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QUALITATIVE INVESTIGATIONS OF A HOMOGENEOUS LINEAR SYMMETRIC t-HYPERBOLIC SYSTEM

Introduction

This paper deals with the behaviour of the solutions of mixed problems for a homogeneous linear system of n partial differential equations of first order with unknown functions depending on two independent variables.

Let us introduce the following notations: $\bar{R}_+ = \langle 0, +\infty \rangle$, $X = \langle 0, \alpha \rangle$, $\Omega = X \times \bar{R}_+$ and definitions

$$u(x, t) := \text{col}[u^{(1)}(x, t), u^{(2)}(x, t)],$$

where

$$\begin{cases} u^{(1)}(x, t) := \text{col}[u_1(x, t), \dots, u_{n_0}(x, t)], \\ u^{(2)}(x, t) := \text{col}[u_{n_0+1}(x, t), \dots, u_n(x, t)], \end{cases}$$

and $u(x, t) = u^{(1)}(x, t)$ if $n_0 = n$, while $u(x, t) = u^{(2)}(x, t)$ if $n_0 = 0$;

$$(1) \quad v(x, t) := \text{col}[u(x, t), D_t u(x, t)],$$

where

$$(2) \quad \begin{cases} v^{(1)}(x, t) := \text{col}[u^{(1)}(x, t), D_t u^{(1)}(x, t)], \\ v^{(2)}(x, t) := \text{col}[u^{(2)}(x, t), D_t u^{(2)}(x, t)], \end{cases}$$

and $v(x,t) = v^{(1)}(x,t)$ if $n_0 = n$, while $v(x,t) = v^{(2)}(x,t)$ if $n_0 = 0$;

$$(3) \quad p(x) := \text{col}[p^{(1)}(x), p^{(2)}(x)],$$

where

$$(4) \quad \begin{cases} p^{(1)}(x) := \text{col}[p_1(x), \dots, p_{n_0}(x)], \\ p^{(2)}(x) := \text{col}[p_{n_0+1}(x), \dots, p_n(x)]; \end{cases}$$

$$A(x,t) := [a_{ij}(x,t)]_{(n,n)};$$

$$B(x,t) := [b_{ij}(x,t)]_{(n,n)};$$

$$C(x,t) := [c_{ij}(x,t)]_{(n,n)};$$

$$B_{11}(x,t) := [b_{ij}(x,t)]_{(n_0, n_0)}, \quad i, j = 1, 2, \dots, n_0;$$

$$B_{12}(x,t) := [b_{ij}(x,t)]_{(n_0, n-n_0)}, \quad \begin{aligned} i &= 1, 2, \dots, n_0; \\ j &= n_0+1, n_0+2, \dots, n; \end{aligned}$$

$$B_{21}(x,t) := [b_{ij}(x,t)]_{(n-n_0, n_0)}, \quad \begin{aligned} i &= n_0+1, n_0+2, \dots, n; \\ j &= 1, 2, \dots, n_0; \end{aligned}$$

$$B_{22}(x,t) := [b_{ij}(x,t)]_{(n-n_0, n-n_0)}, \quad i, j = n_0+1, n_0+2, \dots, n;$$

$$(5) \quad A(x,t) := \begin{bmatrix} A(x,t) & 0_n \\ 0_n & A(x,t) \end{bmatrix}, \quad B(x,t) := \begin{bmatrix} B(x,t) & 0_n \\ 0_n & B(x,t) \end{bmatrix},$$

where 0_n is a zero matrix of degree n . Finally, we denote by I_n the unit matrix of degree n .

Consider the linear homogeneous system of partial differential equations of first order with the unknown functions $u_i : \Omega \rightarrow \mathbb{R}$, $i = 1, 2, \dots, n$, represented in the following form

$$(6) \quad A(x,t)D_t u + B(x,t)D_x u + C(x,t)u = 0_{(n,1)}.$$

Onto the equation (6) we impose the initial condition

$$(7) \quad u(x, 0) = p(x) \quad \text{for } x \in X$$

with the given vector-function $p \in C^2(X)$ and the homogeneous boundary conditions

$$(8) \quad \begin{cases} u^{(1)}(0, t) = 0_{(n_0, 1)} \\ u^{(2)}(\alpha, t) = 0_{(n-n_0, 1)} \end{cases} \quad \text{for } t \in \bar{R}_+$$

assuming that the compatibility conditions

$$(9) \quad \begin{cases} p^{(1)}(0) = 0_{(n_0, 1)} \\ p^{(2)}(\alpha) = 0_{(n-n_0, 1)} \end{cases}$$

are satisfied.

Let us suppose that the following conditions hold:

- C.1. The matrices $A(x, t)$, $B(x, t)$ are symmetric.
- C.2. The matrix $A(x, t)$ is positively defined on the set Ω .
- C.3. There exists a positive number $\tilde{\mu}$ such that for any $(x, t) \in \Omega$ it holds $|\det B(x, t)| \geq \tilde{\mu}$.
- C.4. For any $t \in \bar{R}_+$ the matrix $B_{11}(\alpha, t)$ is positively defined, while the matrix $B_{22}(0, t)$ is negatively half-defined.
- C.5. The matrices $A(x, t)$, $B(x, t)$, $C(x, t)$ are bounded on the set Ω .
- C.6. $A, C \in C^{0,1}(\Omega)$, $B \in C^1(\Omega)$, $p \in C^2(X)$.

Let us now adopt the following definitions:

$$(10) \quad G(x, t) := D_t C(x, t) - [D_t B(x, t)] B^{-1}(x, t) C(x, t),$$

$$(11) \quad H(x, t) := C(x, t) - [D_t B(x, t)] B^{-1}(x, t) A(x, t)$$

for any $(x, t) \in \Omega$ and make the following assumptions:

- A.1. There exist classical solutions of the problem (6)-(8), defined on the set Ω ([2], p.263).
- A.2. $u \in C^2(\Omega)$.

A.3. There exists a numerical matrix $\tilde{A} := [\alpha_{ij}]_{(n,n)}$ positively defined and such that at any point $(x,t) \in \Omega$ the matrix $A(x,t) - \tilde{A}$ is positively half-defined.

A.4. At any point $(x,t) \in \Omega$ the matrix

$$\begin{bmatrix} C + C^T - [D_t A + D_x B] & G^T \\ G & H + H^T + D_t A - D_x B \end{bmatrix} (x,t)$$

is positively half-defined, the matrices $G(x,t)$, $H(x,t)$ being given by formulas (10), (11).

A.4'. There exists a positive constant β such that at any point $(x,t) \in \Omega$ the matrix

$$\begin{bmatrix} C + C^T - [D_t A + D_x B + 2\beta I_n] & G^T \\ G & H + H^T + D_t A - [D_x B + 2\beta I_n] \end{bmatrix} (x,t)$$

is positively half-defined.

R e m a r k . If the matrices $A(x,t)$, $B(x,t)$, $C(x,t)$ are constant on the set Ω , then the assumptions A.4, A.4' take on the following form:

$\bar{A}.4.$ The matrix $C + C^T$ is positively half-defined.

$\bar{A}.4'.$ There exists a positive constant β such that the matrix $C + C^T - 2\beta I_n$ is positively half-defined.

From conditions C.1, C.2 it follows that the system (6) is a symmetric t-hyperbolic system in the sense of Friedrichs ([2], p.86).

Differentiating both sides of equation (6) with respect to the variable t we obtain the equation

$$(12) \quad A(x,t)D_t^2 u + B(x,t)D_t D_x u + [D_t A(x,t) + C(x,t)] D_t u + \\ + D_t B(x,t) D_x u + [D_t C(x,t)] u = 0_{(n,1)}.$$

Under the condition C.3, we find $D_x u$ from equation (6), put it into equation (12) and, taking account of definitions (10), (11), we get the equation

$$(6') \quad A(x,t)D_t^2 u + B(x,t)D_t D_x u + [D_t A(x,t) + H(x,t)]D_t u + \\ + G(x,t)u = 0_{(n,1)}.$$

Let us now consider the system (6), (6') written in the following form

$$(13) \quad \begin{bmatrix} A(x,t) & 0_n \\ 0_n & A(x,t) \end{bmatrix} D_t \begin{bmatrix} u \\ D_t u \end{bmatrix} + \begin{bmatrix} B(x,t) & 0_n \\ 0_n & B(x,t) \end{bmatrix} D_x \begin{bmatrix} u \\ D_t u \end{bmatrix} + \\ + \begin{bmatrix} C(x,t) & 0_n \\ G(x,t) & D_t A(x,t) + H(x,t) \end{bmatrix} \begin{bmatrix} u \\ D_t u \end{bmatrix} = \begin{bmatrix} 0_{(n,1)} \\ 0_{(n,1)} \end{bmatrix}.$$

From equation (6) we obtain under condition C.2, the equality

$$D_t u = -A^{-1}(x,t)[B(x,t)D_x u + C(x,t)u],$$

whence it follows that the condition

$$D_t u(x,0) = -A^{-1}(x,0)[B(x,0)D_x u(x,0) + C(x,0)u(x,0)]$$

holds for any $x \in X$.

Assume now the following definitions:

$$(14) \quad q(x) := -A^{-1}(x,0)[B(x,0)p'(x) + C(x,0)p(x)]$$

for any $x \in X$,

$$(15) \quad \begin{cases} q^{(1)}(x) := \text{col}[q_1(x), \dots, q_{n_0}(x)] \\ q^{(2)}(x) := \text{col}[q_{n_0+1}(x), \dots, q_n(x)] \end{cases} \quad \text{for any } x \in X,$$

where $q_i(x)$, $i = 1, \dots, n_0$, $n_0+1, \dots, n$ denote the coordinates of the vector-function $q(x)$;

$$(16) \quad s(x) := \text{col}[p(x), q(x)] \quad \text{for any } x \in X;$$

$$(17) \quad \begin{cases} s^{(1)}(x) := \text{col}[p^{(1)}(x), q^{(1)}(x)] \\ s^{(2)}(x) := \text{col}[p^{(2)}(x), q^{(2)}(x)] \end{cases} \quad \text{for any } x \in X;$$

$$(18) \quad P(x, t) := \begin{bmatrix} C(x, t) & 0_n \\ G(x, t) & D_t A(x, t) + H(x, t) \end{bmatrix} \quad \text{for any } (x, t) \in \Omega.$$

Taking into account the conditions (7)-(9) imposed on equation (6), the corresponding conditions for equation (13) can be written in the form:
initial conditions

$$u(x, 0) = p(x), \quad D_t u(x, 0) = q(x) \quad \text{for } x \in X,$$

boundary conditions

$$\begin{cases} u^{(1)}(0, t) = D_t u^{(1)}(0, t) = 0_{(n_0, 1)}, \\ u^{(2)}(\alpha, t) = D_t u^{(2)}(\alpha, t) = 0_{(n-n_0, 1)}, \end{cases} \quad \text{for } t \in \bar{R}_+$$

compatibility conditions

$$\begin{cases} p^{(1)}(0) = q^{(1)}(0) = 0_{(n_0, 1)}, \\ p^{(2)}(\alpha) = q^{(2)}(\alpha) = 0_{(n-n_0, 1)}. \end{cases}$$

Thus, making use of definitions (1)-(5) and (14)-(18) we find that to the equation (6) with conditions (7)-(9) there corresponds the equation

$$(19) \quad A(x, t) D_t v + B(x, t) D_x v + P(x, t) v = 0_{(2n, 1)}$$

with the initial condition

$$(20) \quad v(x, 0) = s(x) \quad \text{for } x \in X,$$

boundary conditions

$$(21) \quad \begin{cases} v^{(1)}(0, t) = 0_{(2n_0, 1)} \\ v^{(2)}(\alpha, t) = 0_{(2(n-n_0), 1)} \end{cases} \quad \text{for } t \in \bar{R}_+$$

and compatibility conditions

$$(22) \quad \begin{cases} s^{(1)}(0) = 0_{(2n_0, 1)} \\ s^{(2)}(\alpha) = 0_{(2(n-n_0), 1)} \end{cases}$$

L e m m a . If the matrix $A(x, t)$ is a non zero bounded matrix on the set Ω , then there exists a positive number μ such that for any $(x, t) \in \Omega$ and any vector $v := \text{col}[v^{(1)}, v^{(2)}]$, where $v^{(1)} := \text{col}[v_1, \dots, v_n]$, $v^{(2)} := \text{col}[v_{n+1}, \dots, v_{2n}]$ it holds

$$|(A(x, t)v, v)| \leq \mu(v, v).$$

P r o o f . From definition (5) and the definition of the vector v there follows

$$(23) \quad \begin{aligned} |(A(x, t)v, v)| &\leq |(A(x, t)v^{(1)}, v^{(1)})| + \\ &+ |(A(x, t)v^{(2)}, v^{(2)})|. \end{aligned}$$

Let us set

$$(24) \quad \mu_{ij} := \sup_{(x, t) \in \Omega} |a_{ij}(x, t)|, \quad i, j=1, 2, \dots, n,$$

$$(25) \quad \mu := n \max_{1 \leq i, j \leq n} \mu_{ij}.$$

Then we find that $\mu > 0$, since $A(x, t)$ is a non zero matrix on the set Ω .

By virtue of definitions (24), (25) for any $(x, t) \in \Omega$ and $k = 1, 2$, we have the inequality

$$|(A(x, t)v^{(k)}, v^{(k)})| \leq \frac{1}{2} \sum_{i, j=1}^n \mu_{ij} [(v_i^{(k)})^2 + (v_j^{(k)})^2] \leq \mu(v^{(k)}, v^{(k)}).$$

Hence and from inequality (23) there follows

$$|(A(x,t)v,v)| \leq \mu(v,v), \quad \mu > 0,$$

for any $(x,t) \in \Omega$ and for any vector v which completes the proof of the lemma.

Let now U be a linear space of real vector-functions $u: \Omega \rightarrow \mathbb{R}^n$ of class $C^2(\Omega)$ and equibounded on this set.

In the space U we define the norm $\|u(t)\|_1$ of the vector-function u in the section $t = \text{const} \geq 0$ by the equality

$$(26) \quad \|u(t)\|_1 := \sup_{x \in X} [(u(x,t), u(x,t))]^{\frac{1}{2}}$$

where

$$(u(x,t), u(x,t)) := \sum_{i=1}^n u_i^2(x,t).$$

The norm $\|u\|_1$ of the vector-function u is defined by the formula

$$(27) \quad \|u\|_1 := \sup_{t \in \bar{R}_+} (\|u(t)\|_1).$$

Let now $\tilde{W}$ be a linear space of real vector-functions $\tilde{w}: \Omega \rightarrow \mathbb{R}^{3n}$ of class $C^2(\Omega)$ and satisfying the condition

$$(\forall t \in \bar{R}_+) (\forall \tilde{w}_1, \tilde{w}_2 \in \tilde{W}) \left(\int_0^\alpha \left(\sum_{i=1}^{3n} \tilde{w}_1^2(x,t) \tilde{w}_2^2(x,t) \right) dx < \infty \right).$$

Denote by W_0, W_t, W_x the subspaces of the space $\tilde{W}$ whose elements are, respectively, the vector-functions of the form

$$\begin{aligned} w_0(x,t) &:= \text{col}[\tilde{w}_1(x,t), \dots, \tilde{w}_n(x,t), 0, \dots, 0, 0, \dots, 0], \\ w_t(x,t) &:= \text{col}[0, \dots, 0, D_t \tilde{w}_1(x,t), \dots, D_t \tilde{w}_n(x,t), 0, \dots, 0], \\ w_x(x,t) &:= \text{col}[0, \dots, 0, 0, \dots, 0, D_x \tilde{w}_1(x,t), \dots, D_x \tilde{w}_n(x,t)]. \end{aligned}$$

The set W of all the vector-functions of the form

$$w = w_0 + w_t + w_x$$

is the simple sum of the spaces W_0, W_t, W_x .

In the space W we define the norm $\|w(t)\|_2$ of the vector-function w in the section $t = \text{const} \geq 0$ by the equality

$$(28) \quad \|w(t)\|_2 := \left(\int_0^\alpha (w(x,t), w(x,t)) dx \right)^{\frac{1}{2}},$$

where

$$(w(x,t), w(x,t)) := \sum_{i=1}^{3n} w_i^2(x,t).$$

The norm $\|w\|_2$ of the vector-function w is defined by the formula

$$(29) \quad \|w\|_2 := \sup_{t \in \bar{R}_+} (\|w(t)\|_2).$$

D e f i n i t i o n . A classical solution of the problem (6)-(8) is said to be:

a) bounded in norm $\|\cdot\|_1$ ($\|\cdot\|_2$) on the set Ω if there exists a positive constant μ_1 such that it holds

$$\|u\|_1 \leq \mu_1 \quad (\|u\|_2 \leq \mu_1);$$

b) ε -bounded in norm $\|\cdot\|_1$ ($\|\cdot\|_2$) on the set Ω if there exist positive constants $\varepsilon, \tilde{\mu}_1$ such that it holds

$$\|u^{(\varepsilon)}\|_1 \leq \tilde{\mu}_1 \quad (\|u^{(\varepsilon)}\|_2 \leq \tilde{\mu}_1),$$

where

$$(30) \quad u^{(\varepsilon)}(x,t) := u(x,t) \exp(\varepsilon t) \quad \text{for any} \quad (x,t) \in \Omega.$$

The aim of the present paper is to study the boundedness and the ε -boundedness in norm $\|\cdot\|_2$ of the classical solutions of the problem (6)-(8) and of their partial derivatives of first order, as well as the boundedness and ε -boundedness in norm $\|\cdot\|_1$ of the classical solutions of this problem. The results will be presented in sections 1 and 2.

1. The boundedness of the classical solutions of the problem (6)-(8)

Theorem 1. If the conditions C.1-C.6 and the assumptions A.1-A.4 are satisfied, then the classical solutions of the problem (6)-(8) are bounded in norm $\|\cdot\|_1$ on the set Ω and their partial derivatives of first order are bounded in norm $\|\cdot\|_2$ on this set.

Proof. Denote by u an arbitrary classical solution on the set Ω of the problem (6)-(8), by v the corresponding classical solution of the problem (19)-(21) on this set.

Let us build, as in paper [1], for the solution v the function of Lapunov type $k: \bar{R}_+ \rightarrow R$ defined by the formula

$$(1.1) \quad k(t) := \int_0^\alpha (A(x,t)v(x,t), v(x,t)) dx.$$

By virtue of conditions C.2, C.6 and assumption A.2 the function $k \in C^1(\bar{R}_+)$ and satisfies for any $t \in \bar{R}_+$ the condition $k(t) \geq 0$. The derivative of this function is given by the formula

$$\begin{aligned} \dot{k}(t) = & \int_0^\alpha ([D_t A(x,t)] v(x,t), v(x,t)) dx + \\ & + \int_0^\alpha (2A(x,t) D_t v(x,t), v(x,t)) dx. \end{aligned}$$

From equation (19) there follows the equality

$$A(x,t)D_t v(x,t) = -P(x,t)v(x,t) - B(x,t)D_x v(x,t).$$

Hence the following condition is satisfied

$$(1.2) \quad -\dot{k}(t) = \int_0^\alpha ((P(x,t) + P^T(x,t) - [D_t A(x,t) + \\ + D_x B(x,t)])v(x,t), v(x,t))dx + [(B(x,t)u(x,t), u(x,t))]_0^\alpha + \\ + [(B(x,t)D_t u(x,t), D_t u(x,t))]_0^\alpha.$$

Moreover, we have the equality

$$[(B(x,t)u(x,t), u(x,t))]_0^\alpha = (B_{11}(\alpha, t)u^{(1)}(\alpha, t), u^{(1)}(\alpha, t)) + \\ + (B_{12}(\alpha, t)u^{(2)}(\alpha, t), u^{(1)}(\alpha, t)) + (B_{21}(\alpha, t)u^{(1)}(\alpha, t), u^{(2)}(\alpha, t)) + \\ + (B_{22}(\alpha, t)u^{(2)}(\alpha, t), u^{(2)}(\alpha, t)) - [(B_{11}(0, t)u^{(1)}(0, t), u^{(1)}(0, t)) + \\ + (B_{12}(0, t)u^{(2)}(0, t), u^{(1)}(0, t)) + (B_{21}(0, t)u^{(1)}(0, t), u^{(2)}(0, t)) + \\ + (B_{22}(0, t)u^{(2)}(0, t), u^{(2)}(0, t))].$$

This equality, together with the boundary conditions (21) and the condition C.4, imply that for any $t \in \bar{R}_+$ we have the inequality

$$(1.3) \quad [(B(x,t)u(x,t), u(x,t))]_0^\alpha > 0.$$

Transforming in the same way the expression $[(B(x,t)D_t u(x,t), D_t u(x,t))]_0^\alpha$ and making use of the boundary conditions (21) and of condition C.4 we find that for any $t \in \bar{R}_+$ it holds

$$(1.4) \quad [(B(x,t)D_t u(x,t), D_t u(x,t))]_0^\alpha > 0.$$

From formulas (1.2)-(1.4) we obtain for $t \in \bar{R}_+$ the condition

$$(1.5) \quad -\dot{k}(t) > \int_0^\alpha ((P(x,t) + P^T(x,t) + \\ -[D_t A(x,t) + D_x B(x,t)] v(x,t), v(x,t)) dx.$$

Making use of inequality (1.5), definition (18), condition C.1 and assumption A.4, we find that for any $t \in \bar{R}_+$ it holds $\dot{k}(t) < 0$, whence there follows the condition

$$(1.6) \quad k(t) < k(0) \quad \text{for any } t \in \bar{R}_+,$$

where

$$k(0) = \int_0^\alpha (A(x,0) v(x,0), v(x,0)) dx,$$

by virtue of definition (1.1).

Taking account of the condition (20) and making use of definition (16) we obtain the equality

$$(1.7) \quad k(0) = \int_0^\alpha [(A(x,0) p(x), p(x)) + (A(x,0) q(x), q(x))] dx,$$

where $q(x)$ is given by formula (14).

Assume now the definition

$$(1.8) \quad \tilde{A} := \begin{bmatrix} \tilde{A} & 0_n \\ 0_n & \tilde{A} \end{bmatrix}.$$

From assumption A.3 and definitions (1.1), (1.8) there follows the condition

$$(1.9) \quad \int_0^\alpha (\tilde{A} v(x,t), v(x,t)) dx \leq k(t) \quad \text{for any } t \in \bar{R}_+.$$

Let

$$(1.10) \quad \lambda(\tilde{A}) := \min_{1 \leq i \leq n} \lambda_i(\tilde{A}),$$

where $\lambda_i(\tilde{A})$, $i = 1, 2, \dots, n$, denote the eigenvalues of the matrix $\tilde{A}$.

By virtue of formula (1.9) and definitions (1.8), (1.10), we find that for any $t \in \bar{R}_+$ it holds

$$\lambda(\tilde{A}) \int_0^t (v(x, t), v(x, t)) dx \leq k(t),$$

whence it follows, by (1.6), (1.7), (28), that for any $t \in \bar{R}_+$ it holds

$$(1.11) \quad \|v(t)\|_2 < \mu_1,$$

where

$$(1.12) \quad \mu_1 := \sqrt{\frac{k(0)}{\lambda(\tilde{A})}}.$$

From (1.11) and (29) we get the inequality

$$(1.13) \quad \|v\|_2 \leq \mu_1.$$

From inequality (1.13) and definition (1) it follows that the conditions

$$\|u\|_2 \leq \mu_1, \quad \|D_t u\|_2 \leq \mu_1$$

are satisfied.

On account of condition C.3 let us now introduce the following definitions

$$(1.14) \quad \begin{cases} [\tilde{b}_{ij}(x, t)]_{(n, n)} := B^{-1}(x, t), \\ [d_{ij}(x, t)]_{(n, n)} := B^{-1}(x, t)A(x, t), \\ [l_{ij}(x, t)]_{(n, n)} := B^{-1}(x, t)C(x, t). \end{cases}$$

Moreover, making use of condition C.5, we assume the following definitions

$$(1.15) \quad \left\{ \begin{array}{l} \mu_A := \max_{1 \leq i, j \leq n} \sup_{(x,t) \in \Omega} |a_{ij}(x,t)|, \\ \mu_{B^{-1}} := \max_{1 \leq i, j \leq n} \sup_{(x,t) \in \Omega} |\tilde{b}_{ij}(x,t)|, \\ \mu_C := \max_{1 \leq i, j \leq n} \sup_{(x,t) \in \Omega} |c_{ij}(x,t)|. \end{array} \right.$$

The definitions (1.14), (1.15) imply that for any $(x,t) \in \Omega$ and any $i, j \in \{1, 2, \dots, n\}$ we have the inequalities

$$(1.16) \quad \left\{ \begin{array}{l} |d_{ij}(x,t)| \leq n \mu_{B^{-1}} \mu_A, \\ |l_{ij}(x,t)| \leq n \mu_{B^{-1}} \mu_C. \end{array} \right.$$

From equation (6) we draw, by condition C.3, the equality

$$D_x u(x,t) = -B^{-1}(x,t)A(x,t)D_t u(x,t) - B^{-1}(x,t)C(x,t)u(x,t),$$

whence we obtain for any $(x,t) \in \Omega$, according to definition (1.14), the inequalities

$$\begin{aligned} |D_x u_i(x,t)| &\leq \sum_{j=1}^n |d_{ij}(x,t)| |D_t u_j(x,t)| + \\ &+ \sum_{j=1}^n |l_{ij}(x,t)| |u_j(x,t)|, \end{aligned}$$

where $i = 1, 2, \dots, n$.

From these inequalities and (1.16) it follows that we have for any $(x,t) \in \Omega$

$$\sum_{i=1}^n (D_x u_i(x,t))^2 \leq 2\mu_2^2(v(x,t), v(x,t)),$$

where

$$(1.17) \quad \mu_2 := n^2 \mu_{B^{-1}} \max(\mu_A, \mu_C),$$

and, further, we get

$$\int_0^{\alpha} (D_x u(x, t), D_x u(x, t)) dx \leq 2\mu_2^2 \int_0^{\alpha} (v(x, t), v(x, t)) dx.$$

Hence we obtain, by definition (28) and condition (1.11), for any $t \in \bar{R}_+$ the inequality

$$\|D_x u(t)\|_2 < \mu_3,$$

where

$$(1.18) \quad \mu_3 := \sqrt{2} \mu_1 \mu_2.$$

By virtue of definition (29), we obtain the condition

$$(1.19) \quad \|D_x u\|_2 \leq \mu_3.$$

We shall now prove that the considered solution of the problem (6)-(8) is bounded in norm $\|\cdot\|_1$.

Making use of the boundary conditions (8) and of the inequality of Bunyakovski-Schwarz we obtain for any $(x, t) \in \Omega$ the following formulas:

$$\begin{aligned} u_1^2(x, t) &= \left(\int_0^x D_y u_1(y, t) dy \right)^2 \leq \int_0^x dy \int_0^x [D_y u_1(y, t)]^2 dy \leq \\ &\leq \alpha \int_0^{\alpha} [D_x u_1(x, t)]^2 dx, \quad i = 1, 2, \dots, n_0, \end{aligned}$$

$$\begin{aligned} u_1^2(x, t) &= \left(\int_x^{\alpha} D_y u_1(y, t) dy \right)^2 \leq \int_x^{\alpha} dy \int_x^{\alpha} [D_y u_1(y, t)]^2 dy \leq \\ &\leq \alpha \int_0^{\alpha} [D_x u_1(x, t)]^2 dx, \quad i = n_0+1, n_0+2, \dots, n. \end{aligned}$$

Therefore, we have the inequality

$$\sum_{i=1}^n u_i^2(x, t) \leq \alpha \int_0^{\alpha} (D_x u(x, t), D_x u(x, t)) dx \quad \text{for } (x, t) \in \Omega,$$

which implies for any $(x, t) \in \Omega$, according to definitions (28), (29) and condition (1.19), the inequality

$$\left[\sum_{i=1}^n u_i^2(x, t) \right]^{\frac{1}{2}} \leq \mu_4,$$

where

$$(1.20) \quad \mu_4 := \mu_3 \sqrt{\alpha}.$$

By virtue of definition (26), we get for any $t \in \bar{R}_+$ the condition

$$\|u(t)\|_1 \leq \mu_4,$$

whence we draw, on account of definition (27), the inequality $\|u\|_1 \leq \mu_4$, which achieves the proof of Theorem 1.

2. The ε -boundedness of classical solutions of the problem (6)-(8)

Theorem 2. If the conditions C.1 - C.6 and the assumptions A.1 - A.3, A.4' are satisfied, then the classical solutions of the problem (6)-(8) are ε -bounded in norm $\|\cdot\|_1$ and their partial derivatives of first order are ε -bounded in norm $\|\cdot\|_2$ on the set Ω .

Proof. By virtue of assumption A.1, the problem (6)-(8) has classical solutions. Let u be an arbitrary classical solution of this problem. Taking account of assumption A.2 and condition C.6 denote by v the classical solution of the problem (19)-(21) corresponding to the solution u . Proceeding as in the proof of Theorem 1 let us build for the solution v the function of Lapunov type $k: \bar{R}_+ \rightarrow R$ defined by formula (1.1). From conditions C.2, C.6 and assumption A.2 it follows that the function k is of class $C^1(\bar{R}_+)$ and satisfies for any $t \in \bar{R}_+$ the condition $k(t) \geq 0$. The boundary conditions (21) and the condition C.4 imply that for any $t \in \bar{R}_+$ the derivative of the function k satisfies the inequality (1.5). Proceeding further as in the proof of Theorem 1 and making use

of definition (18), condition C.1 and assumption A.4', we obtain for any $t \in \bar{R}_+$ the inequality

$$(2.1) \quad -\dot{k}(t) > 2\beta \int_0^\alpha (v(x,t), v(x,t)) dx.$$

Making now use of definition (1.1) and of the previously proved lemma, that follows from conditions C.2 and C.5, we obtain for any $t \in \bar{R}_+$ the inequality

$$(2.2) \quad k(t) \leq \mu \int_0^\alpha (v(x,t), v(x,t)) dx,$$

where, by definitions (24), (25), (1.15), the positive constant μ is given by the formula

$$\mu = n \mu_A.$$

From the inequalities (2.1) and (2.2) we get for any $t \in \bar{R}_+$ the condition

$$\dot{k}(t) < -\frac{2\beta}{\mu} k(t)$$

which implies the inequality

$$k(t) < k(0) \exp \left(-\frac{2\beta}{\mu} t \right) \quad \text{for } t \in \bar{R}_+,$$

where $k(0)$ is given by the formula (1.7).

Making now use of assumption A.3 and definition (1.1) and proceeding as in the proof of Theorem 1 we find that for any $t \in \bar{R}_+$ we have the inequality

$$(2.3) \quad \|v(t)\|_2 < \mu_1 \exp \left(-\frac{\beta}{\mu} t \right),$$

where the positive constant μ_1 is given by the formula (1.12).

Assume the definition

$$(2.4) \quad \varepsilon := \frac{\beta}{\mu}.$$

From formula (2.3) and definitions (2.4), (30), (28), (29) we obtain the inequality

$$\|v^{(\varepsilon)}\|_2 \leq \mu_1.$$

This inequality, together with definitions (28), (29), (1) imply that we have

$$\|u^{(\varepsilon)}\|_2 \leq \mu_1, \quad \|(D_t u)^{(\varepsilon)}\|_2 \leq \mu_1.$$

Applying the same argument as in the proof of Theorem 1 we obtain for any $t \in \bar{R}_+$ the inequality

$$(2.5) \quad \|D_x u(t)\|_2 \leq \sqrt{2} \mu_2 \|v(t)\|_2,$$

where the positive constant μ_2 is given by the formula (1.17).

From inequalities (2.5) and (2.3) it follows that we have

$$(2.6) \quad \|D_x u(t)\|_2 < \mu_3 \exp\left(-\frac{\beta}{\mu} t\right) \quad \text{for } t \in \bar{R}_+,$$

where the constant μ_3 is given by formula (1.18).

By virtue of definitions (2.4), (30), (29) and condition (2.6) we get the inequality

$$\|(D_x u)^{(\varepsilon)}\|_2 \leq \mu_3.$$

We shall now prove that the considered solution of the problem (6)-(8) is ε -bounded in norm $\|\cdot\|_1$. Proceeding as in the proof of Theorem 1 we obtain for any $x \in X$ and any $t \in \bar{R}_+$ the inequality

$$(2.7) \quad \left(\sum_{i=1}^n u_i^2(x, t) \right)^{\frac{1}{2}} \leq \sqrt{\alpha} \|D_x u(t)\|_2.$$

By virtue of inequalities (2.6) and (2.7), we get for any $x \in X$ and any $t \in \bar{R}_+$ the inequality

$$\left(\sum_{i=1}^n u_i^2(x, t) \right)^{\frac{1}{2}} < \mu_4 \exp\left(-\frac{\beta}{\mu} t\right),$$

where the positive constant μ_4 is given by formula (1.20). From this latter inequality and from definitions (2.4), (30), (26) we obtain for any $t \in \bar{R}_+$ the condition

$$\|u^{(\varepsilon)}(t)\|_1 \leq \mu_4.$$

Hence it follows, by definition (27), that the inequality

$$\|u^{(\varepsilon)}\|_1 \leq \mu_4$$

is true, which completes the proof of Theorem 2.

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