appeared in work [4] (p.18) at determination of certain subsemigroups of the differential group L_2^1 , and of the equation

$$(2) \quad f(x + yf(x)) = f(x)f(y)$$

which S. Gołąb and A. Schinzel obtained in the paper [2] while searching for three-parameter subgroups of centre-affinity transformations group of a plane.

In particular the paper [3] contains the construction of the general solution of the equation (2) in the class of functions $f: E \rightarrow \Phi$, where E is a vector space over a field Φ .

The research on the equation (2), was also done in papers [1], [3], [8].

2. Results

In this section k and l will signify positive integers. Some property of solutions of the equation (1) may be found in [5], [6], [7]. In [7] we proved the following theorem.

Theorem 1. The equation (1) in the class of continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}_0$ and such that there exists $x_0 \neq 0$ for which $f(x_0) = 1$, has the unique solution $f(x) = 1$.

From the above theorem and from the fact that $f(0) = 1$ implies what follows.

Corollary. Let f be a continuous solution of (1). Now, the following cases are possible:

- I. $f(x) > 1$, for $x < 0$,
 $0 < f(x) < 1$, for $x > 0$,
 $f(0) = 1$;
- II. $0 < f(x) < 1$, for $x < 0$,
 $f(x) > 1$, for $x > 0$,
 $f(0) = 1$;
- III. $f(x) > 1$, for $x \neq 0$,
 $f(0) = 1$;
- IV. $0 < f(x) < 1$, for $x \neq 0$,
 $f(0) = 1$;
- V. $f(x) = 1$, for all $x \in \mathbb{R}$.

Let us suppose that the continuous function f satisfies the equation (1). From the form of the equation (1) it follows that: if $f((-\infty, 0)) \subset (1, \infty)$ or $f((0, \infty)) \subset (1, \infty)$, then for every $x, y \in (-\infty, 0)$ or respectively $x, y \in (0, \infty)$

$$(3) \quad f(xf^k(y) + yf^l(x)) > f(x),$$

but, if $f((-\infty, 0)) \subset (0, 1)$ or $f((0, \infty)) \subset (0, 1)$, then for every $x, y \in (-\infty, 0)$ or $x, y \in (0, \infty)$

$$(4) \quad f(xf^k(y) + yf^l(x)) < f(x).$$

For $x > 0$ we have

$$\lim_{y \rightarrow \infty} (xf^k(y) + yf^l(x)) = \infty,$$

however, for $x < 0$, we obtain

$$\lim_{y \rightarrow -\infty} (xf^k(y) + yf^l(x)) = -\infty.$$

From continuity of the function f we also have

$$(5) \quad \lim_{y \rightarrow 0} (xf^k(y) + yf^l(x)) = xf^k(0) = x.$$

From the last three relations and from Darboux's property for the function $y \rightarrow xf^k(y) + yf^l(x)$, we get

$$(6) \quad \forall x > 0 \quad \forall y > x \quad \exists y_1 > 0 \quad y = xf^k(y_1) + y_1f^l(x)$$

and

$$(7) \quad \forall x < 0 \quad \forall y < x \quad \exists y_2 < 0 \quad y = xf^k(y_2) + y_2f^l(x).$$

(3) and (6) imply that if $f((0, \infty)) \subset (1, \infty)$, then f is strictly increasing function in interval $(0, \infty)$. (3) and (7) imply that if $f((-\infty, 0)) \subset (1, \infty)$, then f is strictly decreasing function in interval $(-\infty, 0)$. (4) and (7) imply that if $f((-\infty, 0)) \subset (0, 1)$, then f is strictly increasing in interval $(-\infty, 0)$. At last (4) and (6) imply that if $f((0, \infty)) \subset (0, 1)$, then f is strictly decreasing function in interval $(0, \infty)$.

Hence, the function f is invertible in each of these sets: $(-\infty, 0)$ and $(0, \infty)$.

Let us assume that $f((0, \infty)) \subset (1, \infty)$; then $f_1 := f|_{(0, \infty)}$. For the function f_1 , the equation (1) has a form

$$f_1(xf_1^k(y) + yf_1^l(x)) = f_1(x)f_1(y).$$

Letting $a := f_1(x)$, $b := f_1(y)$, from the invertibility of f_1 we obtain

$$(8) \quad f_1^{-1}(a)b^k + f_1^{-1}(b)a^1 = f_1^{-1}(ab).$$

From the right-hand side symmetry of (8) we have

$$f_1^{-1}(a)b^k + f_1^{-1}(b)a^1 = f_1^{-1}(b)a^k + f_1^{-1}(a)b^1.$$

If we fix $b \neq 1$, then under assumption $k \neq 1$ we get

$$f_1^{-1}(a) = \frac{f_1^{-1}(b)}{b^k - b^1} (a^k - a^1).$$

Hence

$$(9) \quad f_1^{-1}(a) = p(a^k - a^1),$$

where $a \in (1, \infty)$ and if $k > 1$, then p is a certain positive number, however if $k < 1$, then p is a certain negative number. It is easy to verify that functions in the form (9) for arbitrary $p > 0$ or $p < 0$ satisfy the equation (8).

Alike at assumption $f((-\infty, 0)) \subset (1, \infty)$ for the function $f_2 := f|_{(-\infty, 0)}$, we obtain

$$(10) \quad f_2^{-1}(a) = q(a^k - a^1),$$

where $a \in (1, \infty)$ and if $k > 1$, then q is a certain positive number, but if $k < 1$, then q is a certain negative number.

Likewise, if $f((-\infty, 0)) \subset (0, 1)$ for $f_3 := f|_{(-\infty, 0)}$, we get

$$(11) \quad f_3^{-1}(a) = r(a^k - a^1),$$

where $a \in (0, 1)$ and if $k > 1$, then r is a certain positive number, but if $k < 1$, then r is a certain negative number. However, from (11) it follows that f_3^{-1} is not injective into interval $(0, 1)$ and so, since f_3^{-1} has to be the inverse function of f_3 , we have contradiction.

Also, if $f((0, \infty)) \subset (0, 1)$, then in the case of function $f_4 := f|_{(0, \infty)}$ we obtain a similar contradiction.

From the above consideration the lemma follows.

L e m m a 1. If f is a continuous solution of (1) for $k \neq 1$, then none of cases I, II and IV may exist.

Thus it is sufficient to consider the case III. At first we shall prove the following lemma.

L e m m a 2. If f is a continuous solution of the equation (1) for $k \neq 1$ and satisfies the case III, p being constant in (9), then for every $x \in (-|p|, 0)$ there exists $z \in (x, 0)$ such that $f(z) > f(x)$.

P r o o f . Let $x \in (-|p|, 0)$. We will calculate

$$\lim_{y \rightarrow \infty} (xf^k(y) + yf^1(x)).$$

For this purpose we consider the quotient $\frac{f^k(y)}{y}$. From the form of the equation (1) and from the fact that f_1 is strictly increasing it follows that $\lim_{y \rightarrow \infty} f(y) = \infty$. Moreover, because the equation (9) is true, hence for $y > 0$ the function f is differentiable at a point y . From de l'Hospital's theorem and from of inverse function theorem for the function (9), we have

$$\begin{aligned} \lim_{y \rightarrow \infty} \frac{f^k(y)}{y} &= \lim_{y \rightarrow \infty} kf^{k-1}(y)f'(y) = \lim_{y \rightarrow \infty} \frac{kf^{k-1}(y)}{p(kf^{k-1}(y) - lf^{1-k}(y))} = \\ &= \lim_{y \rightarrow \infty} \frac{1}{p - \frac{p1}{k} f^{1-k}(y)} = \begin{cases} 0, & \text{for } k < 1, \\ \frac{1}{p}, & \text{for } k > 1. \end{cases} \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{y \rightarrow \infty} (xf^k(y) + yf^1(x)) &= \lim_{y \rightarrow \infty} y \left(x \frac{f^k(y)}{y} + f^1(x) \right) = \\ &= \begin{cases} \lim_{y \rightarrow \infty} yf^1(x), & \text{for } k < 1, \\ \lim_{y \rightarrow \infty} y \left(\frac{x}{p} + f^1(x) \right), & \text{for } k > 1. \end{cases} \end{aligned}$$

Thus

$$\lim_{y \rightarrow \infty} (x f^k(y) + y f^1(x)) = \infty, \text{ because } f^1(x) > 1 \text{ and } \frac{x}{p} \in (-1, 0).$$

From (5) and the above equation it follows that there exists $\bar{y} > 0$ such that $x < x f^k(\bar{y}) + \bar{y} f^1(x) < 0$.

Taking $z := x f^k(\bar{y}) + \bar{y} f^1(x)$ from (3) we obtain $f(z) > f(x)$, where $x < z < 0$, which was to be proved.

Lemma 2 contradicts the fact that arbitrary continuous solution of the equation (1) satisfying the case III is strictly decreasing in interval $(-\infty, 0)$. Thus we have the following lemma.

L e m m a 3. For $k \neq 1$ the equation (1) has not continuous solutions satisfying the case III.

From Corollary, Lemma 1 and Lemma 3 we obtain the theorem.

T h e o r e m 2. The equation (1) for $k \neq 1$, in the class of continuous functions $f: R \rightarrow R_0$, has the unique solution $f(x) \equiv 1$.

3. Remarks

a) When $k = 0$ and $l = 1$, from (1) we obtain (2). Its solution in the class of continuous functions $f: R \rightarrow R$ was published in [2]. Among the continuous functions satisfying the equation (2) only $f(x) \equiv 1$ assumes values from R_0 . Let us notice that restriction of the range of the function f is natural (see [2] or [9]).

b) From the theorem 2 it follows that in the theorem 2 from [7] the assumption " $\bar{\alpha}_2 \neq 0$ and $\bar{\alpha}_3 = 0$ exist such that $F(\bar{\alpha}_2, 0) = F(0, \bar{\alpha}_3) = 1$ " may be omitted.

c) Knowledge of solutions of the equation (1) is important at determining subsemigroups of the group L_n^1 ($n > 2$) such that first parameter depends on remaining $n-1$ parameters (see [7]).

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