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QUALITATIVE INVESTIGATIONS OF SOLUTIONS OF SOME HYPERBOLIC EQUATIONS WITH PERIODIC COEFFICIENTS

1. In this paper, using the results in [3], [4] we shall consider behaviour of solutions of hyperbolic equations with periodic coefficients.

Consider in the domain $G = \{(x,y): 0 < x < 1, t > 0\}$ the equation

$$(1) \quad u_{tt} + a(t)u_t + b(t)u = u_{xx} + c(x,t),$$

where $a(t+\pi) = a(t)$, $b(t+\pi) = b(t)$, $a, b \in L^2(0,\pi)$, $c(x,t+\pi) = c(x,t)$, $c \in L^2([0,1] \times (0,\pi))$, with the initial conditions

$$(IC) \quad u(x,0) = f(x), \quad u_t(x,0) = F(x)$$

and the boundary conditions

$$(BC) \quad u(0,t) = u(1,t) = 0.$$

In [2] this problem is considered when the involved equation has continuous coefficients and the assumption $a(t) \geq \alpha > 0$ is imposed to assure the boundedness and stability of the solutions. In the present paper, by the periodicity of coefficients this assumption can be replaced by

$$\int_0^\pi a(t)dt > 0.$$

Moreover the case $a(t) \equiv 0$ is also investigated. As in [2] being not interested in consideration of the uniqueness of solution we always assume this property of the problem.

In the sequel all derivatives are understood in the Sobolev sense and the boundary values of functions in Sobolev spaces are referred to as the traces (see [1]).

Denote $G_T = \{(x,y) : x \in (0,1), t \in (0,T)\}$.

D e f i n i t i o n 1. Let a sequence of functions u_k be given so that

(i) u_k is an almost everywhere solution of equation

$$u_{ktt} + a(t)u_{kt} + b(t)u_k = u_{kxx} + c_k(x,t)$$

and satisfy (BC) (see [1]),

(ii) $u_k|_{t=0} \rightarrow f$ as $k \rightarrow \infty$ in $H^1(0,1)$,

$$\frac{\partial u_k}{\partial t}|_{t=0} \rightarrow F$$
 as $k \rightarrow \infty$ in $L^2(0,1)$,

$$c_k \rightarrow c \text{ as } k \rightarrow \infty \text{ in } L^2(G_T) .$$

We shall say that a function u is a generalized solution of the problem (1), (IC), (BC) in the domain G if it is the limit of u_k in $H^1(G_T)$ for every $t > 0$.

D e f i n i t i o n 2. A generalized solution u of the problem (1), (IC), (BC) is said to be stable if for each $\epsilon > 0$ there exists $\delta > 0$ such that for every $\bar{f}$ and for every $\bar{F}$ satisfying

$$\|f - \bar{f}\|_{H^1(0,1)} \leq \delta , \quad \|F - \bar{F}\|_{L^2(0,1)} \leq \delta$$

and if $\bar{u}$ is a generalized solution of the problem (1), (IC),

(BC), where

$$(IC) \quad \bar{u}(x, 0) = \bar{f}(x) \quad , \quad \bar{u}_t(x, 0) = \bar{F}(x) \quad ,$$

we have for all $t \geq 0$

$$\|u - \bar{u}\|_{H^1(0,1)}^2 + \left\| \frac{\delta}{\delta t}(u - \bar{u}) \right\|_{L^2(0,1)}^2 \leq \varepsilon \quad .$$

Definition 3. A generalized solution of the problem (1), (IC) and (BC) is said to be asymptotically stable if it is stable and

$$\lim_{t \rightarrow \infty} \left(\|u - \bar{u}\|_{H^1(0,1)}^2 + \left\| \frac{\delta}{\delta t}(u - \bar{u}) \right\|_{L^2(0,1)}^2 \right) = 0 \quad .$$

We shall say that a function φ satisfies $B(r)$, $r \geq 0$, if

$$(i) \quad \varphi \in H^r(0,1) \quad ,$$

$$(ii) \quad \varphi(0) = \varphi(1) = 0 \quad ,$$

$$(iii) \quad \varphi^{2k}(0) = \varphi^{2k}(1) = 0 \quad \text{for } 0 < 2k < r \quad , \quad k \in \mathbb{N} \quad .$$

Lemma 1. If φ satisfies $B(r)$, then the series

$$\sum_{n=1}^{\infty} |p_n| n^{r-1} \quad , \quad \sum_{n=1}^{\infty} p_n^2 n^{2r}$$

$$\sum_{n=1}^{\infty} p_n^2 n^{2r} \leq \frac{2^{1-2r-1}}{\pi^{2r}} \left\| \varphi^{(r)}(x) \right\|_{L^2(0,1)}^2 \quad ,$$

where

$$p_n = \frac{2}{1} \int_0^1 \varphi(x) \sin \frac{n\pi}{1} x dx \quad .$$

Proof. Integrating r times the expression p_n by parts we get

$$p_n = R(r) \frac{1^r}{n^r \pi^r} \int_0^1 \varphi^{(r)}(x) S(x) dx \quad ,$$

where $R(r) = \frac{1}{1}$ and

$$S(x) = \begin{cases} \sin \frac{n\pi}{1} x & \text{for } r = 2k, \quad k \in \mathbb{N}, \\ \cos \frac{n\pi}{1} x & \text{for } r = 2k-1, \quad k \in \mathbb{N}. \end{cases}$$

Therefore

$$p_n \frac{n^r \pi^r}{1^r} R(r) = \frac{2}{1} \int_0^1 \varphi^{(r)}(x) S(x) dx.$$

Regarding the right-hand side as coefficient of the Fourier expansion of the function $\varphi^{(r)}$ we get the desired result.

L e m m a 2. Let c belong to $H^{r,0}(G_\Pi)$ satisfy r (t is regarded as a parameter) and

$c(x, t+\pi) = c(x, t)$. Then the series $\sum_{n=1}^{\infty} C_n n^{r-1}$, $\sum_{n=1}^{\infty} C_n^2 n^{2r}$ are convergent, where

$$C_n = \int_0^\pi |c_n(t)| dt, \quad c_n(t) = \frac{2}{1} \int_0^1 c(x, t) \sin \frac{n\pi}{1} x dx.$$

P r o o f. By Lemma 1, for almost all $t \in (0, \pi)$ the series $\sum_{n=1}^{\infty} c_n^2(t) n^{2r}$ is convergent and

$$\sum_{n=1}^{\infty} c_n^2(t) n^{2r} \leq 2 \frac{1^{2r-1}}{\pi^{2r}} \left\| \frac{\partial^r}{\partial x^r} c(x, t) \right\|_{L^2(0,1)}^2.$$

From this we have

$$\sum_{n=1}^{\infty} \int_0^\pi c_n^2(t) n^{2r} dt \leq 2 \frac{1^{2r-1}}{\pi^{2r}} \left\| \frac{\partial^r}{\partial x^r} c(x, t) \right\|_{L^2(G_\Pi)}^2.$$

Applying Cauchy inequality we get

$$\sum_{n=1}^{\infty} C_n n^{r-1} \leq \frac{\pi^{\frac{3}{2}}}{\sqrt{6}} \left(\sum_{n=1}^{\infty} \int_0^\pi c_n^2(t) n^{2r} dt \right)^{\frac{1}{2}},$$

$$\sum_{n=1}^{\infty} c_n^2 2^n \leq \pi \sum_{n=1}^{\infty} \int_0^{\pi} c_n^2(t) n^{2n} dt .$$

2. Consider the following homogeneous equation

$$(2) \quad v_{tt} + a(t)v_t + b(t)v = v_{xx}$$

with the initial conditions

$$(IC) \quad v(x,0) = f(x) \quad , \quad v_t(x,0) = F(x) \quad ,$$

and the boundary conditions

$$(BC) \quad v(0,t) = v(1,t) = 0 \quad .$$

Theorem 1. If the functions a , b satisfy the hypotheses of Theorem 1 in [4], f satisfies $BC(1)$ and F satisfies $BC(0)$, then for the problem (2), (IC), (BC) there exists a generalized solution, which is continuous, bounded in $\bar{G}$ and asymptotically stable.

Proof. We shall seek a solution of (2) in the form

$$(3) \quad v(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)$$

so that each function $X_n T_n$ is an almost everywhere solution of (1), (BC). From this

$$(4) \quad X_n(x) = \sin \frac{n\pi}{1} x \quad , \quad T_n(t) = p_{n1} T_{n1}(t) + p_{n2} T_{n2}(t) \quad ,$$

where T_{n1} , T_{n2} are the solutions of the equation

$$(5) \quad T'' + a(t)T' + \left[\frac{n^2 \pi^2}{1^2} + b(t) \right] T = 0$$

satisfying

$$(6) \quad T_{n1}(0) = 1 \quad , \quad T'_{n1}(0) = 0 \quad , \quad T_{n2}(0) = 0 \quad , \quad T'_{n2}(0) = 1 \quad .$$

By Theorem 1 in [4] there exists a constant $D > 0$ independent of n such that the solutions T_{n1} , T_{n2} can be estimated as follows

$$(7) \quad \begin{cases} |T_{n1}(t)| \leq D e^{-st} & , \quad |T'_{n1}(t)| \leq D n e^{-st} \\ |T_{n2}(t)| \leq D n^{-1} e^{-st} & , \quad |T'_{n2}(t)| \leq D e^{-st} \end{cases}$$

Choose p_{n1} , p_{n2} so that v satisfies (IC). By (3)-(6) we have

$$p_{n1} = \frac{2}{1} \int_0^1 f(x) \sin \frac{n\pi}{1} x \, dx \quad , \quad p_{n2} = \frac{2}{1} \int_0^1 F(x) \sin \frac{n\pi}{1} x \, dx$$

Put $v_k(x, t) = \sum_{n=1}^k X_n(x) T_n(t)$. It is easy to check that v_k is an almost everywhere solution of (2) satisfying (BC) and

$$v_k|_{t=0} \longrightarrow f \quad \text{as } k \longrightarrow \infty \quad \text{in } H^1(0,1) \quad ,$$

$$\frac{\delta v_k}{\delta t} \Big|_{t=0} \longrightarrow F \quad \text{as } k \longrightarrow \infty \quad \text{in } L^2(0,1) \quad ,$$

By (3), (4), (7) we have

$$\begin{aligned} \|v_k - v_j\|_{H^1(0,1)}^2 &= \int_0^1 \left\{ \left(\sum_{n=j+1}^k X_n T_n \right)^2 + \left(\sum_{n=j+1}^k X'_n T_n \right)^2 \right\} dx = \\ &= \int_0^1 \sum_{n=j+1}^k T_n^2 (X_n^2 + X_n'^2) dx \leq \\ &\leq 1 D^2 \sum_{n=j+1}^k (p_{n1}^2 + \frac{1}{n^2} p_{n2}^2) (1 + \frac{n^2 \pi^2}{1^2}) \quad , \end{aligned}$$

$$\left\| \frac{\delta}{\delta t} (v_k - v_j) \right\|_{L^2(0,1)}^2 = \int_0^1 \left(\sum_{n=j+1}^k X_n T'_n \right)^2 dx = \sum_{n=j+1}^k \int_0^1 X_n^2 T_n'^2 \leq$$

$$\leq D^2 \sum_{n=j+1}^k (p_{n1}^2 n^2 + p_{n2}^2) .$$

Hence for arbitrary positive T we have

$$\|v_k - v_j\|_{H^1(G_T)}^2 \leq D^2 \sum_{n=j+1}^k (n^2 p_{n1}^2 + p_{n2}^2) \left(1 + \frac{1}{n^2} + \frac{\pi^2}{1^2}\right).$$

By Lemma 1 the sequence v_k is convergent in $H^1(G_T)$. On the other hand, for all $(x, t) \in \bar{G}_T$,

$$|X_n(x) T_n(t)| \leq D \left(|p_{n1}| + \frac{1}{n} |p_{n2}| \right).$$

So, the sequence v_k is uniformly convergent in $\bar{G}_T$ and therefore the function v is continuous and bounded by the constant $D \sum_{n=1}^{\infty} \left(|p_{n1}| + \frac{1}{n} |p_{n2}| \right)$. This constant is independent of T , so v is bounded in $\bar{G}$.

Let $\bar{v}$ be a generalized solution of (2), (IC), (BC). Similarly we have

$$\begin{aligned} & \|v - \bar{v}\|_{H^1(0,1)}^2 + \left\| \frac{\delta}{\delta t} (v - \bar{v}) \right\|_{L^2(0,1)}^2 \leq \\ & \leq D^2 \left(2 + \frac{\pi^2}{1^2} \right) \left\{ \left\| (v - \bar{v}) \right\|_{t=0}^2_{H^1(0,1)} + \right. \\ & \quad \left. + \left\| \frac{\delta}{\delta t} (v - \bar{v}) \right\|_{t=0}^2_{L^2(0,1)} \right\} e^{-\epsilon t}, \end{aligned}$$

which implies the asymptotically stability of the solution v .

3. Now we consider the following nonhomogeneous equation

$$(1) \quad u_{tt} + a(t)u_t + b(t)u = u_{xx} + c(x, t) .$$

Theorem 2. If the functions a , b , f and F satisfy the hypotheses of Theorem 1 and c satisfies $BC0$ (t is regarded as a parameter), then for the problem (1), (IC), (BC) there exists a generalized solution, which is continuous, bounded in $\bar{G}$ and asymptotically stable.

Proof. We shall seek a solution of this problem in the form $u(x,t) = v(x,t) + U(x,t)$, where v is the solution of (2), (IC), (BC) and U is the solution of (1) satisfying

$$(IC)' \quad U(x,0) = U_t(x,0) = 0,$$

$$(BC) \quad U(0,t) = U(1,t) = 0.$$

By Theorem 1 it suffices to prove that there exists a solution of (1), (IC)', (BC) which is continuous and bounded in $\bar{G}$. We seek a solution of (1) in the form

$$U(x,t) = \sum_{n=1}^{\infty} T_{n0}(t) \sin \frac{n\pi}{1}x,$$

where T_{n0} is the solution of the equation

$$T'' + a(t)T' + \left(\frac{n^2\pi^2}{1^2} + b(t) \right) T = c_n(t)$$

$$\text{with } c_n(t) = \frac{2}{1} \int_0^1 c(x,t) \sin \frac{n\pi}{1}x \, dx, \text{ satisfying}$$

$$T_{n0}(0) = T'_{n0}(0) = 0. \text{ It is an easy matter to see that}$$

$U_k(x,t) = \sum_{n=1}^k T_{n0}(t) \sin \frac{n\pi}{1}x$ is an almost everywhere solution of the equation

$$u_{tt} + a(t)u_t + b(t)u = u_{xx} + \sum_{n=1}^k c_n(t) \sin \frac{n\pi}{1}x$$

satisfying (IC)', (BC) and $\sum_{n=1}^k c_n(t) \sin \frac{n\pi}{l}x \rightarrow c(x,t)$ as $k \rightarrow \infty$ in $L^2(G_\Pi)$.

Due to Theorem 2 in [4] we have

$$|T_{n0}(t)| \leq KC_n n^{-1}, \quad |T'_{n0}(t)| \leq KC_n,$$

where K is a constant independent of n and

$$C_n = \int_0^\Pi |c_n(t)| dt.$$

By Lemma 2 the series

$$\sum_{n=1}^{\infty} C_n n^{-1}, \quad \sum_{n=1}^{\infty} C_n^2$$

are convergent. The remainder is proved as in the proof of Theorem 1.

4. Consider the cases where $a \equiv 0$:

$$(8) \quad u_{tt} + b(t)u = u_{xx},$$

$$(9) \quad u_{tt} + b(t)u = u_{xx} + c(x,t).$$

We shall say that a function b satisfies $A(r)$, $r \geq 0$, if there exists a constant $K > 0$ independent of n such that the solutions y_{n1} , y_{n2} , satisfying

$$y_{n1}(0) = 1, \quad y'_{n1}(0) = 0,$$

$$y_{n2}(0) = 0, \quad y'_{n2}(0) = 1$$

of the equations $y'' + \left[\frac{n^2\pi^2}{l^2} + b(t) \right] y = 0$ can be estimated as follows

$$|y_{n1}(t)| \leq K n^r, \quad |y'_{n1}(t)| \leq K n^{r+1},$$

$$|y_{n2}(t)| \leq K n^{r-1}, \quad |y'_{n2}(t)| \leq K n^r.$$

We shall say that a function b satisfies $A(r, s)$, $r \geq 0$, $s \geq -1$, if it satisfies $A(r)$ and the solutions z_{n0} satisfying

$$z_{n0}(0) = z'_{n0}(0) = 0$$

of the equations $z'' + \left(\frac{n^2 \pi^2}{l^2} + b(t) \right) z = c_n(t)$ can be estimated as follows

$$|z_{n0}(t)| \leq K C_n n^{r+s}, \quad |z'_{n0}(t)| \leq K C_n n^{r+s+1},$$

$$\text{where } C_n = \int_0^\pi |c_n(t)| dt.$$

It is added that in [3] sufficient conditions for b to satisfy $A(r)$ and $A(r, s)$ are established.

Theorem 3. Let b satisfy $A(r)$, f satisfy $B(r+1)$ and F satisfy $B(r)$. Then the problem (8), (IC), (BC) has a generalized solution which is continuous and bounded in $\bar{G}$. Moreover, if $r = 0$, it is stable.

Theorem 4. Let b satisfy $A(r, s)$, f satisfy $B(r+1)$ and F satisfy $B(r)$, c satisfy $B(r+s+1)$ (t is regarded as a parameter) and $c \in H^{r+s+1, 0}(G_n)$. Then the problem (9), (IC), (BC) has a generalized solution which is continuous and bounded in $\bar{G}$. Moreover, if $r = 0$ it is stable.

The proofs of Theorems 3-4 are similar to the proofs of Theorems 1-2.

In a similar way we can consider the problems with the

boundary conditions

$$u(0,t) = u_x(1,t) = 0$$

or

$$u_x(0,t) = u_x(1,t) = 0.$$

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