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PRIME IDEAL STRUCTURE IN ADDITIVE CONSERVATIVE SYSTEMS

In this paper we prove that the set of prime ideals in an additive conservative system is a spectral space and that every morphism of additive conservative systems induces a spectral map.

Throughout this paper all rings are commutative with identity and all ring homomorphisms preserve the identity.

An additive conservative system $([4], [7])$ is a pair $(R, \mathcal{M})$, where R is a ring and $\mathcal{M}$ is a set of ideals of R satisfying the following conditions:

A1. The intersection of any set of elements of $\mathcal{M}$ is an element of $\mathcal{M}$.

A2. The union of any nonempty set, totally ordered by inclusion, of elements of $\mathcal{M}$ is an element of $\mathcal{M}$.

A3. The sum of any two ideals of $\mathcal{M}$ is an element of $\mathcal{M}$.

Let $(R, \mathcal{M})$ be an additive system. The elements of $\mathcal{M}$ are called $\mathcal{M}$ -ideals. By A1 applied to the empty set of elements of $\mathcal{M}$, R is an $\mathcal{M}$ -ideal. If E is a subset of R , then by $[E]$ we denote the smallest $\mathcal{M}$ -ideal containing E . It is clear, that if A and B are ideals in R then $[A+B] = [A] + [B]$.

If $(R, \mathcal{M})$ and $(S, \mathcal{N})$ are additive conservative systems then a ring homomorphism $f: R \rightarrow S$ will be called a morphism of additive conservative systems $(R, \mathcal{M})$ and $(S, \mathcal{N})$ iff the inverse images of $\mathcal{N}$ -ideals are $\mathcal{M}$ -ideals.

L e m m a 1. Let $(R, \mathcal{M})$ be an additive conservative system and let E be a subset of R . If $x \in [E]$, then there exists a finite subset $T \subset E$ such that $x \in [T]$.

P r o o f . See [7] or [4].

L e m m a 2. Let $(R, \mathcal{M})$ be an additive conservative system and let A be an ideal of R . Then A is an $\mathcal{M}$ -ideal if and only if for every $x \in R$ the condition $x \in A$ implies $[x] \subset A$.

P r o o f . The necessity of the condition is obvious. To prove sufficiency we show that $[A] \subset A$. Let $x \in [A]$, then, by Lemma 1, we have $x \in [a_1, \dots, a_n]$, for some $a_1, \dots, a_n \in A$. Since $[a_1, \dots, a_n] = [a_1] + \dots + [a_n]$ and $[a_1] \subset A$, for $i = 1, \dots, n$, thus $x \in A$.

For any ring R , $\text{Spec}(R)$ will denote the set of prime ideals in R with the Zariski topology ([1] p.125). If E is a subset of R , then by $V(E)$ we denote the set of prime ideals of R containing E and by $D(E)$ we denote the set $\text{Spec}(R) \setminus V(E)$.

Let $(R, \mathcal{M})$ be an additive conservative system. The set of prime $\mathcal{M}$ -ideals in R will be denoted by $\text{Spec}(R, \mathcal{M})$ and will be called the prime spectrum of $(R, \mathcal{M})$. As a topological space, it has the subspace topology from $\text{Spec}(R)$, so that the closed sets in $\text{Spec}(R, \mathcal{M})$ are of the form $V(E) \cap \text{Spec}(R, \mathcal{M})$, where E is a subset of R .

L e m m a 3. If $(R, \mathcal{M})$ is an additive conservative system, then

$$\text{Spec}(R, \mathcal{M}) = \bigcap \{D(r) \cup V([r]), r \in R\}.$$

P r o o f . Let $P \in \text{Spec}(R, \mathcal{M})$, $r \in R$. If $r \notin P$ then $P \in D(r)$. If $r \in P$, then $[r] \subset P$. Therefore, for every $r \in R$, we have $\text{Spec}(R, \mathcal{M}) \subset D(r) \cup V([r])$. Conversely, assume that $P \in D(r) \cup V([r])$, for any $r \in R$. We prove that $P \in \mathcal{M}$. Let $r \in P$. Then $P \notin D(r)$ and thus $P \in V([r])$. Therefore $[r] \subset P$ and, by Lemma 2, we have $P \in \mathcal{M}$.

Let $f: (R, \mathcal{M}) \rightarrow (S, \mathcal{N})$ be a morphism of additive conservative systems. Then f induces a continuous mapping $a_f: \text{Spec}(S) \rightarrow \text{Spec}(R)$ via $a_f(P) = f^{-1}(P)$, where $P \in \text{Spec}(S)$. Since $a_f(\text{Spec}(S, \mathcal{N})) \subset \text{Spec}(R, \mathcal{M})$, we have a continuous mapping

$$b_f: \text{Spec}(S, \mathcal{N}) \rightarrow \text{Spec}(R, \mathcal{M}),$$

given by $b_f(P) = a_f(P)$, for any $P \in \text{Spec}(S, \mathcal{N})$.

Recall from ([5]) that a topological space X is called spectral if it is T_0 and quasi-compact, the quasi-compact open subsets are closed under finite intersection and form an open basis, and every nonempty irreducible closed subset has a generic point. Moreover, a continuous map of spectral spaces is called spectral if the inverse image of quasi-compact open sets are quasi-compact. It is well known, that for every ring R , $\text{Spec}(R)$ is a spectral space, and, for every ring homomorphism f , a_f is a spectral map.

L e m m a 4. Let Y be a subspace of a spectral space X . Every quasi-compact open set in Y is of the form $U \cap Y$, where U is a quasi-compact open set in X .

P r o o f . Let V be a open set in X such that $V \cap Y$ is a quasi-compact set in Y , and let $V = \bigcup U_i$, where U_i are quasi-compact open sets in X . Since $V \cap Y$ is a quasi-compact set, we have $V \cap Y = (U_{i_1} \cup \dots \cup U_{i_n}) \cap Y$. The set $U_{i_1} \cup \dots \cup U_{i_n}$ is quasi-compact open in X , because, for every topological space, the quasi-compact open subsets are closed under finite union.

T h e o r e m . If $(R, \mathcal{M})$ is an additive conservative system then $\text{Spec}(R, \mathcal{M})$ is a spectral space. If $f : (R, \mathcal{M}) \longrightarrow (S, \mathcal{N})$ is a morphism of additive conservative systems then b_f is a spectral map.

P r o o f . Lemma 3 implies that $\text{Spec}(R, \mathcal{M})$ is a closed set in the patch topology on $\text{Spec}(R)$ (see [5] p.45). It follows that $\text{Spec}(R, \mathcal{M})$ is a spectral subobject of $\text{Spec}(R)$ (see [5] p.45), and therefore $\text{Spec}(R, \mathcal{M})$ is a spectral space.

Let $f : (R, \mathcal{M}) \longrightarrow (S, \mathcal{N})$ be a morphism of additive conservative systems and let G be a quasi-compact open set in $\text{Spec}(S, \mathcal{N})$. By Lemma 4, there exists a quasi-compact open set U in $\text{Spec}(S)$, such that $G = U \cap \text{Spec}(S, \mathcal{N})$. Since $b_f^{-1}(G) = (a_f^{-1})(U) \cap \text{Spec}(R, \mathcal{M})$ and $a_f^{-1}(U)$ is a quasi-compact open set in $\text{Spec}(R)$, and since $\text{Spec}(R, \mathcal{M})$ is a spectral subobject of $\text{Spec}(R)$, then $b_f^{-1}(G)$ is a quasi-compact open set in $\text{Spec}(R, \mathcal{M})$.

C o r o l l a r y . Let $f : (R, \mathcal{M}) \longrightarrow (S, \mathcal{N})$ be a morphism of additive conservative systems. Then there exist rings R' and S' and a ring homomorphism $f' : R' \longrightarrow S'$ such that $\text{Spec}(R, \mathcal{M})$ is homeomorphic to $\text{Spec}(R')$, $\text{Spec}(S, \mathcal{N})$ is homeomorphic to $\text{Spec}(S')$, and b_f is equivalent to $a_{f'}$ i.e. $b_f = g a_{f'} p$, where $p : \text{Spec}(S, \mathcal{N}) \longrightarrow \text{Spec}(S')$, $q : \text{Spec}(R') \longrightarrow \text{Spec}(R, \mathcal{M})$ are homeomorphisms.

P r o o f . This follows from Theorem and from [5] Th.6. p.51.

E x a m p l e s : 1. Let R be a ring, F a set of endomorphisms of the additive group of R , and let $\mathcal{M}$ be the set of ideals A in R such that $f(A) \subset A$ for any $f \in F$. Then $(R, \mathcal{M})$ is an additive conservative system.

2. Let D be a set of derivations of R and let $\mathcal{M}$ be the set of differential ideals of R ([7]). Then $(R, \mathcal{M})$ is an additive conservative system. Every differential homomorphism is a morphism of additive conservative systems. The theorem, for special differential rings, was proved by W.F. Keigher in [6], and, for arbitrary differential rings, was proved by G. Carra in [3].

3. Let D be a set of higher derivations of R and let $\mathcal{M}$ be a set of D -ideals ([2], [4]). Then $(R, \mathcal{M})$ is an additive conservative system and every D -homomorphism is a morphism of additive conservative systems.

4. Let R be a graded ring and let $\mathcal{M}$ be the set of homogeneous ideals of R . Then $(R, \mathcal{M})$ is an additive conservative system and every homomorphism of graded rings is a morphism of additive conservative systems.

R e m a r k s 1. Lemma 1 is also true without the assumption that $\mathcal{M}$ satisfies A3 (see [7], [4]).

2. If $\mathcal{M}$ is a set of ideals of R satisfying only A1 and A2 then Lemma 2 is not necessarily true e.g. Let $R = \mathbb{Z}_2[X, Y]/A$, where A is an ideal in $\mathbb{Z}_2[X, Y]$ generated by X^2, XY, Y^2 . Let $x = X + A$, $y = Y + A$ and $\mathcal{M} = \{0, (x), (y), (x+y), R\}$. Then the ideal (x, y) satisfies the condition of Lemma 2, and it is not $\mathcal{M}$ -ideal.

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