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ON SOME PROPERTIES OF THE SOLUTION OF THE DIRICHLET PROBLEM FOR THE HELMHOLTZ EQUATION IN THE INTERIOR AND EXTERIOR OF A CIRCLE

1. Consider the Helmholtz equation

$$(1) \quad (\Delta - c^2)u(x, y) = 0, \quad c > 0,$$

and the domain $K = \{(x, y): x^2 + y^2 \leq R^2\}$.

Let us denote by $L_{2\pi}^p$, $1 \leq p < \infty$, the class of 2π -periodic functions f with a finite norm defined as follows

$$\|f\| = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^p dx \right)^{\frac{1}{p}},$$

by $C_{2\pi}$ the class of 2π -periodic continuous functions f with the norm $\|f\| = \sup_{-\pi \leq x \leq \pi} |f(x)|$, and by $X_{2\pi}$ the class $L_{2\pi}^p$ or $C_{2\pi}$.

Let $f \in X_{2\pi}$ and $(a_n)_{n=0,1,2,\dots}$, $(b_n)_{n=1,2,3,\dots}$ denote the Fourier coefficients of the function f .

Consider the following functions

$$\begin{aligned} V(r, \varphi) &= \frac{a_0}{2} \frac{I_0(cr)}{I_0(cR)} + \\ &+ \sum_{n=1}^{\infty} \frac{I_n(cr)}{I_n(cR)} (a_n \cos n\varphi + b_n \sin n\varphi) = \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\varphi - s) \left[\frac{I_0(cr)}{I_0(cR)} + 2 \sum_{n=1}^{\infty} \frac{I_n(cr)}{I_n(cR)} \cos ns \right] ds, \quad 0 \leq r < R, \end{aligned}$$

$$\begin{aligned}
W(r, \varphi) &= \frac{a_0}{2} \frac{K_0(cr)}{K_0(cR)} + \\
&+ \sum_{n=1}^{\infty} \frac{K_n(cr)}{K_n(cR)} (a_n \cos n\varphi + b_n \sin n\varphi) = \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\varphi - s) \left[\frac{K_0(cr)}{K_0(cR)} + 2 \sum_{n=1}^{\infty} \frac{K_n(cr)}{K_n(cR)} \cos ns \right] ds, \quad R < r < \infty,
\end{aligned}$$

where I_n and K_n are the modified Bessel functions and Mac-Donald functions, respectively (see [7]).

In the papers [1], [6] it has been proved that the functions $V(r, \varphi)$ and $W(r, \varphi)$ are solutions of equation (1) in the interior and exterior of the disc K , respectively, and that the conditions

$$\lim_{r \rightarrow R^-} \|V(r, \cdot) - f(\cdot)\| = 0, \quad \lim_{r \rightarrow R^+} \|W(r, \cdot) - f(\cdot)\| = 0$$

are fulfilled. In this paper we shall prove theorems concerning the estimate of $A_f(r) = \|V(r, \cdot) - f(\cdot)\|$, $B_f(r) = \|W(r, \cdot) - f(\cdot)\|$, according to the properties of the function f , and inverse theorems which give certain characterization of f according to $A_f(r)$ and $B_f(r)$. These problems for the Laplace equation are treated in [2]-[5], [8], [11]-[14].

2. In this section we are going to prove some auxiliary results. Write

$$\alpha(n, r) = [R^n I_n(cr) - r^n I_n(cR)] (R^n I_n(cR))^{-1}.$$

We remark that

$$(2) \quad \frac{I_n(cr)}{I_n(cR)} = \left(\frac{r}{R}\right)^n + \alpha(n, r), \quad n = 0, 1, 2, \dots$$

L e m m a 1. For every positive integer n and for every real number $r \in (0, R)$ there exist a number $\theta \in (r, R)$ and a function $\psi(n, r)$ such that

$$(a) \quad \alpha(n, r) = \frac{(r-R) c^{2\theta}}{2(n+1)} \left(\frac{r}{R}\right)^n + \psi(n, r),$$

$$(b) \quad |\psi(n, r)| \leq \frac{1}{2} c^{4R^3(R-r)} \left(\frac{r}{R}\right)^n (n+1)^{-2}.$$

P r o o f . Applying the formula ([7], p.115)

$$I_n(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} z\right)^{n+2k}}{k! (n+k)!}$$

and the mean-value theorem, we have

$$\begin{aligned} \alpha(n, r) &= \left(\frac{1}{2} crR\right)^n \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} cr\right)^{2k} - \left(\frac{1}{2} cR\right)^{2k}}{k! (n+k)!} (R^n I_n(cR))^{-1} = \\ &= \left(\frac{1}{2} crR\right)^n (r-R) \sum_{k=1}^{\infty} \frac{ck \left(\frac{1}{2} c\theta\right)^{2k-1}}{k! (n+k)!} (R^n I_n(cR))^{-1} = \\ &= \frac{(r-R) c^{2\theta}}{2(n+1)} \left(\frac{r}{R}\right)^n + \psi(n, r), \end{aligned}$$

where

$$\begin{aligned} \psi(n, r) &= (r-R) c^{2\theta} \left[\left(\frac{1}{2} crR\right)^n \frac{1}{2} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} c\theta\right)^{2k}}{(k+1)! (n+k+1)!} + \right. \\ &\quad \left. - \frac{1}{2(n+1)} r^n I_n(cR) \right] (R^n I_n(cR))^{-1}. \end{aligned}$$

It is easy to see that the function ψ satisfies condition (b) of Lemma 1 what ends the proof of Lemma 1.

L e m m a 2 ([6]). For every positive integer n and every positive number z the inequality

$$K_n(z) \geq \Gamma\left(2n - \frac{1}{2}\right) \sqrt{\frac{2}{z}} (2z)^{1-n} ((n-1)! \exp z)^{-1}$$

holds. Let

$$\beta(n, r) = \left[(cr)^n K_n(cr) - (cR)^n K_n(cR) \right] \left((cr)^n K_n(cR) \right)^{-1}.$$

We remark that

$$(3) \quad \frac{K_n(cr)}{K_n(cR)} = \left(\frac{R}{r} \right)^n + \beta(n, r), \quad n = 0, 1, 2, \dots$$

L e m m a 3. For every number $n \geq 3$ and every real number $r \in (R, \infty)$ there exist a number $\bar{\theta} \in (R, r)$ and a function ψ_1 such that

$$(a) \quad \beta(n, r) = (r-R) \frac{c^2 \bar{\theta}}{2(n-1)} \left(\frac{R}{r} \right)^n + \psi_1(n, r),$$

$$(b) \quad |\psi_1(n, r)| \leq C(r-R) \left(\frac{R}{r} \right)^n (n-2)^{-2},$$

where

$$C = \left(2c^4 \left(\frac{r}{R} \right)^2 + c^4 r \right) \sqrt{\pi c R} \exp(1+cR) (16 \sqrt{2} R)^{-1}.$$

P r o o f . Applying the formulas ([7] p.117)

$$\frac{d}{dz} \left[z^v K_v(z) \right] = -z^v K_{v-1}(z),$$

$$K_v(z) = K_{v-2}(z) + \frac{2(v-1)}{z} K_{v-1}(z)$$

and the mean-value theorem we obtain

$$\begin{aligned} (4) \quad \beta(n, r) &= -c(r-R)(c\bar{\theta})^n K_{n-1}(c\bar{\theta}) \left((cr)^n K_n(cR) \right)^{-1} = \\ &= (r-R) \left[\frac{1}{2(n-1)} c^2 \bar{\theta} (cR)^n K_n(cR) + c(c\bar{\theta})^n K_{n-1}(c\bar{\theta}) + \right. \\ &\quad \left. - \frac{1}{2(n-1)} c^2 \bar{\theta} (cR)^n K_n(cR) \right] \left((cr)^n K_n(cR) \right)^{-1} = \\ &= (r-R) \frac{c^2 \bar{\theta}}{2(n-1)} \left(\frac{R}{r} \right)^n + (r-R) \left[c^2 \bar{\theta} \left((c\bar{\theta})^{n-1} K_{n-1}(c\bar{\theta}) + \right. \right. \end{aligned}$$

$$\begin{aligned}
& - (cR)^{n-1} K_{n-1}(cR) - \frac{(cR)^n}{2(n-1)} K_{n-2}(cR) \Big] \left((cr)^n K_n(cR) \right)^{-1} = \\
& = (r-R) \frac{c^2 \bar{\theta}}{2(n-1)} \left(\frac{R}{r} \right)^n + \left[(r-R) c^3 \bar{\theta} (R-\bar{\theta}) (c\bar{\theta})^{n-1} K_{n-2}(c\bar{\theta}) + \right. \\
& \left. + c^2 \bar{\theta} (R-r) \frac{(cR)^n}{2(n-1)} K_{n-2}(cR) \right] \left((cr)^n K_n(cR) \right)^{-1},
\end{aligned}$$

where $\bar{\theta} \in (R, \bar{\theta})$. By the formulas ([10] p.169,365)

$$K_p(xz) = \frac{\Gamma(p + \frac{1}{2})(2z)^p}{x^p \Gamma(\frac{1}{2})} \int_0^\infty \frac{\cos xt}{(t^2 + z^2)^{p + \frac{1}{2}}} dt,$$

$$p \geq -\frac{1}{2}, \quad x > 0, \quad z > 0,$$

$$\int_0^\infty \frac{x^{g-1}}{(p+qx^c)^{h+1}} dx =$$

$$= \Gamma\left(h + 1 - \frac{g}{c}\right) \Gamma\left(\frac{p}{qc} + q\right) \Gamma\left(\frac{g}{c}\right) c^{g-1} p^{-h-1} \left(\Gamma\left(\frac{p}{qc}\right) \Gamma(h+1)\right)^{-1}$$

$$0 < \frac{g}{c} < h+1,$$

and by Lemma 2, we get

$$\begin{aligned}
(5) \quad & \frac{K_{n-2}(cR)}{K_n(cR)} = \\
& = \left(K_n(cR)\right)^{-1} \frac{\Gamma\left(n - \frac{3}{2}\right)(2c)^{n-2}}{R^{n-2} \Gamma\left(\frac{1}{2}\right)} \int_0^\infty \frac{\cos Rt}{(t^2 + c^2)^{n - \frac{3}{2}}} dt \leq \\
& \leq \frac{\Gamma\left(n - \frac{3}{2}\right)(2c)^{n-2}}{\Gamma\left(\frac{1}{2}\right) R^{n-2} K_n(cR)} \int_0^\infty \frac{dt}{(t^2 + c^2)^{n - \frac{3}{2}}} \leq \\
& \leq \frac{2^{2n-4} (n-1)! (n-3)! c^2 \sqrt{cR} \exp(cR)}{\sqrt{2} R \Gamma\left(2n - \frac{1}{2}\right)}.
\end{aligned}$$

By virtue of the equality ([7] p.9)

$$2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) = \sqrt{\pi} \Gamma(2z)$$

and of the Stirling formula

$$n! = n^n \sqrt{2\pi n} \exp\left(-n + \frac{\tilde{\theta}}{12n}\right), \quad 0 < \tilde{\theta} < 1,$$

we have

$$(6) \quad \frac{(n-1)! (n-3)!}{\Gamma\left(2n - \frac{1}{2}\right)} \leq e 2^{-2n} (n-2)^{-2}.$$

Further,

$$(7) \quad \frac{K_{n-2}(cR)}{K_n(cR)} \leq \bar{C} (n-2)^{-2},$$

where $\bar{C} = c^2 \sqrt{\pi cR} \exp(1+cR) (16 \sqrt{2} R)^{-1}$.

The function $z^n K_n(z)$ is decreasing in the interval $(0, \infty)$. Hence, by (4) and (7), we get

$$\beta(n, r) = (r-R) \frac{c^2 \bar{C}}{2(n-1)} \left(\frac{R}{r}\right)^n + \psi_1(n, r),$$

where $\psi_1(n, r)$ satisfies the inequality

$$|\psi_1(n, r)| \leq (r-R) \left[2c^2 \left(\frac{r}{R}\right)^2 \left(\frac{R}{r}\right)^n + c^2 r \left(\frac{R}{r}\right)^n \right] \bar{C} (n-2)^{-2},$$

what ends the proof of Lemma 3.

It is easy to verify the successive lemmas.

L e m m a 4. The following inequalities hold:

$$\left| \frac{I_0(cr)}{I_0(cR)} - 1 \right| \leq (R-r) \frac{1}{2} c^2 R \quad \text{for } r \in (0, R),$$

$$\left| \frac{K_0(cr)}{K_0(cR)} - 1 \right| \leq (r-R) \circ \frac{K_1(cR)}{K_0(cR)} \quad \text{for } r \in (R, \infty),$$

$$\left| \frac{K_1(cr)}{K_1(cR)} - \frac{R}{r} \right| \leq (r-R) c \frac{R}{r} \quad \text{for } r \in (R, \infty),$$

$$\left| \frac{K_2(cr)}{K_2(cR)} - \left(\frac{R}{r} \right)^2 \right| \leq (r-R) c \left(\frac{R}{r} \right)^2 \quad \text{for } r \in (R, \infty),$$

L e m m a 5 ([9]). If $(a_n)_{n=0,1,\dots}$ is a convex sequence (i.e. $a_n - 2a_{n+1} + a_{n+2} \geq 0$ for $n = 0, 1, \dots$) and $\lim_{n \rightarrow \infty} a_n = 0$, then

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \geq 0$$

for every $x \neq 0 \pmod{2\pi}$.

By Lemma 5 and the Hölder-Minkowski inequality, we get

L e m m a 6. If $f \in X_{2\pi}$, then

$$\left\| \frac{1}{\pi} \int_0^{2\pi} f(\cdot - s) \sum_{n=1}^{\infty} \frac{(r-R)c^{2\bar{\theta}}}{2(n+1)} \left(\frac{r}{R} \right)^n \cos ns \, ds \right\| \leq 2 \|f\| c^{2\bar{\theta}}(R-r)$$

for $r \in (0, R)$.

L e m m a 7. If $f \in X_{2\pi}$, then

$$\left\| \frac{1}{\pi} \int_0^{2\pi} f(\cdot - s) \sum_{n=3}^{\infty} \frac{(r-R)c^{2\bar{\theta}}}{2(n-1)} \left(\frac{r}{R} \right)^n \cos ns \, ds \right\| \leq 9 \|f\| c^{2\bar{\theta}}(r-R)$$

for $r \in (R, \infty)$.

P r o o f . Let $b_0 = 1$, $b_1 = \frac{3}{4}$, $b_n = \frac{1}{2(n-1)}$, $n=2, 3, \dots$.

Then the sequence $\left(c^{2\bar{\theta}} \left(\frac{R}{r} \right)^n b_n \right)_{n=0,1,\dots}$ is convex.

By Lemma 5 and the Hölder-Minkowski inequality, we have

$$\left\| \frac{1}{\pi} \int_0^{2\pi} f(\cdot - s) \sum_{n=3}^{\infty} \frac{(r-R)c^{2\bar{\theta}}}{2(n-1)} \left(\frac{R}{r} \right)^n \cos ns \, ds \right\| =$$

$$= c^{2\bar{\theta}} \left\| \frac{1}{\pi} \int_0^{2\pi} f(\cdot - s) \left[b_0 + 2 \sum_{n=1}^{\infty} b_n \left(\frac{R}{r} \right)^n \cos ns \right] ds + \right. \\ \left. - \frac{1}{\pi} \int_0^{2\pi} f(\cdot - s) \left[b_0 + 2 \sum_{n=1}^2 b_n \left(\frac{R}{r} \right)^n \cos ns \right] ds \right\| \leq 9 \|f\| c^{2\bar{\theta}}(r-R).$$

This ends the proof of Lemma 7.

3. Let $f \in X_{2\pi}$ and $w_2(f, t)$ denote the second modulus of smoothness of the function f with respect to metric $X_{2\pi}$ and, further, let w denote a function of the second modulus of smoothness type satisfying the Dini condition (see [15] p.115)

$$(8) \quad \int_0^{1-\varphi} \frac{w(t)}{t} dt < \infty, \quad 0 < \varphi < 1.$$

Let $X_{2\pi}^w$ denote the set of all functions $f \in X_{2\pi}$ for which

$$(9) \quad w_2(f, t) \leq w(t).$$

Theorem 1. If $f \in X_{2\pi}^w$, then

$$\|V(r, \cdot) - f(\cdot)\| \leq \int_0^{1-\varphi} \frac{w(t)}{t} dt + \frac{\pi}{4\varphi} (1-\varphi) \int_{1-\varphi}^{\pi} \frac{w(t)}{t^2} dt + C_1(1-\varphi),$$

where $\varphi = \frac{r}{R}$ and $C_1 = \left[\frac{5}{2} R^2 c^2 + \frac{1}{2} c^4 R^4 \left(\frac{\pi^2}{6} - 1 \right) \right] \|f\|$.

Proof. By the simple inequalities

$$\frac{1}{2\pi} \frac{1 - \varphi^2}{1 - 2\varphi \cos t + \varphi^2} \leq \frac{1}{t} \quad \text{for } t \in (0, 1-\varphi), \\ \frac{1}{2\pi} \frac{1 - \varphi^2}{1 - 2\varphi \cos t + \varphi^2} \leq \frac{\pi(1-\varphi)}{4\varphi t^2} \quad \text{for } t \in (1-\varphi, \pi)$$

and by (2), we have

$$\begin{aligned}
 (10) \quad \|V(r, \cdot) - f(\cdot)\| &= \left\| \frac{1}{2\pi} \int_0^{2\pi} f(\cdot - t) \frac{1 - \varrho^2}{1 - 2\varrho \cos t + \varrho^2} dt + \right. \\
 &\quad + \frac{1}{2\pi} \int_0^{2\pi} f(\cdot - t) \sum_{n=1}^{\infty} \alpha(n, r) \cos nt \, dt + \\
 &\quad \left. + \frac{1}{2\pi} \int_0^{2\pi} f(\cdot - t) \left(\frac{I_0(cr)}{I_0(cR)} - 1 \right) dt - f(\cdot) \right\| \leq \\
 &\leq \left\| \frac{1}{2\pi} \int_0^{\pi} (f(\cdot - t) - 2f(\cdot) + f(\cdot + t)) \frac{1 - \varrho^2}{1 - 2\varrho \cos t + \varrho^2} dt \right\| + M(r) \leq \\
 &\leq \int_0^{1-\varrho} \frac{w(t)}{t} dt + \frac{\pi(1-\varrho)}{4\varrho} \int_{1-\varrho}^{\pi} \frac{w(t)}{t^2} dt + M(r),
 \end{aligned}$$

where

$$\begin{aligned}
 M(r) &= \left\| \frac{1}{2\pi} \int_0^{2\pi} f(\cdot - t) \sum_{n=1}^{\infty} \alpha(n, r) \cos nt \, dt \right\| + \\
 &\quad + \left\| \frac{1}{2\pi} \int_0^{2\pi} f(\cdot - t) \left(\frac{I_0(cr)}{I_0(cR)} - 1 \right) dt \right\|.
 \end{aligned}$$

Applying Lemmas 1, 4, 6 we obtain

$$M(r) \leq (1-\varrho) \|f\| \left(\frac{1}{2} c^4 R^4 \left(\frac{\pi^2}{6} - 1 \right) + \frac{5}{2} c^2 R^2 \right),$$

what ends the proof of Theorem 1.

Similarly, by (3) and Lemmas 3, 4, 7, one can prove the following theorem.

Theorem 2. If $f \in X_{2\pi}^w$ and $\bar{\varphi} = \frac{R}{r}$, then

$$\|w(r, \cdot) - f(\cdot)\| \leq \int_0^{1-\bar{\varphi}} \frac{w(t)}{t} dt + \frac{\pi(1-\bar{\varphi})}{4\bar{\varphi}} \int_{1-\bar{\varphi}}^{\pi} \frac{w(t)}{t^2} dt + C_2(r)(1-\bar{\varphi}),$$

where

$$C_2(r) = \left[9c^2r + \left(2c^2 \left(\frac{r}{R} \right)^2 + c^2r \right) \frac{c^2 \sqrt{\pi cR} \exp(1+cR)}{16\sqrt{2} R} \frac{\pi^2}{6} + c \frac{K_1(cR)}{K_0(cR)} + 2c \right] r \|f\|.$$

4. In this section we are going to give the inverse approximation theorems. First, we prove an inequality of the Bernstein type.

Lemma 8. If $f \in X_{2\pi}$, then

$$\left\| \frac{\partial^2}{\partial \varphi^2} v(r, \cdot) \right\| \leq \frac{C_3 \|f\|}{(1-\varphi)^2},$$

where $\varphi = \frac{r}{R}$, $C_3 = 4 + \frac{(cR)^2}{2} + \frac{(cR)^4}{2}$.

Proof. By the inequality ([2] p.110)

$$\left\| \frac{\partial^2}{\partial u^2} \frac{1}{2\pi} \int_0^{2\pi} f(\cdot - t) \frac{1 - \varphi^2}{1 - 2\varphi \cos t + \varphi^2} dt \right\| \leq \frac{4 \|f\|}{(1-\varphi)^2}$$

and by formula (2) and Lemma 1, we have

$$\left\| \frac{\partial^2}{\partial \varphi^2} v(r, \cdot) \right\| \leq \frac{4 \|f\|}{(1-\varphi)^2} + \frac{R^2 c^2 \|f\|}{2(1-\varphi)} + \frac{1}{2} c^4 R^4 \|f\| \leq \frac{C_3 \|f\|}{(1-\varphi)^2}.$$

This ends the proof of Lemma 8.

Similarly we can prove the following lemma.

Lemma 9. If $f \in X_{2\pi}$, then

$$\left\| \frac{\partial^2}{\partial \varphi^2} w(r, \cdot) \right\| \leq \frac{C_4(r) \|f\|}{(1-\bar{\varphi})^2},$$

where $\bar{\varphi} = \frac{R}{r}$, $C_4(r) = 4 + r^2 c^2 + rc + 4c + \left(2c^2 \left(\frac{r}{R}\right)^2 + c^2 r\right) \times$
 $\times \frac{c^2 \sqrt{\pi c R} \exp(1+cR)}{16 \sqrt{2} R} 3r$.

Applying the results of paper [4] and Lemma 8 we obtain

Theorem 3. If $f \in X_{2\pi}$ and w is a function of the second modulus of smoothness type satisfying the inequality

$$\|U(r, \cdot) - f(\cdot)\| \leq w(1-\varphi), \quad \varphi = \frac{r}{R},$$

then the second modulus of smoothness $w_2(f, t)$ of the function f satisfies the inequality

$$w_2(f, t) \leq At^2 \int_t^{\pi} \frac{w(u)}{u^2} du, \quad 0 < t < \frac{1}{2},$$

where A is a positive constant independent of t .

Theorem 4. If $f \in X_{2\pi}$ and w is a function of the second modulus of smoothness type satisfying the inequality

$$\|W(r, \cdot) - f(\cdot)\| \leq w(1-\bar{\varphi}), \quad \bar{\varphi} = \frac{R}{r},$$

then the second modulus of smoothness $w_2(f, t)$ of the function f satisfies the inequality

$$w_2(f, t) \leq Bt^2 \int_t^{\pi} \frac{w(u)}{u^2} du, \quad 0 < t < \frac{1}{2},$$

where B is positive constant independent of t .

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