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A NOTE ON THE LEAST UPPER BOUND OPERATION IN THE PARTIALLY ORDERED SET

1. Introductory remarks

Let P be a set which is partially ordered by a relation $\leq$. We denote the least upper bound of the subset $D \subseteq P$ by $\cup D$. D is cofinal with P if

$$(\forall x \in P)(\exists y \in D) x \leq y.$$

Let $|X|$ be the cardinality of the set X . $\text{cf}(P)$ is the smallest element of the set $\{|D| : D \subseteq P \text{ \& } D \text{ is cofinal with } P\}$. Let $\underline{m}$ be an infinite cardinal. The subset D of P is $\underline{m}$ -directed if every subset $X \subseteq D$ of the cardinality less than $\underline{m}$ has an upper bound in D . An $\aleph_0$ -directed set is called directed. A chain is a linearly ordered set.

$\underline{m}^+$ is a cardinal successor of the cardinal $\underline{m}$ i.e. the smallest cardinal greater than $\underline{m}$. We identify ordinal with the set of all its predecessors and cardinals with the initial ordinals. A cardinal $\underline{m}$ is regular if $\text{cf}(\underline{m}) = \underline{m}$ where $\text{cf}(\underline{m})$ concerns an natural order ϵ on the ordinal $\underline{m}$. The first infinite ordinal i.e. the set of all integers we design by ω as usual. The following fact is well known:

if C is a chain, then $\text{cf}(C)$ is a regular cardinal and there is a subchain C_1 such that

1. C_1 is well ordered and its ordinal type is $\text{cf}(C)$
2. C_1 is cofinal with C .

For an infinite cardinal $\underline{m}$ and partially ordered set P let $\underline{C_m}$ and $\underline{D_m}$ be the following statements:

$\underline{C_m}$: for each chain $C \subseteq P$ such that $\text{cf}(C) \geq \underline{m}$ there exists $\bigcup C$ in P

$\underline{D_m}$: for each $\underline{m}$ -directed subset D of P there exists $\bigcup D$ in P .

An equivalence $\underline{D_{\aleph_0}} \equiv \underline{C_{\aleph_0}}$ is well known. It implies the question on the relation between $\underline{D_m}$ and $\underline{C_m}$ for uncountable $\underline{m}$. Trivially $\underline{D_m} \Rightarrow \underline{C_m}$. Investigations of the problem $\underline{C_m} \Rightarrow \underline{D_m}$ for various cardinals $\underline{m}$ and various partially ordered sets P is the aim of this note.

Observe finally that if $\underline{m}$ is a singular cardinal (i.e. a cardinal which is not regular) then $\underline{C_m} \equiv \underline{C_m}^+$ and $\underline{D_m} \equiv \underline{D_m}^+$.

2. List of the results

2.1. **Theorem 1.** For each uncountable cardinal $\underline{m}$ there exists a set P , a partial ordering $\leq$ of P and a subset D of P such that

1. P satisfies $\underline{C_m}$,
2. D is $\underline{m}^+$ -directed,
3. D has no upper bound in P .

(in particular: $\underline{C_m} \Rightarrow \underline{D_m}$ is not satisfied in P).

2.2. Let $n(P)$ be the smallest cardinal $\underline{n}$ such that $\text{cf}(C) < \underline{n}$ for every chain $C \subseteq P$.

Theorem 2. 1. If $\underline{m} > n(P)$ then P satisfies $\underline{D_m}$ (moreover every $\underline{m}$ -directed subset D of P has the greatest element). 2. If $\underline{m} < n(P)$ and there are only finitely many cardinals between $\underline{m}$ and $n(P)$ then $\underline{C_m} \Rightarrow \underline{D_m}$ is satisfied in P .

2.3. A partially ordered set P is called $\underline{m}$ -semilattice if for every nonempty subset X of P of cardinality less than $\underline{m}$ there exists $\bigcup X$ in P .

Theorem 3. If P is an $\underline{m}$ -semilattice satisfying the condition $\underline{C_m}$, then P is a complete semilattice (i.e. every subset of P has the least upper bound in P , in particular: for each cardinal $\underline{n}$, $\underline{D_n}$ is satisfied in P).

3. Proofs of the results

3.1. **P r o o f** of Theorem 1. Let $\underline{m} = \aleph_r$ for some ordinal r and let P be a set $\{f: \text{dom}(f) = \omega \text{ \& } (\forall n \in \omega) f(n) \leq \aleph_{r+n+1} \text{ \& } (\exists n \in \omega) f(n) < \aleph_{r+n+1}\}$ $f \leq g$ if $(\forall n \in \omega) f(n) \leq g(n)$.
 $D = \{f \in P: (\forall n \in \omega) f(n) < \aleph_{r+n+1}\}$.

3.1.1. D is an $\underline{m}^+$ -directed set.

P r o o f. Let $X \subseteq D$ be a set of cardinality less than $\underline{m}^+$. Then for each $n \in \omega$

$$|\{f(n) : f \in X\}| \leq |X| \leq \underline{m} < \aleph_{r+n+1}.$$

Thus $g(n) = \sup\{f(n): f \in X\} < \aleph_{r+n+1}$ i.e. $g \in D$. Obviously g is an upper bound of X (moreover $g = \bigcup X$).

3.1.2. Trivially, D has no upper bound in P .

3.1.3. **L e m m a**. If $C \subseteq P$ is a chain such that $\text{cf}(C) \geq \aleph_1$ then

$$(\exists n_0 \in \omega)(\forall f \in C) f(n_0) < \aleph_{r+n_0+1}.$$

P r o o f. Suppose on the contrary that $(\forall n \in \omega)(\exists f_n \in C) f_n(n) = \aleph_{r+n+1}$. By the assumption about $\text{cf}(C)$ there exists $f \in C$ such that $(\forall n \in \omega) f_n \leq f$. Thus $f(n) \geq f_n(n) = \aleph_{r+n+1}$ for each $n \in \omega$, which contradicts the fact that $f \in C$.

3.1.4. P satisfies $\underline{C}_m$.

P r o o f. Let $C \subseteq P$ be a chain such that $\text{cf}(C) \geq \underline{m}$ and let C_1 be its well ordered subchain whose ordinal type is $\text{cf}(C)$ and which is cofinal with C . Let $f_0 = \bigcup C_1$ i.e. $f_0(n) = \sup\{f(n): f \in C_1\}$ for every $n \in \omega$. We should prove that $f_0 \in P$ i.e. that $f_0(n) < \aleph_{r+n+1}$ for some n . Let n_0 be an integer from Lemma 3.1.3. We consider two cases:

a. $\text{cf}(C) \leq \aleph_{r+n_0+1}$. Thus

$$|\{f(n_0+1): f \in C_1\}| \leq |C_1| = \text{cf}(C) \leq \aleph_{r+n_0+1} < \aleph_{r+n_0+2}.$$

It implies that $\sup\{f(n_0+1): f \in C_1\} < \aleph_{r+n_0+2}$.

b. $\text{cf}(C) > \aleph_{r+n_0+1}$. Then $\sup\{f(n_0) : f \in C_1\} < \aleph_{r+n_0+1}$.

Indeed, suppose on the contrary that

$$(\forall \alpha < \aleph_{r+n_0+1})(\exists f_\alpha \in C_1) \alpha \leq f_\alpha(n_0).$$

But the set $\{f_\alpha : \alpha < \aleph_{r+n_0+1}\}$ is bounded in C_1 . Let $f \in C_1$ be its upper bound. There holds

$$(\forall \alpha < \aleph_{r+n_0+1}) \alpha \leq f_\alpha(n_0) \leq f(n_0).$$

Thus $f(n_0) = \aleph_{r+n_0+1}$, a contradiction with the property of n_0 .

3.2. P r o o f of the Theorem 2.

3.2.1. Let $\underline{m} \geq n(P)$, let D be an arbitrary $\underline{m}$ -directed subset of P and C - an arbitrary chain in D . Let C_1 be a well ordered subchain of C of the type $\text{cf}(C)$ such that C_1 is cofinal with C . Observe that $|C_1| = \text{cf}(C) < n(P) \leq \underline{m}$. This implies that C_1 (i.e. also C , by cofinality) has an upper bound in D . By the Kuratowski-Zorn lemma D has a maximal which is also the greatest element of D (since D is directed!).

3.2.2. Let $\underline{m}$ be a regular cardinal, Cm is satisfied in P and let D be an $\underline{m}$ -directed subset of P . We design by D^+ the following set

$$U\{UC : C \subseteq D \text{ and } C \text{ is a well ordered chain of type } \underline{m}\}$$

L e m m a . D^+ is $\underline{m}^+$ -directed. If $x = UD^+$ then $x = UD$.

P r o o f . The second part of the lemma follows immediately from the definition of D^+ . Let $X = \{x_\xi : \xi < \underline{m}\}$ be an arbitrary subset of D^+ such that $|X| < \underline{m}^+$. For $\xi < \underline{m}$ let $(x_{\xi\eta} : \eta < \underline{m})$ be an non-decreasing sequence of elements of D which is convergent to x_ξ (if $x_\xi \in D$ let $x_{\xi\eta} = x_\xi$ for $\eta < \underline{m}$). We define, by transfinite induction, an increasing sequence $\{y_\alpha : \alpha < \underline{m}\}$ of elements of D .

y_α = an arbitrary upper bound in D of the set $\{x_\xi: \xi < \alpha & \eta < \alpha\} \cup \{y_\eta: \eta < \alpha\}$. Observe that if the chain $C = (y_\alpha: \alpha < \underline{m})$ stabilizes in ordinal $\alpha_0 < \underline{m}$ then $x_{\alpha_0} = UC$, $x_{\alpha_0} \in D \subseteq D^+$ and x_{α_0} is an upper bound of the set $\{x_\xi: \xi < \underline{m}\}$ as well as of the set $X = \{x_\xi: \xi < \underline{m}\}$. If C is a strictly increasing chain then it is of ordinal type $\underline{m}$, which is the regular cardinal. Thus $\text{cf}(C) = \underline{m}$ and by the condition $C\underline{m}$ there exists $y = UC$ and $y \in D^+$. Obviously y is an upper bound of the set X .

3.2.3. P r o o f of the part 2 of Theorem 2. Let $\underline{m}$ be a regular cardinal and $n(P) = \underline{m}^{+\dots+}$ for some $k \in \omega$. Let D be an $\underline{m}$ -directed subset of P and let $C\underline{m}$ holds in P . Observe that $C\underline{m}$ implies $C\underline{m}^+$, $C\underline{m}^{++}$ etc. Iterating k times the above lemma we obtain that $D\underline{m}^{+\dots+}$ is $n(P)$ -directed. So it has a least upper bound by the part 1 of this Theorem. Thus D has also the least upper bound by the lemma. If $\underline{m}$ is a singular cardinal then $\underline{m}^+$ is a regular one and it satisfies the assumption of the part 2 of Theorem 2. Thus $C\underline{m}^+ \Rightarrow D\underline{m}^+$ holds in P . By the final remark in chapter 1, an implication $C\underline{m} \Rightarrow D\underline{m}$ is also satisfied in P .

3.2.4. Observe that by point 3.1 for every $n \in \omega$ one can easily find in the set P a chain of type $\aleph_{\aleph_n + 1}$. Thus $n(P) \geq \aleph_{\aleph_n + \omega}$. It shows that the assumption of the second part of Theorem 2 is essential.

3.3. P r o o f of Theorem 3. Let $\underline{n}$ be a cardinal such that $\underline{n} \geq \underline{m}$ and such that each subset of P of cardinality less than $\underline{n}$ has the least upper bound in P . Let $X = \{x_\xi: \xi < \underline{n}\}$ be an arbitrary subset of P of cardinality $\underline{n}$. For $\alpha < \underline{n}$ let $y_\alpha = U\{x_\xi: \xi < \alpha\}$ and $C = (y_\alpha: \alpha < \underline{n})$. C is a non-decreasing chain. If $\text{cf}(C) < \underline{m}$ then C has a well ordered sub-chain C_1 of the type less than $\underline{m}$ and C_1 is cofinal with C . C_1 has the least upper bound because P is the $\underline{m}$ -semilattice. Thus C has the least upper bound too. If $\text{cf}(C) \geq \underline{m}$ then UC exists by the condition $C\underline{m}$. Obviously UC is also the least upper bound of the set X . Thus P is $\aleph^+$ -semilattice.

Finally observe that the set P from the point 3.1 is not an $\mathcal{H}_1$ -semilattice.

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