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INTEGRABILITY CONDITIONS OF A STRUCTURE SATISFYING $f^5 - a^2f = 0$

Andreou [1], [2] has studied the structure defined by a tensor field f of type $(1,1)$ satisfying $f^5 + f = 0$, and also obtained its integrability conditions. Upadhyay and Gupta [3] have obtained the integrability conditions of a structure f_λ satisfying $f^3 - \lambda^2f = 0$. The aim of the present paper is to obtain the integrability conditions of a structure satisfying $f^5 - a^2f = 0$, where a is a complex number not equal to zero.

1. Preliminaries

Let M^n be an n -dimensional differentiable manifold of class C^∞ endowed with a non-null tensor field f of type $(1,1)$ and of class C^∞ such that

$$(1.1) \quad f^5 - a^2f = 0,$$

where a is a complex number not equal to zero.

Let us define the operators l and m by

$$(1.2) \quad l = \frac{f^4}{a^2}, \quad m = I - \frac{f^4}{a^2},$$

I denoting the identity operator.

T h e o r e m 1.1. For a tensor field f ($\neq 0$) satisfying (1.1) and the operators l , m defined by (1.2) and applied

to the tangent space at each point of the manifold are complementary projection operators.

P r o o f . From (1.1) and (1.2), we have

$$(1.3) \quad l + m = I,$$

$$(1.4) \quad l^2 = \frac{f^8}{a^4} = \frac{f^5 \cdot f^3}{a^4} = \frac{a^2 f^4}{a^4} = \frac{f^4}{a^2} = l,$$

$$(1.5) \quad m^2 = I + \frac{f^8}{a^4} - \frac{2f^4}{a^2} = I + \frac{f^4}{a^2} - \frac{2f^4}{a^2} = I - \frac{f^4}{a^2} = m,$$

$$(1.6) \quad lm = ml = \frac{f^4}{a^2} - \frac{f^8}{a^4} = \frac{f^4}{a^2} - \frac{f^4}{a^2} = 0.$$

This proves the theorem.

Thus if there is given a tensor field f ($\neq 0$) satisfying (1.1), then there exist two complementary distributions L and M corresponding to the projection operators l and m respectively. If the rank of f is constant and is equal to r every where, then $r = 4q$ [1] and the dimensions of L and M are r and $n-r$ respectively. We call such a structure a $f_{(5,1)}$ -structure of rank r .

T h e o r e m 1.2. For a tensor field f ($\neq 0$) satisfying (1.1) and the operators l, m defined by (1.2), we have

$$(1.7) \quad fl = lf = f,$$

$$(1.8) \quad fm = mf = 0,$$

$$(1.9) \quad f^2 l = lf^2 = f^2,$$

$$(1.10) \quad f^2 m = mf^2 = 0,$$

$$(1.11) \quad f^4 l = lf^4 = a^2 l,$$

$$(1.12) \quad f^4 m = mf^4 = 0.$$

P r o o f .

$$f_1 = f \frac{f^4}{a^2} = \frac{f^5}{a^2} = \frac{a^2 f}{a^2} = f.$$

Similarly,

$$1f = f.$$

$$f_m = mf = f \left(I - \frac{f^4}{a^2} \right) = f - \frac{f^5}{a^2} = f - f = 0,$$

$$f^2_1 = 1f^2 = f^2 \frac{f^4}{a^2} = \frac{f^5 \cdot f}{a^2} = f \cdot f = f^2,$$

$$f^2_m = mf^2 = f^2 \left(I - \frac{f^4}{a^2} \right) = f^2 - \frac{f^2 f^4}{a^2} = f^2 - f^2 = 0.$$

$$f^4_1 = \frac{f^4 \cdot f^4}{a^2} = \frac{f^5 \cdot f^3}{a^2} = \frac{a^2 f^4}{a^2} = a^2_1.,$$

$$f^4_m = f^4 \left(I - \frac{f^4}{a^2} \right) = f^4 - \frac{a^2 f^4}{a^2} = f^4 - f^4 = 0.$$

If the rank of f is n , then f satisfies $I - \frac{f^4}{a^2} = 0$ or $f^4 - a^2 I = 0$.

2. Nijenhuis tensor

Let f be an $f_a(5,1)$ -structure of rank r , then the Nijenhuis tensor $N(X,Y)$ of f is [4]

$$(2.1) \quad N(X,Y) = [fX, fY] - f[fX, Y] - f[X, fY] + f^2[X, Y].$$

T h e o r e m 2.1. We have the following identities:

$$(2.2) \quad N(mX, mY) = lN(mX, mY) = l[mX, mY],$$

$$(2.3) \quad mN(X, Y) = m[fX, fY],$$

$$(2.4) \quad mN(lX, lY) = m[fX, fY],$$

$$(2.5) \quad mN(f^3X, f^3Y) = m[a^2lX, a^2lY].$$

P r o o f . From (2.1) and using theorem (1.2), we have

$$N(mX, mY) = f^2[mX, mY],$$

$$lN(mX, mY) = lf^2[mX, mY] = f^2[mX, mY] = N(mX, mY) \text{ by (1.9).}$$

From (2.1) we also have

$$f^2N(mX, mY) = f^4[mX, mY] \text{ or } f^2N(mX, mY) = a^2l[mX, mY].$$

Also

$$f^4N(mX, mY) = a^2lf^2[mX, mY],$$

$$a^2lN(mX, mY) = a^2l[mX, mY],$$

or

$$lN(mX, mY) = l[mX, mY].$$

Thus

$$N(mX, mY) = lN(mX, mY) = l[mX, mY].$$

Similarly, we can easily prove the identities (2.3), (2.4) and (2.5).

We have, $l = \frac{f^4}{a^2}$, $fl = f$. Now if for all vector fields X and Y

$$N(lX, lY) = 0,$$

then

$$N(f^3X, f^3Y) = 0 \text{ and conversely.}$$

T h e o r e m 2.2. The following conditions are equivalent:

$$(2.6) \quad mN(X, Y) = 0,$$

$$(2.7) \quad mN(lX, lY) = 0,$$

$$(2.8) \quad mN(f^3X, f^3Y) = 0.$$

P r o o f . By virtue of (2.3), (2.4) and (2.5), the theorem follows.

The Lie derivative $\mathcal{L}_Y f$ of the tensor field f with respect to a vector field Y is a tensor field of the same type as f given by [4],

$$(2.9) \quad (\mathcal{L}_Y f)X \stackrel{\text{def}}{=} f[X, Y] - [fX, Y].$$

By virtue of (2.1), (2.9) and theorem (1.2), we have

$$(2.10) \quad N(lX, mY) = f(\mathcal{L}_{mY} f)lX = f\{l(\mathcal{L}_{mY} f)lX\},$$

$$(2.11) \quad f^3 N(lX, mY) = a^2 l(\mathcal{L}_{mY} f)lX,$$

or

$$lN(lX, mY) = l(\mathcal{L}_{mY} f)lX.$$

3. Integrability conditions

A necessary and sufficient condition for the distribution M to be integrable is that $l[mX, mY] = 0$ for any vector fields X and Y [4].

T h e o r e m 3.1. A necessary and sufficient condition for the distribution M to be integrable is that

$$(3.1) \quad N(mX, mY) = 0$$

or equivalently

$$(3.2) \quad lN(mX, mY) = 0$$

for all vector fields X and Y .

P r o o f . By virtue of (2.2), the result follows.

T h e o r e m 3.2. In order that the distribution L be integrable, it is necessary and sufficient that (2.6), (2.7) or (2.8) is satisfied.

P r o o f . For any vector fields X and Y , the distribution L is integrable if and only if $m[lX, lY] = 0$. Thus by virtue of (2.5), the theorem follows.

In view of $l+m = I$, the Nijenhuis tensor $N(X,Y)$ can be written in the form

$$(3.3) \quad N(X,Y) = N(lX,lY) + N(lX,mY) + N(mX,lY) + N(mX,mY)$$

or equivalently,

$$(3.4) \quad N(X,Y) = lN(lX,lY) + mN(lX,lY) + N(lX,mY) + \\ + N(mX,lY) + N(mX,mY).$$

T h e o r e m 3.3. A necessary and sufficient condition for both distributions L and M to be integrable is that

$$N(X,Y) = lN(lX,lY) + N(lX,mY) + N(mX,lY),$$

for any vector fields X and Y .

P r o o f . In consequence of Theorems 3.1 and 3.2 and taking account of (3.4), the theorem follows.

Suppose that the distribution L is integrable, then f induces on each integral manifold of L a structure f_* satisfying $(f_*)^4 = -I$. The Nijenhuis tensor of this structure is exactly $N(lX,lY)$.

D e f i n i t i o n 3.1. If the distribution L is integrable and the structure $f_* = f/L$ induced from f on each integral manifold of L is also integrable; then we say that the $f_a(5,1)$ -structure is partially integrable.

T h e o r e m 3.4. For any two vector fields X and Y a necessary and sufficient condition for a $f_a(5,1)$ -structure to be partially integrable is that one of the following equivalent conditions be satisfied

$$N(lX,lY) = 0 \quad \text{and} \quad N(f^3X, f^3Y) = 0.$$

P r o o f . Since the conditions

$$N(lX,lY) = 0 \quad \text{and} \quad N(f^3X, f^3Y) = 0$$

are equivalent, therefore from Theorem 3.2, the theorem follows.

Theorem 3.5. A necessary and sufficient condition for the distribution M to be integrable and the $f_a(5,1)$ -structure to be partially integrable, is that

$$N(X, Y) = N(lX, mY) + N(mX, lY)$$

for any two vector fields X and Y .

Proof. The theorem follows from Theorems 3.1, 3.3 and 3.4.

Theorem 4.1. The tensor field $l(\mathcal{L}_{mY}f)l$ vanishes identically for any vector field Y if and only if $N(lX, mY) = 0$.

Proof. The theorem follows from equation (2.4).

Let us suppose that the two distributions L and M are integrable. Then, we can choose a local coordinate system in such a way that L'^S are represented by putting $n-r$ local coordinates constant and M'^S by putting the other r coordinates constant. We call such a coordinate system in adapted coordinate system. The projection operators l and m can be supposed to have the components of the form

$$(4.1) \quad l = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad m = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix}$$

respectively, where I_r and I_{n-r} are unit matrices of orders r and $n-r$ respectively.

Since f satisfies (1.2), the tensor f has the components of the form

$$(4.2) \quad f = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$$

in an adapted coordinate system, where f_r is an $r \times r$ square matrix. Thus for any vector field mY on M , the Lie derivative $\mathcal{L}_{mY}f$ has components of the form

$$\mathcal{L}_{mY}f = \begin{pmatrix} L' & 0 \\ 0 & 0 \end{pmatrix}.$$

T h e o r e m 4.2. Let us suppose that L and M are both integrable and that an adapted coordinate system has been chosen. In order that the components of f be independent of the coordinates which are constant along the integral manifold of L it is necessary and sufficient condition that

$$N(lX, mY) = 0$$

for any two vector fields X and Y .

P r o o f . Let us suppose that the tensor field $l(\mathcal{L}_{mY}f)$ vanishes identically for any vector field Y , then we have $\mathcal{L}_{mY}f = 0$ which implies that the components of f are independent of the coordinates which are constant along the integral manifolds of the distribution L in an adapted coordinate system. Conversely, if the components of f are independent of these coordinates, then we can show easily that $l(\mathcal{L}_{mY}f)$ vanishes identically for any vector field Y . Thus from Theorem 4.1 the theorem follows.

T h e o r e m 4.3. Suppose that the distributions L and M are both integrable and that an adapted coordinate system has been chosen. A necessary and sufficient condition for the components of f to be independent of the coordinates which are constant along the integral manifolds of L is that

$$N(X, Y) = lN(lX, lY)$$

for any two vector fields X and Y .

P r o o f . The theorem follows from Theorems 3.3 and 4.2.

D e f i n i t i o n 4.1. The structure $f_a(5,1)$ is said to be integrable if:

(a) $f_a(5,1)$ is partially integrable; that is

$$N(X, Y) = N(lX, mY) + N(mX, lY).$$

(b) The distribution M is integrable, that is,

$$N(mX, mY) = 0.$$

(c) The components of the structure $f_a(5,1)$ are independent of the coordinates which are constant along the integral manifold of L in an adapted coordinate system.

Theorem 4.4. In order that the structure $f_g(5,1)$ to be integrable, it is necessary and sufficient condition that $N(X,Y) = 0$ for any vector fields X and Y .

Proof. In consequence of Theorems 3.1, 3.4 and 4.3, the theorem follows.

Acknowledgement. We are thankful to Dr. V.C. Gupta for his kind guidance throughout the preparation of this paper.

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Received January 10, 1984.