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CLOSE-TO-SPIRALLIKE AND CLOSE-TO-ROBERTSON FUNCTIONS WITH FIXED $(n+1)$ th COEFFICIENT

1. Introduction

Let S denote the class of functions

$$s(z) = z + b_2 z^2 + \dots$$

which are analytic and univalent in the unit disc $E \equiv \{|z| < 1\}$.

A function $s(z)$ is said to be in $S(\lambda, \gamma)$ if

$$(1.1) \quad \operatorname{Re} \left\{ e^{i\lambda} z \frac{s'(z)}{s(z)} \right\} > \gamma \cos \lambda \quad (|\lambda| < \frac{\pi}{2}, 0 \leq \gamma < 1, z \in E).$$

The class $S(\lambda, \gamma)$ of λ -spirallike functions of order γ was introduced by Libera [6]. We note from (1.1) that $S(\lambda, 0) \equiv S(\lambda)$ is the class of so-called spirallike functions defined and shown to be in S by Spac̆ek [13]; and $S(0, 0) \equiv S^*$ is the class of starlike functions in E .

In [3], Chichra introduced an associated class, $M(\lambda, \gamma)$, consisting of those functions $s(z)$ for which $zs'(z)$ is in $S(\lambda, \gamma)$. In view of (1.1), a function $s(z)$ is in $M(\lambda, \gamma)$ if and only if

$$(1.2) \quad \operatorname{Re} \left\{ e^{i\lambda} \left(1 + z \frac{s''(z)}{s'(z)} \right) \right\} > \gamma \cos \lambda \quad (|\lambda| < \frac{\pi}{2}, 0 \leq \gamma < 1, z \in E).$$

The functions of the class $M(\lambda, \gamma)$ are called Robertson functions of order γ . We observe from (1.2) that $M(\lambda, 0) \equiv M(\lambda)$ is

the class of so-called Robertson functions introduced in [10] while $M(0,0) \equiv C$ is the class of convex functions in E . It is well-known [10] that the functions in $M(\lambda)$ need not be univalent in E .

A function $s(z) = z + b_{n+1} z^{n+1} + \dots$, analytic in E , is said to be in $S_n^q(\lambda, \gamma)$ ($|\lambda| < \frac{\pi}{2}$, $|b_{n+1}| = \frac{2q}{n}$, $0 \leq q \leq (1-\gamma)\cos\lambda$) if it satisfies (1.1); and a function $s(z)$ is in $M_n^q(\lambda, \gamma)$ if $zs'(z)$ is in $S_n^q(\lambda, \gamma)$.

In [8], Reade introduced the class of close-to-starlike functions. According to him, a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in E is said to be close-to-star if there exists a function $s(z)$ in S^* such that

$$(1.3) \quad \operatorname{Re} \left\{ \frac{f(z)}{s(z)} \right\} > 0, \quad z \in E.$$

We denote this class by G . Goel [4] generalized the concept of close-to-starlikeness by introducing the class G^λ of close-to-spirallike functions; by replacing $s(z)$ in (1.3) with a spirallike function. Accordingly, a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in E is said to be close-to-spiral-like if there exists a function $s(z)$ in $S(\lambda)$ such that

$$\operatorname{Re} \left\{ \frac{f(z)}{s(z)} \right\} > 0, \quad z \in E.$$

We say that an analytic function $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is said to be in H^λ if there exists a function $h(z)$ in $M(\lambda)$ such that

$$\operatorname{Re} \left\{ \frac{g'(z)}{h'(z)} \right\} > 0, \quad z \in E.$$

The functions of this class we call close-to-Robertson in E . Note that $H^0 \equiv C^*$ is the class of close-to-convex functions introduced by Kaplan [5].

Recently, considerable attention has been paid to the study of various subclasses of close-to-starlike and close-to-convex functions where second coefficient in the power series representation of either one or both the functions under consideration has been kept fixed (see [2], [7], [11] etc.). The aim of this paper is to study certain subclasses of close-to-spirallike and close-to-Robertson functions where we take $(n+1)$ th coefficient fixed in the functions under consideration. Thus we introduce the following classes:

$$D_n(\alpha, \beta) = \left\{ F(z) = 1 + c_n z^n + \dots : |(F(z)-1)/\{2\beta(F(z)-\alpha) - (F(z)-1)\}| < 1, \right. \\ \left. 0 \leq \alpha < 1, 0 < \beta \leq 1, z \in E \right\}$$

$$D_{n,p}(\alpha, \beta) = \left\{ F(z) \in D_n(\alpha, \beta) : F^{(n)}(0) = n!(2\beta(1-\alpha)p), 0 \leq p < 1, z \in E \right\}$$

$$G_{n,p;q}^\lambda(\alpha, \beta, \gamma) = \left\{ f(z) = z + 2(\beta(1-\alpha)p + \frac{q}{n})z^{n+1} + \dots : \frac{f(z)}{s(z)} \in D_{n,p}(\alpha, \beta), \right.$$

$$\left. s(z) \in S_n^q(\lambda, \gamma), 0 \leq p \leq 1, 0 \leq q \leq (1-\gamma)\cos \lambda, z \in E \right\}$$

$$H_{n,p;q}^\lambda(\alpha, \beta, \gamma) = \left\{ g(z) = z + \frac{2}{n+1} (\beta(1-\alpha)p + \frac{q}{n})z^{n+1} + \dots : \frac{g'(z)}{h'(z)} \in D_{n,p}(\alpha, \beta), \right.$$

$$\left. h(z) \in h_n^q(\lambda, \gamma), 0 \leq p \leq 1, 0 \leq q \leq (1-\gamma)\cos \lambda, z \in E \right\}$$

and determine sharp estimates for radii of starlikeness for functions in $G_{n,p;q}^\lambda(\alpha, \beta, \gamma)$ and radii of convexity for functions in $H_{n,p;q}^\lambda(\alpha, \beta, \gamma)$. Note that $G_{n,p;q}^\lambda(0, 1, 0) \equiv G_{n,p;q}^\lambda$ and $H_{n,p;q}^\lambda(0, 1, 0) \equiv H_{n,p;q}^\lambda$ are respectively the classes of close-to-spirallike and close-to-Robertson functions with fixed $(n+1)$ th coefficient.

2. Preliminaries

We need the following lemmas.

L e m m a 1. For all real μ and ν , the following inequality holds

$$(2.1) \quad \frac{(1-\rho^2)\cos\mu + 2\rho\sin\mu\sin\nu}{1 - 2\rho\cos\nu + \rho^2} \geq \frac{(1+\rho^2)\cos\mu - 2\rho}{1 - \rho^2} \quad (0 \leq \rho < 1).$$

The above inequality is due to Robertson [9].

L e m m a 2. Let $w(z) = d_n z^n + d_{n+1} z^{n+1} + \dots$ be analytic in E and $|w(z)| < 1$. Then $|d_n| \leq 1$ and

$$(2.2) \quad |w(z)| \leq r^n \left(\frac{r + |d_n|}{1 + |d_n|r} \right), \quad |z| = r.$$

P r o o f . Let $w_1(z) = \frac{w(z)}{z^n} = d_n + \dots$. Then $w_1(z)$ is analytic and satisfies $|w_1(z)| \leq 1$ with $|d_n| \leq 1$. Define

$$(2.3) \quad w_2(z) = \frac{w_1(z) - d_n}{1 - \bar{d}_n w_1(z)}.$$

Then $w_2(0) = 0$ and $|w_2(z)| \leq 1$ in E . Thus, an application of Schwarz's lemma gives

$$(2.4) \quad |w_2(z)| \leq |z|.$$

By (2.3), we have

$$(2.5) \quad w_1(z) = \frac{w_2(z) + d_n}{1 + \bar{d}_n w_2(z)}.$$

Therefore, by (2.4) and (2.5), we have

$$\left| \frac{w(z)}{z^n} \right| = |w_1(z)| = \left| \frac{w_2(z) + d_n}{1 + \bar{d}_n w_2(z)} \right| \leq \frac{|w_2(z)| + |d_n|}{1 + |d_n||w_2(z)|} \leq \frac{r + |d_n|}{1 + |d_n|r}$$

which gives the required result.

L e m m a 3. Suppose $s(z) = z + b_{n+1} z^{n+1} + \dots$ is in $S_n^q(\lambda, \gamma)$ and let

$$u = qr + (1-\gamma)\cos\lambda, \quad v = (1-\gamma)\cos\lambda \cdot r + q.$$

Then for $|z| = r < 1$ and for all γ, λ, q ($0 \leq \gamma < 1$, $-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$, $0 \leq q \leq (1-\gamma)\cos \lambda$)

$$(2.6) \quad \operatorname{Re} \left(\frac{zs'(z)}{s(z)} \right) \geq \frac{u^2 - 2uv(1-\gamma)\cos \lambda \cdot r^n + (2(1-\gamma)\cos^2 \lambda - 1)v^2 r^{2n}}{u^2 - v^2 r^{2n}}.$$

The result is sharp.

P r o o f . Since $s(z) \in S_n^q(\lambda, \gamma)$, we can write

$$(2.7) \quad \frac{e^{i\lambda}(zs'(z)/s(z)) - i\sin \lambda - \gamma \cos \lambda}{(1-\gamma)\cos \lambda} = \frac{1+w(z)}{1-\overline{w(z)}}$$

where $w(z) = d_n z^n + \dots$ ($|d_n| = \frac{q}{((1-\gamma)\cos \lambda)}$) satisfies the hypothesis of Lemma 2. Therefore

$$(2.8) \quad |w(z)| \leq r^n \left(\frac{r + |d_n|}{1 + |d_n|r} \right) = r^n \frac{v}{u}$$

where u and v are as given in the statement of the lemma. Equation (2.7) may be written as

$$z \frac{s'(z)}{s(z)} = e^{-i\lambda} \left[(1-\gamma)\cos \lambda \left\{ \frac{1+w(z)}{1-\overline{w(z)}} \right\} + \gamma \cos \lambda + i\sin \lambda \right]$$

and

$$(2.9) \quad \operatorname{Re} \left\{ z \frac{s'(z)}{s(z)} \right\} = (1-\gamma)\cos^2 \lambda \operatorname{Re} \left(\frac{1+w(z)}{1-\overline{w(z)}} \right) + \\ + (1-\gamma)\sin \lambda \cos \lambda \operatorname{Im} \left(\frac{1+w(z)}{1-\overline{w(z)}} \right) + \gamma \cos^2 \lambda + \sin^2 \lambda.$$

Setting $w(z) = \rho e^{i\varphi}$ in (2.9) and simplifying, we get

$$(2.10) \quad \operatorname{Re} z \left[\frac{s'(z)}{s(z)} \right] = (1-\gamma)\cos \lambda \left(\frac{(1-\rho^2)\cos \lambda + 2\rho \sin \varphi \sin \lambda}{1 - 2\rho \cos \varphi + \rho^2} \right) + \\ + \gamma \cos^2 \lambda + \sin^2 \lambda.$$

Applying Lemma 1 with $\mu = \lambda$ and $\nu = \varphi$ to the equation (2.10), we obtain

$$(2.11) \quad \operatorname{Re} \left\{ z \frac{s'(z)}{s(z)} \right\} \geq (1-\gamma) \cos \lambda \left(\frac{(1+\rho^2) \cos \lambda - 2\rho}{1-\rho^2} \right) + \gamma \cos^2 \lambda + \sin^2 \lambda.$$

Now using (2.8) in (2.11) and simplifying, we get the result easily.

The bound in (2.6) is sharp for the function $s(z)$ given by

$$z \frac{s'(z)}{s(z)} = \frac{u + (e^{i\theta} - 2(1-\gamma) \cos \lambda \cdot e^{i(\theta-\lambda)})vz}{u + e^{i\theta} vz}$$

where θ satisfies the relation

$$\theta = 2 \arctan \left[\left(\frac{u-vr}{u+vr} \right) \cot \frac{\lambda}{2} \right].$$

3. Radii of starlikeness for $G_{n,p;q}^\lambda(\alpha, \beta, \gamma)$.

Theorem 1. Let $f(z) \in G_{n,p;q}^\lambda(\alpha, \beta, \gamma)$ and let

$$c = (1+pr)^2 + str^{2n}(p+r)^2, \quad d = 2(s-t)(1-\gamma) \cos \lambda, \quad e = u - \cos \lambda \cdot vr^n$$

where $s = 2\beta - 1$, $t = 2\alpha\beta - 1$ and u and v are as in Lemma 3. Then $f(z)$ is starlike in $|z| < r_0$, where r_0 is the smallest positive root of the equation

$$(3.1) \quad (u^2 - v^2 r^{2n})(c(ns+nt-2t)+2pt(2ns-s-t)r^n(1+r^2)+(4nst(1+p^2)+p^2(s^2-t^2-4st)-s^2-3t^2)r^{n+1}) - cdev r^n - dev(s+t)(p+r)(1+pr)r^{2n} = 0$$

if $R_{p,n} \geq R_n^*$, and r_0 is the smallest positive root of the equation

$$(3.2) \quad (u^2 - v^2 r^{2n})(2\sqrt{N} - 2(1-str^{2n}) - (s+t)r^{n-1}(1-r^2)) + (s-t)r^{n-1}(1-r^2)(u^2 - 2uv \cdot (1-\gamma) \cos \lambda \cdot r^n + (2(1-\gamma) \cos^2 \lambda - 1)v^2 r^{2n}) = 0,$$

N being given by

$$N = (1 + nsr^{n-1}(1-r^2) - s^2r^{2n})(1 + ntr^{n-1}(1-r^2) - t^2r^{2n})$$

if $R_{p,n} \leq R_n^*$, where

$$R_{p,n} = \frac{1 + pr + ptr^n + tr^{n+1}}{1 + pr + psr^n + sr^{n+1}}, \quad R_n^* = \left(\frac{1 + ntr^{n-1}(1-r^2) - t^2r^{2n}}{1 + nsr^{n-1}(1-r^2) - s^2r^{2n}} \right)^{\frac{1}{2}}$$

and $r = |z| < 1$. The estimates are sharp for all admissible values of $\alpha, \beta, \gamma, p, q$ and n .

P r o o f . Since $f(z) \in G_{n,p;q}(\alpha, \beta, \gamma)$, there exists a function $s(z) \in S_n^q(\lambda, \gamma)$ such that $f(z)/s(z) \in D_{n,p}(\alpha, \beta)$ and hence an application of Schwarz lemma gives

$$(3.3) \quad \frac{f(z)}{s(z)} = \frac{1 + tz^{n-1}w(z)}{1 + sz^{n-1}w(z)}$$

where $s = 2\beta - 1$, $t = 2\alpha\beta - 1$ and $w(z) = \sum_{n=1}^{\infty} d_n z^n$ is analytic in E and satisfies the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$. It is well-known (see [12]) that such $w(z)$ satisfies the inequality

$$(3.4) \quad |zw'(z) - w(z)| \leq \frac{|z|^2 - |w(z)|^2}{1 - |z|^2}.$$

Differentiating equation (3.3) logarithmically and equating the real parts, we get

$$(3.5) \quad \operatorname{Re} \left(z \frac{f'(z)}{f(z)} \right) = \operatorname{Re} \left(z \frac{s'(z)}{s(z)} \right) - (s-t) \operatorname{Re} \left\{ \frac{z^n w'(z) + (n-1)z^{n-1}w(z)}{1 + sz^{n-1}w(z); (1 + tz^{n-1}w(z))} \right\}$$

Since $s(z) \in S_n^q(\lambda, \gamma)$, by Lemma 3, we have

$$(3.6) \quad \operatorname{Re} \left(z \frac{s'(z)}{s(z)} \right) \geq \frac{u^2 - 2uv(1-\gamma)\cos\lambda \cdot r^n + (2(1-\gamma)\cos^2\lambda - 1)v^2 r^{2n}}{u^2 - v^2 r^{2n}}$$

where $u = (1-\gamma)\cos\lambda + qr$ and $v = q + (1-\gamma)\cos\lambda.r$. Using the estimates (3.4) and (3.6) to the equation (3.5), we get

$$(3.7) \operatorname{Re} \left(z \frac{f'(z)}{f(z)} \right) \geq \frac{u^2 - 2uv(1-\gamma)\cos\lambda.r^n + (2(1-\gamma)\cos^2\lambda - 1)v^2r^{2n}}{u^2 - v^2r^{2n}} - \frac{s+t}{s-t} + \\ + \frac{1}{s-t} \left[\operatorname{Re} \left(nsF(z) + \frac{nt}{F(z)} \right) - \frac{r^{2n} |sF(z) - t|^2 - |1 - F(z)|^2}{r^{n-1}(1-r^2) |F(z)|} \right]$$

where

$$(3.8) \quad F(z) = \frac{1 + tz^{n-1}w(z)}{1 + sz^{n-1}w(z)}.$$

From equation (3.8), we have

$$w(z) = \frac{1 - F(z)}{z^{n-1}(sF(z) - t)} = - [pz + \dots] = -z\delta(z)$$

where $\delta(z)$ is analytic and $|\delta(z)| \leq 1$ for $z \in E$ with $\delta(0) = p$. Further, since $(\delta(z) - p)/(1 - p\delta(z))$ is subordinate to z , it follows that $\delta(z)$ is subordinate to $(z+p)/(1+pz)$ and so

$$(3.9) \quad \left| \frac{1 - F(z)}{sF(z) - t} \right| \leq r^n \left(\frac{p+r}{1+pr} \right), \quad |z| = r.$$

On simple calculations (3.9) gives

$$|F(z) - a_p| < d_p$$

where

$$a_p = \frac{(1+pr)^2 - str^{2n}(p+r)^2}{(1+pr)^2 - s^2r^{2n}(p+r)^2} \quad \text{and} \quad d_p = \frac{(s-t)r^n(p+r)(1+pr)}{(1+pr)^2 - s^2r^{2n}(p+r)^2}.$$

Let $F(z) = a_p + l + im$ and $R^2 = (a_p + l)^2 + m^2$. Denoting the right hand side of (3.7) by $I_p(l, m)$ we get

$$(3.10) \quad I_p(t, m) = \frac{u^2 - 2uv(1-\tau)\cos\lambda \cdot r^n + (2(1-\tau)\cos^2\lambda - 1)v^2 r^{2n}}{u^2 - v^2 r^{2n}} - \frac{s+t}{s-t} +$$

$$+ \frac{1}{s-t} \left[ns(a_p + t) + nt(a_p + t)R^{-2} + \right.$$

$$\left. + \frac{(1-s^2 r^{2n})((a_p + t) - a_1)^2 + m^2 - d_1^2}{r^{n-1}(1-r^2)} R^{-1} \right]$$

and

$$(3.11) \quad \frac{\partial I_p}{\partial m} = \frac{m}{s-t} R^{-4} J_p(t, m)$$

where

$$(3.12) \quad J_p(t, m) = -2nt(a_p + t) + (d_1^2 + 2a_1(a_p + t) - a_1^2) \left(\frac{1-s^2 r^{2n}}{r^{n-1}(1-r^2)} \right) R +$$

$$+ \left(\frac{1-s^2 r^{2n}}{r^{n-1}(1-r^2)} \right) R^3.$$

Denoting the right hand side of (3.12) by $K_p(R)$ with $a_p + t = R \cos \psi$ and $|\psi| < \frac{\pi}{2}$ we get

$$K_p(R) = 2 \left[a_1 R \left(\frac{1-s^2 r^{2n}}{r^{n-1}(1-r^2)} \right) - nt \right] R \cos \psi +$$

$$+ \left(\frac{1-s^2 r^{2n}}{r^{n-1}(1-r^2)} \right) (d_1^2 - a_1^2 + R^2) R.$$

Since for a fixed r , $0 \leq r < 1$, $a_p - d_p$ decreases as p increases over the interval $[0, 1]$, it follows that

$$R \geq R \cos \psi \geq a_p - d_p \geq a_1 - d_1.$$

Thus, for all p , $0 \leq p \leq 1$,

$$\begin{aligned} K_p(R, \psi) &\geq \left[2 \left(a_1 (a_1 - d_1) \left(\frac{1 - s^2 r^{2n}}{r^{n-1} (1 - r^2)} \right) - nt \right) + \right. \\ &\quad \left. + \left(\frac{1 - s^2 r^{2n}}{r^{n-1} (1 - r^2)} \right) (d_1^2 - a_1^2 (a_1 - d_1)^2) \right] (a_1 - d_1) \geq \\ &\geq 2 \left[\left(\frac{1 - s^2 r^{2n}}{r^{n-1} (1 - r^2)} \right) (a_1 - d_1)^2 - nt \right] (a_1 - d_1) > 0 \end{aligned}$$

for all α, β satisfying $0 < \alpha < 1$, $0 < \beta \leq 1$ and n is a fixed integer greater than or equal to one. Therefore, from (3.11) it follows that for a fixed l , $I_p(l, m)$ is an increasing function of m for positive m and is a decreasing function of m for negative m . Hence, the minimum of $I_p(l, m)$ inside the disc $|F(z) - a_p| < d_p$ is attained on the diameter forming part of the real axis. Setting $m = 0$ in (3.10), we obtain

$$\begin{aligned} (3.13) \quad \min_{-1 \leq m \leq 1} I_p(l, m) &\equiv L_p(R) \\ &= \frac{u^2 - 2uv(1-g)\cos\lambda \cdot r^n + (2(1-g)\cos^2\lambda - 1)v^2 r^{2n}}{u^2 - v^2 r^{2n}} - \\ &- \frac{s+t}{s-t} + \frac{1}{s-t} \left[\left(ns + \frac{1 - s^2 r^{2n}}{r^{n-1} (1 - r^2)} \right) R + \frac{1 + ntr^{n-1}(1-r^2) - t^2 r^{2n}}{r^{n-1} (1 - r^2)} R^{-1} - \right. \\ &\quad \left. - 2a_1 \frac{1 - s^2 r^{2n}}{r^{n-1} (1 - r^2)} \right] \end{aligned}$$

where $R = a_p + l \epsilon [a_p - d_p, a_p + d_p]$. Thus the absolute minimum of $L_p(R)$ in $(0, \infty)$ is attained at

$$R_n^* = \left(\frac{1 + ntr^{n-1}(1-r^2) - t^2 r^{2n}}{1 + nsr^{n-1}(1-r^2) - s^2 r^{2n}} \right)^{\frac{1}{2}}$$

and the value of this minimum is equal to

$$(3.14) \quad L_p(R_n^*) = \frac{u^2 - 2uv(1-\gamma)\cos\lambda \cdot r^n + (2(1-\gamma)\cos^2\lambda - 1)v^2 r^{2n}}{u^2 - v^2 r^{2n}} - \frac{s+t}{s-t} + \\ + \frac{2}{(s-t)r^{n-1}(1-r^2)} \left[((1+nsr^{n-1}(1-r^2) - s^2 r^{2n})(1+ntr^{n-1}(1-r^2) - tr^{2n}))^{\frac{1}{2}} - (1-str^{2n}) \right]$$

where s and t are defined in the statement of the theorem.

It is easy to check that $R_n^* < a_p + d_p$ for all p , $0 \leq p \leq 1$, and $n \geq 1$ a fixed integer, but R_n^* may not be in the interval $[a_p - d_p, a_p + d_p]$. In such a case when $R_n^* \notin [a_p - d_p, a_p + d_p]$, the minimum of $L_p(R)$ on the segment $[a_p - d_p, a_p + d_p]$ is attained at $R_{p,n} = a_p - d_p$ since $L_p(R)$ increases with R on this segment. The value of this minimum equals

$$(3.15) \quad L_p(R_{p,n}) = \\ = \frac{u^2 - 2uv(1-\gamma)\cos\lambda \cdot r^n + (2(1-\gamma)\cos^2\lambda - 1)v^2 r^{2n}}{u^2 - v^2 r^{2n}} + \frac{1}{s-t} \left[-(s+t) + \frac{W}{W^*} \right]$$

where

$$W = n(s+t)(1+pr)^2 + 4npstr^n(1+r^2) + (-(1-p^2)(s-t)^2 + \\ + 4nst(1+p^2))r^{n+1} + nst(s+t)r^{2n}(p+r)^2$$

and

$$W^* = (1+pr+spr^n+sr^{n+1})(1+pr+tp^n+tr^{n+1}).$$

It follows from what has been said that the bound r_0 of star-likeness for the class $G_{n,p;q}^\lambda(\alpha, \beta, \gamma)$ is determined either from the equation

$$(3.16) \quad L_p(R_n^*) = 0$$

or from the equation

$$(3.17) \quad L_p(R_{p,n}) = 0.$$

Now the result follows from equations (3.16) and (3.17) after simplification.

The functions given by

$$f(z) = \frac{1-pz-ptz^n+tz^{n+1}}{1-pz-psz^n+sz^{n+1}} s(z) \quad \text{if } R_{p,n} \geq R_n^*$$

and

$$f(z) = \frac{1-bz-btz^n+tz^{n+1}}{1-bz-bsz^n+sz^{n+1}} s(z) \quad \text{if } R_{p,n} \leq R_n^*$$

where $s(z)$ is the extremal function given in Lemma 3 and b is determined by the relation

$$\frac{1-br-btr^n+tr^{n+1}}{1-br-bsr^n+sr^{n+1}} = R_n^* = \left(\frac{1+ntr^{n-1}(1-r^2)-t^2r^{2n}}{1+nsr^{n-1}(1-r^2)-s^2r^{2n}} \right)^{\frac{1}{2}}$$

can be used to show that the results obtained in the theorem are sharp.

C o r o l l a r y 1. Let $f(z) \in G_{n,p;q}^\lambda$ and let

$$c = (1+pr)^2 - r^{2n}(p+r)^2, \quad d = 4\cos\lambda, \quad e = u - \cos\lambda.vr^n$$

where u and v are same as defined in Lemma 3 at $\gamma = 0$. Then $f(z)$ is starlike in $|z| < r_0$, where r_0 is the smallest positive root of the equation

$$2(u^2-v^2r^{2n})(c-2npr^n(1+r^2) - 2(1+n-p^2+np^2)r^{n+1}) - cdevr^n = 0.$$

The result is sharp.

Remark 1. The result corresponding to $\lambda = 0$, $\beta = p = n = 1$ in Theorem 1 sharpens the result obtained by Silverman [11].

Remark 2. In case $\alpha = \lambda = 0$ and $\beta = p = n = 1$ in Theorem 1 we get the result of Al-Amiri [1].

Remark 3. In case $\lambda = 0$, $p = n = 1$ in Corollary 1, we get the result due to Chase [2].

4. Radii of convexity for $H_{n,p;q}^\lambda(\alpha, \beta, \gamma)$

Using the fact that $s(z) \in L(\lambda, \gamma)$ if and only if $zs'(z) \in S(\lambda, \gamma)$, it is easy to show that $f(z) \in G_{n,p;q}^\lambda(\alpha, \beta, \gamma)$ if and only if $\int_0^z \frac{f(t)}{t} dt \in H_{n,p;q}^\lambda(\alpha, \beta, \gamma)$, and $f(z) \in H_{n,p;q}^\lambda(\alpha, \beta, \gamma)$ if and only if $zf'(z) \in G_{n,p;q}^\lambda(\alpha, \beta, \gamma)$.

Thus from the above observation $q(z) \in H_{n,p;q}^\lambda(\alpha, \beta, \gamma)$ if and only if

$$1 + z \frac{g''(z)}{g'(z)} = z \frac{f'(z)}{f(z)}$$

for some $f(z) \in G_{n,p;q}^\lambda(\alpha, \beta, \gamma)$. Hence a radii of starlikeness theorem in $G_{n,p;q}^\lambda(\alpha, \beta, \gamma)$ will correspond to a radii of convexity theorem in $H_{n,p;q}^\lambda(\alpha, \beta, \gamma)$. Therefore, from Theorem 1 the following result follows.

Theorem 2. Let $g(z) \in H_{n,p;q}^\lambda(\alpha, \beta, \lambda)$ and let R_n^* and $R_{p,n}$ are as given in Theorem 1. Then $g(z)$ maps the disc $|z| < r_0$ onto a convex domain where:

- (i) if $R_{p,n} \geq R_n^*$, then r_0 is the smallest positive root of the equation (3.1), and
- (ii) if $R_{p,n} \leq R_n^*$, then r_0 is the smallest positive root of the equation (3.2).

The result is sharp.

Corollary 2. Let $g(z) \in H_{n,p;q}^\lambda$ and let

$$c = (1+pr)^2 - r^{2n}(p+r)^2, \quad d = 4\cos\lambda, \quad e = u - \cos\lambda \cdot vr^n$$

where u and v are as in Lemma 3. Then $q(z)$ is convex in $|z| < r_0$, where r_0 is the smallest positive root of the equation

$$2(u^2 - v^2 r^{2n})(c - 2np r^n(1 + r^2) - 2(1 + n - p^2 + np^2)r^{n+1}) - cdevr^n = 0.$$

The result is sharp.

R e m a r k 4. By fixing $\lambda = 0$, $\beta = n = 1$ in Theorem 2, we obtain the estimates for the radii of convexity for functions of the class introduced and studied by Padmanabhan [7].

R e m a r k 5. The result corresponding to $\lambda = 0$, $\beta = p = n = 1$ in Theorem 2 sharpens the result obtained by Silverman [11] by using a different technique.

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