Anna Szymanowska

APPLICATION OF INTEGRAL TRANSFORMS TO DETERMINING THE POINTS OF DISCONTINUITY OF THE SOLUTIONS OF LINEAR CONTROL SYSTEMS

The paper contains an estimation of the number of points of discontinuity of the time-optimal control of linear systems with time-varying coefficients. The application of integral transforms brings the system with time-varying coefficients to a system with constant coefficients. The method for determining the points of discontinuity, presented here, is based on our complex-analytical procedure given in [15] and [16].

1. The solving of linear differential equations by means of integral transforms

We obtain an effective solution of the n -th order linear equations

$$(1.1) \quad \sum_{k=0}^n a_{k+1}(t)x^{(k)}(t) = z(t)$$

with given $x^{(k)}(0)$, $k = 0, 1, \dots, n-1$. Functions $a_{k+1}(t)$, $t \geq 0$, $k = 0, 1, \dots, n$, are piecewise continuously differentiable up to order n , while $z(t)$ is for $t \geq 0$ a function of bounded variation in every bounded interval. We suppose that $a_{n+1}(t) \neq 0$ for $t \geq 0$. We reduce the solution of equation (1.1) to that of a linear equation with constant coefficients. From (1.1), by taking a one-sided integral transform, we obtain

$$(1.2) \quad \int_0^{+\infty} \sum_{k=0}^n a_{k+1}(t) x^{(k)}(t) g(s, t) dt = \int_0^{+\infty} z(t) g(s, t) dt$$

where s is a complex number and $g(s, t)$ is a complex function which will be calculated later. After a suitable number of integrations by parts, we get

$$(1.3) \quad E(s)M(s) = P(s) + Z(s)$$

where

$$(1.4) \quad E(s)M(s) = \int_0^{+\infty} x(t) \left\{ \sum_{k=0}^n \left[\sum_{h=k}^n (-1)^h \binom{h}{k} a_{h+1}^{(h-k)}(t) \right] \frac{d^k}{dt^k} g(s, t) \right\} dt,$$

$$P(s) = \sum_{k=1}^n \sum_{h=0}^{k-1} (-1)^{h+1} \frac{d^h}{dt^h} \left[a_{k+1}(t) g(s, t) \right] x^{(k-1-h)}(t) \Big|_{t=0}^{t=+\infty},$$

$$Z(s) = \int_0^{+\infty} z(t) g(s, t) dt,$$

$$M(s) = C_1 + C_2 s + \dots + C_{n+1} s^n,$$

$C_{n+1} = 1$, $C_1, C_2, \dots, C_n$ are real numbers. The relationship between these numbers and the coefficients of equation (1.1) will be determined in the sequel. The unknown function $g(s, t)$ is calculated from the condition that the integral transform $E(s)$ of the function $x(t)$ is the Laplace transform

$$E(s) = \int_0^{+\infty} x(t) e^{-st} dt.$$

Then $g(s, t)$ satisfies the differential equation

$$(1.5) \quad \sum_{k=0}^n \left[\sum_{h=k}^n (-1)^h \binom{h}{k} a_{h+1}^{(h-k)}(t) \right] \frac{d^k}{dt^k} g(s, t) = M(s) e^{-st}.$$

Next, we confine ourselves to the case of equation (1.5) with constant coefficients. Then the conditions

$$(1.6) \quad \frac{1}{a_{n+1}(t)} \sum_{h=k}^n (-1)^h \binom{h}{k} a_{h+1}^{(h-k)}(t) = (-1)^k C_{k+1},$$

$$k = n, n-1, \dots, 0,$$

are satisfied. The real constants $C_1, C_2, \dots, C_{n+1}$ are the same as the coefficients of the polynomial $M(s)$ defined in (1.4). The coefficients $a_{h+1}(t)$, $h = 0, 1, \dots, n$, satisfying (1.6), have the following form

$$(1.7) \quad a_{h+1}(t) = \sum_{i=h}^n C_{i+1} \binom{i}{h} a_{n+1}^{(i-h)}(t).$$

The functions $g(s, t)$ is a solution of the linear differential equation with constant coefficients

$$(1.8) \quad \sum_{i=0}^n (-1)^i C_{i+1} \frac{d^i}{dt^i} g(s, t) = \frac{M(s)e^{-st}}{a_{n+1}(t)}.$$

In addition to the above hypotheses, we suppose that:

(i) transforms $E(s)$ and $Z(s)$ converge absolutely for some $s = \sigma > 0$;

(ii) the function $\frac{P(s)+Z(s)}{M(s)}$ has a finite number of poles $s_1, s_2, \dots, s_l$ of order, respectively $\gamma_1, \gamma_2, \dots, \gamma_l$, $\operatorname{re} s_i \leq \sigma$, $i = 1, 2, \dots, l$;

(iii) the integral

$$\int_{\Gamma(R)} \frac{P(s)+Z(s)}{M(s)} e^{ts} ds, \quad \Gamma(R) : s = \sigma - R e^{i\varphi}, \quad -\frac{1}{2}\pi \leq \varphi \leq \frac{1}{2}\pi,$$

converges for sufficiently large $R > 0$ and tends to 0 as $R \rightarrow +\infty$.

Now, we formulate

Theorem 1.1. Under the above hypotheses, the solution of equation (1.1) with coefficients (1.7) is given by the formula

$$(1.9) \quad x(t) = \sum_{i=1}^1 \sum_{h=0}^{\gamma_i-1} \alpha_{i,h} t^h e^{ts_i}, \quad t \geq 0,$$

where

$$(1.10) \quad \alpha_{i,h} = \frac{1}{(\gamma_i - 1)!} \left(\gamma_i - 1 \right)_h \frac{d^{\gamma_i-1-h}}{ds^{\gamma_i-1-h}} \times \\ \times \left\{ (s - s_i)^{\gamma_i} \frac{[P(s) + Z(s)]}{M(s)} \right\} \Big|_{s=s_i},$$

$i = 1, 2, \dots, l, h = 0, 1, \dots, \gamma_i - 1.$

Proof. It is clear that when assuming the above hypotheses, for $t \geq 0$, both sides of relation (1.3) are holomorphic in the half-plane $\operatorname{re} s < \sigma$, except, perhaps, for isolated singularities at $s_1, s_2, \dots, s_l$, since the convergence of a Laplace transform in an open half-plane implies its holomorphy there ([2], vol. 2, p.144). From (1.3), by taking the inverse Laplace transform, we obtain

$$(1.11) \quad \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} S(s) e^{ts} ds = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{Z(s) + P(s)}{M(s)} e^{ts} ds.$$

The left-hand side of (1.11), by the Laplace theorem ([3]), is $x(t)$, $t \geq 0$. Its right-hand side, by assumption (iii) and the theorem on residues, becomes

$$\frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{Z(s) + P(s)}{M(s)} e^{ts} ds = \lim_{R \rightarrow +\infty} \frac{1}{2\pi j} \int_{\Gamma(R) + \Gamma^*(R)} \frac{Z(s) + P(s)}{M(s)} e^{ts} ds = \\ = \sum_{i=1}^l \operatorname{res}_{s=s_i} \left\{ \frac{[Z(s) + P(s)]}{M(s)} e^{ts} \right\},$$

where $\Gamma^*(R)$ is a line segment joining $\sigma - j \cdot R$ to $\sigma + j \cdot R$. Now, we calculate

$$\begin{aligned}
& \operatorname{res}_{s=s_1} \left\{ \frac{[P(s) + Z(s)]}{M(s)} e^{ts} \right\} = \\
& = \frac{1}{(\gamma_1 - 1)!} \frac{d^{\gamma_1-1}}{ds^{\gamma_1-1}} \left\{ (s - s_1)^{\gamma_1} \frac{[P(s) + Z(s)]}{M(s)} e^{ts} \right\} = \\
& = \frac{1}{(\gamma_1 - 1)!} \sum_{h=0}^{\gamma_1-1} \binom{\gamma_1-1}{h} \frac{d^{\gamma_1-1-h}}{ds^{\gamma_1-1-h}} \times \\
& \times \left\{ (s - s_1)^{\gamma_1} \frac{[P(s) + Z(s)]}{M(s)} \right\} t^h e^{ts} \Big|_{s=s_1}.
\end{aligned}$$

Denoting, for brevity, the coefficient at $t^h e^{ts_1}$ by $\alpha_{1,h}$, we get (1.9). This completes the proof.

2. Determination of the points of discontinuity of solutions of linear control systems

Let us consider equation (1.1) with coefficients (1.7) and with the function $z(t) = u(t)$ being piecewise constant, of bang-bang type,

$$|u| = U_{\max},$$

with the points of discontinuity $0 = t_0, t_1, \dots, t_k$ and t_k such that

$$u(t) = 0 \quad \text{for } t > t_k.$$

In this case,

$$U(s) = \int_0^{+\infty} u(t)g(s,t)dt = \sum_{i=0}^{k-1} (-1)^i \delta U_{\max} \int_{t_i}^{t_{i+1}} g(s,t)dt,$$

where δ denotes a sign of $u(t)$ for $t = 0$.

Let us consider a plant described by (1.1) and (1.7) with $z(t) = u(t)$. We want to transfer this plant from the state $x_0 = (x_0^{(0)}, x_0^{(1)}, \dots, x_0^{(n-1)})$ to the state $x_1 = (x_1^{(0)}, x_1^{(1)}, \dots$

..., $x_1^{(n-1)}$) by help of the control u (we assume that the final state is reachable). From the maximum principle of Pontriagin we know that the time-optimal control is piecewise constant and changes its value at times $t_1, t_2, \dots, t_k$. Our theorem gives the relations for $t_1, t_2, \dots, t_k$, depending on the boundary conditions and δ . From Theorem 1.1 we have the formula for the solution

$$(2.1) \quad x(t) = \sum_{i=1}^1 \sum_{h=0}^{\gamma_i-1} \alpha_{i,h} t^h e^{ts_i}, \quad t \geq 0,$$

where

$$\alpha_{i,h} = \frac{1}{(\gamma_i - 1)!} \left(\frac{\gamma_i - 1}{h} \right) \frac{d^{\gamma_i - 1 - h}}{ds^{\gamma_i - 1 - h}} \left\{ (s - s_i)^{\gamma_i} \frac{[P(s) + U(s)]}{M(s)} \right\} \Big|_{s=s_i},$$

$i = 1, 2, \dots, l, h = 0, 1, \dots, \gamma_i - 1$.

After $(n-1)$ differentiations of (2.1) with $t = t_k$ we obtain the system of n equations

$$(2.2) \quad x_1^{(m)} = \sum_{i=1}^1 \sum_{h=0}^{\gamma_i-1} \alpha_{i,h} \sum_{v=0}^m \binom{m}{v} \frac{h!}{(h-v)!} t_k^{h+m-2v} e^{t_k s_i},$$

$m = 0, 1, \dots, n-1$. System (2.2) determines $t_1, t_2, \dots, t_k$ only in the case where $k < n$.

3. Estimation of the number of switching instants in the time-optimal control

Let us consider the minimum-time control for a time-varying plant described by the equation

$$(3.1) \quad \sum_{k=0}^n a_{k+1}(t) x^{(k)}(t) = u(t),$$

where the functions $a_{k+1}(t)$, $t \geq 0$, $k = 0, 1, \dots, n$, satisfy conditions (i)-(iii) and the equations

$$(3.2) \quad a_{i+1}(t) = (-1)^{n-i} \sum_{h=i}^n C_{h+1} \binom{h}{i} a_{n+1}^{(h-i)}(t),$$

$$i = 0, 1, \dots, n;$$

$C_{n+1} = 1$ and $C_1, C_2, \dots, C_n$ are arbitrary real numbers. We suppose that the function $a_{n+1}(t) \neq 0$ for $t \geq 0$ and the control u is a piecewise continuous and bounded function. It is assumed that no constraints on the state variables are imposed. We want to transfer the above plant from the given initial state $x_0 = (x_0^{(0)}, x_0^{(1)}, \dots, x_0^{(n-1)})$ to the desired final state $x_1 = (x_1^{(0)}, x_1^{(1)}, \dots, x_1^{(n-1)})$ by help of control u in the minimum time. We assume that the initial state x_0 is such that the final state x_1 is reachable with some admissible control u . The time-optimal control u will be in the case a bang-bang one ([13]). The piecewise constant control signal $u = \pm U_{\max}$ will change its sign a finite number of times, say $t_1, t_2, \dots, t_n$, and there is an instant t_n^* such that $u(t) = 0$ for $t \geq t_n^*$ (at $t = t_n^*$, the system attains the desired final state). The points of discontinuity of the optimal control are zeros of the function $\eta(t)$ which is a solution of the differential equation

$$(3.3) \quad a_{n+1}(t)\eta^{(n)}(t) - a_n(t)\eta^{(n-1)}(t) + \dots + (-1)^n a_1(t)\eta(t) = 0.$$

Let us solve equation (3.3) with the method given in section 1. According to Theorem 1.1, this solution is given by the formula

$$(3.4) \quad \eta(t) = \sum_{i=1}^1 \sum_{h=0}^{\gamma_i-1} \alpha_{i,h} t^h e^{ts_1}, \quad t \geq 0,$$

where

$$(3.5) \quad \alpha_{i,h} = \frac{1}{(\gamma_i-1)!} \binom{\gamma_i-1}{h} \frac{d^{\gamma_i-1-h}}{ds} \left[(s-s_1)^{\gamma_i} \frac{P(s)}{M(s)} \right] \Big|_{s=s_1},$$

$$i = 1, 2, \dots, l, \quad h = 0, 1, \dots, \gamma_i-1.$$

The function $P(s)$ and $M(s)$ have the forms

$$(3.6) \quad P(s) = \sum_{k=0}^n \sum_{i=0}^{k-1} (-1)^{i+1} \frac{d^i}{dt^i} \left[(-1)^{n-k} a_{k+1}(t) g(s, t) \right] x^{(k-1-i)}(t) \Big|_{t=0}^{t=+\infty},$$

$$M(s) = C_1 + C_2 s + \dots + C_{n+1} s^n.$$

The function $\frac{P(s)}{M(s)}$ has a finite number of poles $s_1, s_2, \dots, s_l$ of order, respectively, $\gamma_1, \gamma_2, \dots, \gamma_l$. The function $g(s, t)$ is a solution, with respect to t , of the linear differential equation with constant coefficients

$$(3.7) \quad \sum_{i=0}^n (-1)^i c_{i+1} \frac{d^i}{dt^i} g(s, t) = \frac{M(s) e^{-st}}{s_{n+1}(t)}.$$

Now, our result may be formulated as follows:

Theorem 3.1. If the poles of the function $\frac{P(s)}{M(s)}$, given in (3.6), are real, then the optimal control u of system (3.1) changes its sign at most $\gamma_1 + \gamma_2 + \dots + \gamma_l - 1$ times.

Proof. By (3.3), the function $\eta(t)$, has the form

$$\eta(t) = Q_1(t) e^{ts_1} + \dots + Q_l(t) e^{ts_l}, \quad t \geq 0,$$

where the polynomials $Q_i(t)$ with real coefficients are of degrees at most $\gamma_i - 1$, $i = 1, 2, \dots, l$. Reasoning analogously as in the case of the system with constant coefficients ([1]), we get that the function $\eta(t)$ has at most $\gamma_1 + \gamma_2 + \dots + \gamma_l - 1$ real zeros. The time-optimal control has the same number of switching instants.

4. Example

Let us consider an integral transform for equation (3.3) in the case of single real zeros $\lambda_1, \lambda_2, \dots, \lambda_n$ of $M(s) =$

$= C_1 + C_2 s + \dots + C_{n+1} s^n$. Then the function $g(s, t)$, which is a solution of (3.7), may be written as follows

$$g(s, t) = \sum_{i=1}^n \beta_i(t) e^{-\lambda_i t},$$

where the function $\beta_i(t)$, $i = 1, 2, \dots, n$, satisfies the system of equations

$$\sum_{i=1}^n \dot{\beta}_i(t) (-\lambda_i)^k e^{-\lambda_i t} = \begin{cases} 0 & \text{for } k = 0, 1, \dots, n-2, \\ \frac{(-1)^n M(s) e^{-st}}{a_{n+1}(t)} & \text{for } k = n-1. \end{cases}$$

After suitable calculations we get

$$(4.1) \quad g(s, t) = \sum_{i=1}^n \frac{(-1)^n M(s) e^{-\lambda_i t}}{\prod_{\substack{k=1 \\ k \neq i}}^n (\lambda_k - \lambda_i)} \int \frac{e^{-(s-\lambda_i)t}}{a_{n+1}(t)} dt.$$

Let $n = 2$, $C_2 = -5$, $C_1 = 4$, $a_{n+1} = \frac{1}{t+1}$. Now, we estimate the number of switching instants of the time-optimal control of the system

$$(4.2) \quad \frac{\ddot{x}(t)}{t+1} + \left[\frac{-C_2}{t+1} + \frac{2}{(t+1)^2} \right] \dot{x}(t) + \left[\frac{C_1}{t+1} - \frac{C_2}{(t+1)^2} + \frac{2}{(t+1)^3} \right] x(t) = u(t)$$

with $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$, $x(T) = x_1$, $\dot{x}(T) = \dot{x}_1$. The coefficients of the above equations satisfy (3.2).

Equation (3.3) is expressed by the formula

$$(4.3) \quad \frac{\ddot{\eta}(t)}{t+1} + \left[\frac{C_2}{t+1} - \frac{2}{(t+1)^2} \right] \dot{\eta}(t) + \left[\frac{C_1}{t+1} - \frac{C_2}{(t+1)^2} + \frac{2}{(t+1)^3} \right] \eta(t) = 0.$$

The coefficients of this equation satisfy (1.7). The correspondent time-invariant equation has the form

$$C_1 g(s, t) - C_2 \frac{d}{dt} g(s, t) + \frac{d^2}{dt^2} g(s, t) = \frac{M(s)e^{-st}}{s_3(t)},$$

$$4g(s, t) + 5 \frac{d}{dt} g(s, t) + \frac{d^2}{dt^2} g(s, t) = (s-1)(s-4)(t+1)e^{-st}.$$

According to (4.1), by putting the integral constants equal to zero, we obtain

$$g(s, t) = e^{-st} \left[\frac{2s-5}{(s-1)(s-4)} + t + 1 \right].$$

In this case, the application of the integral transform of type (1.2) to both sides of (4.3) gives

$$\int_0^{+\infty} \eta(t) e^{-st} dt = \frac{P(s)}{(s-1)(s-4)}$$

where, according to (1.4)

$$\begin{aligned} \frac{P(s)}{(s-1)(s-4)} &= \frac{1}{(s-1)(s-4)} \left\{ - \frac{g(s, t)}{t+1} \dot{\eta}(t) \right|_{t=0}^{t=+\infty} + \\ &+ \frac{d}{dt} \left[\frac{g(s, t)}{t+1} \right] \eta(t) \Big|_{t=0}^{t=+\infty} - \left[\frac{-5}{t+1} - \frac{2}{(t+1)^2} \right] g(s, t) \eta(t) \Big|_{t=0}^{t=+\infty} \right\} = \\ &= \frac{(s^2-3s-1)\dot{\eta}(0) - \eta(0)(-s^3+10s^2-22s-2)}{(s-1)^2(s-4)^2}. \end{aligned}$$

The function $\frac{P(s)}{(s-1)(s-4)}$ has two real poles of order two. By Theorem 3.1, the number of switching instants of the time-optimal control of system (4.2) is at most equal to three. The exact values of these instants can be determined if one knows the solution of equation (4.3). After suitable calculations, by Theorem 1.1, we obtain

$$\begin{aligned}\eta(t) &= \operatorname{res}_{s=1} \frac{P(s)}{M(s)} e^{ts} + \operatorname{res}_{s=4} \frac{P(s)}{M(s)} e^{ts} = \\ &= \frac{1}{3} (t+1) \left[e^{4t} (\dot{\eta}(0) - 2\eta(0)) + e^t (-\dot{\eta}(0) + 5\eta(0)) \right].\end{aligned}$$

The function $\eta(t)$ has at most one positive zero. Then the time-optimal control of systems (4.2) has at most one switching instant.

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF ŁÓDŹ, 90-238 ŁÓDŹ,
POLAND

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