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ON THE HOLOMORPHIC SOLUTIONS OF CERTAIN DIFFERENTIAL EQUATIONS OF FIRST ORDER FOR THE MAPPINGS OF THE UNIT BALL IN $\mathbb{C}^n$ INTO $\mathbb{C}^n$

In this paper we consider the problem of the existence of the holomorphic solutions of the differential equation $Df(z)h(z) = Dh(0)f(z)$ for $z \in B^n$ and the problem of the uniqueness of the solutions for this equation with the initial conditions $f(0) = 0$ and $Df(0) = I$.

1° Introduction

Let $\mathbb{C}^n$ denote the space of n complex variables ($z \in \mathbb{C}^n$ if and only if $z = (z_1, \dots, z_n)$ and $z_j \in \mathbb{C}$ for $j = 1, 2, \dots, n$) with Euclidean inner product $\langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j$ and norm $\|z\| = \sqrt{\langle z, z \rangle}$. The ball $\{z: \|z\| < r\}$ is denoted by B_r^n and open unit ball by B^n . Let $L_s^k(\mathbb{C}^n, \mathbb{C}^n)$ denote the space of k -linear and symmetrical operators from $\mathbb{C}^n$ into $\mathbb{C}^n$ (see [2], p. 6). The letter I represent the identity map on $\mathbb{C}^n$.

The class of holomorphic mappings

$$f(z) = (f_1(z), \dots, f_n(z)), \quad z \in B^n$$

from the unit ball B^n in $\mathbb{C}^n$ into $\mathbb{C}^n$ is denoted by $H(B^n)$.

The following classes of mappings are defined

$$N = \{h \in H(B^n), h(0) = 0, \operatorname{re} \langle h(z), z \rangle > 0 \text{ for } z \in B^n, z \neq 0\},$$

$$N_0 = \{h \in H(B^n), h(z) = (z_1 \tilde{h}_1(z), \dots, z_n \tilde{h}_n(z)), \text{ where } \operatorname{re}(\tilde{h}_j(z)) > 0 \text{ for } j = 1, 2, \dots, n \text{ and } z \in B^n\}.$$

Naturalny, $N_0 \subset N$.

In addition, the following notations are introduced

$$m(h) = \inf_{\|z\|=1} \operatorname{re} \langle Dh(0)z, z \rangle, \quad M(h) = \|Dh(0)\|,$$

$$\alpha(h) = \left[\frac{M(h)}{m(h)} \right] \quad \text{for } h \in H(B^n)$$

where the symbol $\left[\frac{M(h)}{m(h)} \right]$ denotes integer part of the number $\frac{M(h)}{m(h)}$.

A mapping $v \in H(B^n)$ is called a Schwarz function if $v(0) = 0$ and $\|v(z)\| \leq 1$ for all $z \in B^n$.

A differential equation which we shall consider in the fundamental part of this paper is connected with the spirallike mappings (see [6]). In the case of the maps from the unit ball B^n in C^n into C^n a definition of the spirallike mappings assumes the following form

D e f i n i t i o n . Let f be a mapping from B^n into C^n , biholomorphic and such that $f(0) = 0$, $Df(0) = I$, while $A \in L(C^n, C^n)$ is such a linear operator that

$$(1) \quad \operatorname{re} \langle A(z), z \rangle > 0 \quad \text{for } z \in B^n \text{ and } z \neq 0.$$

The mapping f is called a spirallike mapping relative to A , if

$$e^{-tA} f(B^n) \subset f(B^n) \quad \text{for } t \geq 0.$$

By such notations Theorem 11 from [6] assumes the following form

T h e o r e m . Let $A \in L(C^n, C^n)$ and satisfies the condition (1) while $f : B^n \rightarrow C^n$ is a locally biholomorphic mapping such that $f(0) = 0$, $Df(0) = I$. Then the mapping f is spirallike relative to A if and only if there exists $h \in N$ such that $Dh(0) = A$ and

$$(2) \quad Df(z)h(z) = Dh(0)f(z) \quad \text{for } z \in B^n.$$

With such formulated necessary and sufficient condition for f to be spirallike relative to A , a natural question arises if, for every mapping $h \in N$ there exists such a locally biholomorphic mapping f satisfying equation (2) with the conditions $f(0) = 0$, $Df(0) = I$.

The answer (in the most general form) to this question is given in Theorem 5.

$$2^\circ \quad \underline{\text{On the holomorphic solutions of the equation } Df(z)h(z) = Dh(0)f(z)}$$

The considerations relating to equation (2) are preceded by the following lemma.

L e m m a 1. Let $h \in N$. Then for each $z \in B^n$ the equation

$$(3) \quad \frac{\partial v}{\partial t} = -h(v), \quad v(0) = z$$

has a unique solution $v(t) = v(z, t)$ defined for all $t \geq 0$. Furthermore, for all these t the function $v(\cdot, t)$ is the univalent Schwarz function on B^n , infinitely many times differentiable with respect to the pair of variables $(z, t) \in B^n \times [0, \infty)$ and the following inequality takes place as well

$$(4) \quad \frac{\|v(z, t)\|}{(1 - \|v(z, t)\|)^2} \leq e^{-m(h)t} \frac{\|z\|}{(1 - \|z\|)^2} \quad \text{for } z \in B^n \text{ and } t \geq 0.$$

P r o o f . The existence and uniqueness of the solution of equation (3) and a fact that $v(\cdot, t)$ is the univalent Schwarz function on B^n for $t \geq 0$ follows immediately from Lemma 2 from [3]. By applying Theorem IX 5' from [4] to the function v the existence of its derivatives of all orders with respect to $(z, t) \in B^n \times [0, \infty)$ can easily be proved. Hence we have to prove only inequality (4). Let z be a fixed point of the ball B^n . Similarly as in the proof of Lemma 2 from [3] we obtain that for each $t \geq 0$ the following inequality holds

$$\frac{\partial \|v(z, t)\|}{\partial t} \leq -m(h) \frac{1 - \|v(z, t)\|}{1 + \|v(z, t)\|} \|v(z, t)\|.$$

Transforming the above inequality we get that for $t \geq 0$

$$\frac{1 + \|v(z, t)\|}{(1 - \|v(z, t)\|) \|v(z, t)\|} \frac{\partial \|v(z, t)\|}{\partial t} \leq -m(h).$$

Integrating each side of this inequality on the interval $[0, t]$ we obtain

$$\frac{\|v(z, t)\|}{(1 - \|v(z, t)\|)^2} \leq e^{-m(h)t} \frac{\|z\|}{(1 - \|z\|)^2} \quad \text{for } z \in B^n \text{ and } t \geq 0.$$

It ends the proof.

Theorem 1. Let $h \in N$ and $f \in H(B^n)$ be a solution of the equation

$$(5) \quad Df(z)h(z) = Dh(0)f(z) \quad \text{for } z \in B^n.$$

Then there exists a limit

$$(6) \quad f(z) = \lim_{t \rightarrow \infty} e^{Dh(0)t} \sum_{n=1}^{\alpha(h)} \frac{1}{n!} D^n f(0) v^n(z, t) \quad \text{for } z \in B^n$$

where $v = v(z, t)$ for $z \in B^n$ and $t \geq 0$ is a solution of equation (3).

Proof. Let $h \in N$ and v be a solution of equation (3). Since a function f fulfils equation (5) then we obtain

$$(7) \quad Df(v(z, t))h(v(z, t)) = Dh(0)f(v(z, t)).$$

Let us denote $w(z, t) = f(v(z, t))$ for $z \in B^n$ and $t \geq 0$. Using the new notation we can rewrite (7) in the form

$$(7') \quad w'(z, t) = -Dh(0)w(z, t) \quad \text{for } z \in B^n \text{ and } t \geq 0.$$

Since $v(z, 0) = z$, then $w(z, 0) = f(z)$ for $z \in B^n$. Resolving equation (7') we obtain $w(z, t) = e^{-Dh(0)t} f(z)$ for $z \in B^n$ and $t \geq 0$. Combining this fact with our new notation we get

$$f(v(z, t)) = e^{-Dh(0)t} f(z) \quad \text{for } z \in B^n \text{ and } t \geq 0.$$

Let now $P_0(z) = \sum_{n=1}^{\alpha(h)} \frac{1}{n!} D^n f(0) z^n$ for $z \in B^n$. To prove equality (6) it is sufficiently to show that

$$(8) \quad \lim_{t \rightarrow \infty} e^{Dh(0)t} (f(v(z,t)) - P_0(v(z,t))) = 0 \quad \text{for } z \in B^n.$$

From Theorem 8.14.3 from [1] there exist such a ball B^n_r and such a constant $C > 0$ that for $y \in B^n_r$ we have

$$\|f(y) - P_0(y)\| \leq C \|y\|^{\alpha(h)+1}.$$

Let now z be a fixed point of the ball B^n . From inequality (4) there exists $t_0 > 0$ such that for $t > t_0$ we have that $v(z,t) \in B^n_r$. Therefore

$$\|f(v(z,t)) - P_0(v(z,t))\| < C \|v(z,t)\|^{\alpha(h)+1} \quad \text{for } t > t_0.$$

From the above inequality and the fact that $\|e^{Dh(0)t}\| \leq e^{t\|Dh(0)\|}$ for $t > 0$, we obtain

$$\begin{aligned} & \|e^{Dh(0)t} (f(v(z,t)) - P_0(v(z,t)))\| \leq \\ & \leq C e^{tM(h)} \|v(z,t)\|^{\alpha(h)+1} \quad \text{for } t > t_0. \end{aligned}$$

Using inequality (4) we get

$$\begin{aligned} & \|e^{Dh(0)t} (f(v(z,t)) - P_0(v(z,t)))\| \leq \\ & \leq \frac{C \|z\|^{\alpha(h)+1}}{(1-\|z\|)^{2\alpha(h)+1}} e^{t(M(h)-\alpha(h)m(h)-m(h))} \quad \text{for } t > t_0. \end{aligned}$$

Since $M(h) - \alpha(h)m(h) - m(h) < 0$, then

$$\lim_{t \rightarrow \infty} \|e^{Dh(0)t} (f(v(z,t)) - P_0(v(z,t)))\| = 0$$

for all $z \in B^n$. Consequently, in accordance with the notation before inequality (8) we obtain equality (6).

Theorem 2. Let $h \in \mathbb{N}$ and $v = v(z, t)$ for $z \in B^h$ and $t \geq 0$ be a solution of equation (3). Let furthermore F be a mapping from B^h into C^h such that exists a limit

$$(9) \quad \lim_{t \rightarrow \infty} e^{Dh(0)t} F(v(z, t)) = f(z) \quad \text{for } z \in B^h$$

and it belongs to $H(B^h)$. Then the mapping f fulfils the equation

$$(10) \quad Df(z)h(z) = Dh(0)f(z) \quad \text{for } z \in B^h.$$

Proof. Let $h \in \mathbb{N}$ and z be a fixed point of the ball B^h , let furthermore t_0 be any non-negative number. From the form of equation (3) it follows that if v is the solution of (3), then the following equality holds

$$(11) \quad v(v(z, t_0), t) = v(z, t+t_0) \quad \text{for } t \geq 0.$$

This equality implies that

$$(12) \quad \lim_{t \rightarrow \infty} e^{Dh(0)t} F(v(v(z, t_0), t)) = f(v(z, t_0))$$

for $t_0 \geq 0$ and $z \in B^h$.

Using equality (11) we obtain

$$(13) \quad \lim_{t \rightarrow \infty} e^{-Dh(0)t_0} e^{Dh(0)(t+t_0)} F(v(z, t+t_0)) = e^{-Dh(0)t_0} f(z).$$

From (12) and (13) it follows immediately that

$$f(v(z, t_0)) = e^{-Dh(0)t_0} f(z) \quad \text{for } z \in B^h \text{ and } t_0 \geq 0.$$

Differentiating the above equation with respect to the parameter t_0 , we have for $t_0 = 0$

$$Df(z)h(z) = Dh(0)f(z) \quad \text{for } z \in B^h.$$

It ends the proof.

R e m a r k 1. If in Theorem 2 we assume, in addition, that $F \in H(B^n)$, $F(0) = 0$ and $DF(0) = I$, then the function f defined by equality (9) fulfils the conditions $f(0) = 0$ and $Df(0) = I$ and the function f is locally biholomorphic.

The formulation of the next theorem will be preceded by the following lemma.

L e m m a 2. Let $h \in N$ and v be a solution of equation (3). If $\alpha(h) = 1$, then there exists a limit

$$(14) \quad \lim_{t \rightarrow \infty} e^{Dh(0)t} v(z, t) = f(z) \quad \text{for } z \in B^n$$

where the function $f \in H(B^n)$ and $f(0) = 0$, $Df(0) = I$.

P r o o f . Let $u(z, t) = e^{Dh(0)t} v(z, t)$ for $z \in B^n$ and $t \geq 0$ where v is a solution of equation (3). A mapping u satisfies the equation

$$\frac{\partial u(z, t)}{\partial t} = Dh(0)u(z, t) - e^{Dh(0)t} h(e^{-Dh(0)t} u(z, t))$$

and

$$u(z, 0) = z$$

for $z \in B^n$ and $t \geq 0$.

Let us denote $H(z) = h(z) - Dh(0)(z)$ for $z \in B^n$. The function H is holomorphic on B^n and $H(0) = 0$, $DH(0) = 0$. Using the new notation and the above equation we obtain

$$\frac{\partial u(z, t)}{\partial t} = -e^{Dh(0)t} H(e^{-Dh(0)t} u(z, t)),$$

with the condition $u(z, 0) = z$, for $z \in B^n$ and $t \geq 0$. Integrating each side of this equality on the interval $[t_1, t_2]$ (where $0 < t_1 < t_2$) we get

$$(15) \quad u(z, t_2) - u(z, t_1) = - \int_{t_1}^{t_2} e^{Dh(0)\tau} H(e^{-Dh(0)\tau} u(z, \tau)) d\tau.$$

From properties of the function H and Theorem 8.14.3 from [1] it follows that there exist such a constant $C > 0$ and

such a ball B_δ^n ($0 < \delta < 1$) that for $y \in B_\delta^n$ the inequality holds

$$(16) \quad \|H(y)\| \leq C\|y\|^2.$$

Let then r be a fixed number of the interval $(0, 1)$. Considering (4) we get

$$\|e^{-Dh(0)\tau} u(z, \tau)\| = \|v(z, \tau)\| \leq e^{-m(h)} \frac{\|z\|}{(1 - \|z\|)^2}$$

for $z \in B^n$ and $\tau > 0$.

From the above inequality it follows immediately that there exists $t_r^0 > 0$ such that for $\tau > t_r^0$ and $z \in B_r^n$ we have

$$\|e^{-Dh(0)\tau} u(z, \tau)\| \leq \delta.$$

On account of inequality (16) we obtain from (15) that for t_1, t_2 such that $t_2 > t_1 > t_r^0$

$$\|u(z, t_2) - u(z, t_1)\| \leq \frac{C\|z\|^2}{(1 - \|z\|)^4} \int_{t_1}^{t_2} e^{M(h)\tau} e^{-2m(h)\tau} d\tau.$$

Since $M(h) - 2m(h) < 0$ then the integral $\int_0^\infty e^{(M(h)-2m(h))\tau} d\tau$ converges.

Then from Cauchy convergence condition for improper integrals it follows that for any $\varepsilon > 0$ there exists a constant $T_r^0 > 0$ such that for t_1, t_2 where $t_2 > t_1 > T_r^0$ and for $z \in B_r^n$ we have

$$\|u(z, t_2) - u(z, t_1)\| < \varepsilon.$$

Then there exists a limit $\lim_{t \rightarrow \infty} e^{Dh(0)t} v(z, t) = f(z)$ for $z \in B^n$. Since this convergence is almost uniform therefore from Weierstrass theorem (see e.g. [5]) the function f belongs to $H(B^n)$. It is not difficult to show that the function f fulfils the conditions $f(0) = 0$ and $Df(0) = I$.

Theorem 3. If $h \in N$ and $\alpha(h) = 1$, then the equation

$$(17) \quad Df(z)h(z) = Dh(0)f(z) \quad \text{for } z \in B^n$$

has the unique solution $f \in H(B^n)$ which satisfies the conditions $f(0) = 0$, $Df(0) = I$ and

$$(18) \quad f(z) = \lim_{t \rightarrow \infty} e^{Dh(0)t} v(z, t) \quad \text{for } z \in B^n$$

where v is a solution of equation (3).

Proof. Let $h \in N$ and $\alpha(h) = 1$. Let then $f \in H(B^n)$, $f(0) = 0$ and $Df(0) = I$. If the function f satisfies equation (17) then from Theorem 1 it follows the existence of limit (18). From the uniqueness of a limit, Theorem 2 and Remark 1 it follows that equation (17) has at most one solution fulfilling the conditions $f(0) = 0$ and $Df(0) = I$.

From Lemma 2 it follows immediately that limit (18) exists and it is the holomorphic function which satisfies equation (17) and the conditions $f(0) = 0$ and $Df(0) = I$.

Theorem 4. If $h \in N_0$ and $\alpha(h) = 1$, then the solution $f \in H(B^n)$ of the equation $Df(z)h(z) = Dh(0)f(z)$ with the conditions $f(0) = 0$, $Df(0) = I$ satisfies the inequality

$$(19) \quad \|f(z)\| \leq \frac{\|z\|}{(1 - \|z\|)^2 \frac{M(h)}{m(h)}} \quad \text{for } z \in B^n.$$

Proof. At first, we shall notice that since $h \in N_0$, then for every $t \in \mathbb{R}$, a function $\tilde{h}(z) = e^{re Dh(0)t} h(z)$ for $z \in B^n$ belongs to N .

From Lemma 1 and inequality (4) in the paper [3] it follows that

$$re \langle e^{re Dh(0)t} h(z), z \rangle \geq \frac{1 - \|z\|}{1 + \|z\|} re \langle Dh(0) e^{re Dh(0)t} z, z \rangle$$

for $z \in B^n$ and $t \in \mathbb{R}$.

Let z be a non-zero fixed point of a ball B^n . From the form of the matrix $Dh(0)$ and the above inequality we obtain

$$(20) \quad \operatorname{re} \langle e^{Dh(0)t} h(z), e^{Dh(0)t} z \rangle \geq \\ \geq \frac{1-\|z\|}{1+\|z\|} \operatorname{re} \langle Dh(0) e^{\operatorname{re} Dh(0)t} z, e^{Dh(0)t} z \rangle \quad \text{for } t \in \mathbb{R}.$$

Let now $v = v(z, t)$ for $z \in B^n$ and $t \geq 0$ be a solution of equation (3). Since

$$\|e^{Dh(0)t} v(z, t)\| \frac{\partial \|e^{Dh(0)t} v(z, t)\|}{\partial t} = \\ = \operatorname{re} \left\langle \frac{\partial (e^{Dh(0)t} v(z, t))}{\partial t}, e^{Dh(0)t} v(z, t) \right\rangle \quad \text{for } t \geq 0$$

then

$$(21) \quad \|e^{Dh(0)t} v(z, t)\| \frac{\partial \|e^{Dh(0)t} v(z, t)\|}{\partial t} = \\ = \operatorname{re} \langle Dh(0) e^{Dh(0)t} v(z, t) - e^{Dh(0)t} h(v(z, t)), e^{Dh(0)t} v(z, t) \rangle$$

for $t \geq 0$.

From (20) and (21) it appears

$$(22) \quad \frac{\partial \log \|e^{Dh(0)t} v(z, t)\|}{\partial t} \geq \frac{2M(h) \|v(z, t)\|}{1 + \|v(z, t)\|} \quad \text{for } t \geq 0.$$

Let now $C(t) = e^{-m(h)t} \frac{\|z\|}{(1-\|z\|)^2}$ for $t \geq 0$. Then from Lemma 1

$$\|v(z, t)\| \leq \frac{2C(t)}{1 + 2C(t) + \sqrt{1+4C(t)}} \quad \text{for } t \geq 0.$$

Considering the above inequality and a fact that $C(t) = -\frac{1}{m(h)} C'(t)$ for $t \geq 0$ we obtain from (22) that

$$\frac{\partial \log \|e^{Dh(0)t} v(z,t)\|}{\partial t} \leq - \frac{M(h)}{m(h)} \frac{4C'(t)}{1 + 4C(t) + \sqrt{1+4C(t)}}$$

for $t \geq 0$.

Integrating this inequality on the interval $[0, t]$ ($t > 0$) we get

$$\|e^{Dh(0)t} v(z,t)\| \leq \frac{\|z\|}{(1-\|z\|)^2} \frac{M(h)}{m(h)} \frac{1 + \sqrt{1+4C(t)}}{2}$$

for $t \geq 0$.

Since from Theorem 3 we have that $f(z) = \lim_{t \rightarrow \infty} e^{Dh(0)t} v(z,t)$ for $z \in B^n$ then using the above inequality we obtain

$$\|f(z)\| \leq \frac{\|z\|}{(1-\|z\|)^2} \frac{M(h)}{m(h)} \quad \text{for } z \in B^n.$$

The further considerations characterize the sufficient condition for equation (17) have a solution if $\alpha(h) \geq 2$.

Theorem 5. Let $h \in N$ and $\alpha(h) \geq 2$. If there exist mappings $F_k \in L_S^k(C^n, C^n)$ for $k = 2, \dots, n_0$ ($n_0 = \alpha(h)$) such that

$$(23) \quad \frac{1}{k!} D^{(k)} h(0)(z^k) = Dh(0)F_k(z^k) + \\ - \sum_{j=2}^k j F_j \left(\frac{1}{(k-j+1)!} D^{(k-j+1)} h(0)(z^{k-j+1}), z^{j-1} \right)$$

for $k = 2, \dots, n_0$, then there exists a limit

$$(24) \quad \lim_{t \rightarrow \infty} e^{Dh(0)t} (v(z,t) + \sum_{k=2}^{n_0} F_k(v^k(z,t))) = f(z) \quad \text{for } z \in B^n$$

where v is a solution of equation (3) and $f \in H(B^n)$, $f(0) = 0$, $Df(0) = I$, and furthermore

$$(25) \quad Df(z)h(z) = Dh(0)f(z) \quad \text{for } z \in B^n.$$

P r o o f . Let $u(z, t) = e^{Dh(0)t}(v(z, t) + \sum_{k=2}^{n_0} F_k(v^k(z, t)))$ for $z \in B^n$ and $t \geq 0$. Then

$$\begin{aligned} \frac{\partial u(z, t)}{\partial t} &= Dh(0)e^{Dh(0)t}(v(z, t) + \sum_{k=2}^{n_0} F_k(v^k(z, t))) + \\ &- e^{Dh(0)t}h(v(z, t)) - e^{Dh(0)t} \sum_{k=2}^{n_0} kF_k(v^{k-1}(z, t), h(v(z, t))) \end{aligned}$$

for $z \in B^n$ and $t \geq 0$.

Let next $\tilde{h}(z) = h(z) - \sum_{l=1}^{n_0} \frac{1}{l!} D^l h(0)(v^l)$ for $z \in B^n$. Such defined function $\tilde{h}$ belongs naturally to $H(B^n)$. Using the new notations we can rewrite the above equality in the form

$$\begin{aligned} \frac{\partial u(z, t)}{\partial t} &= Dh(0)e^{Dh(0)t}(v(z, t) + \sum_{k=2}^{n_0} F_k(v^k(z, t))) - \\ &- e^{Dh(0)t} \sum_{l=1}^{n_0} \frac{1}{l!} D^{(l)}h(0)(v^l(z, t)) + e^{Dh(0)t}h(v(z, t)) - \\ &- e^{Dh(0)t} \sum_{j=2}^{n_0} jF_j(v^{j-1}(z, t), \sum_{l=1}^{n_0} \frac{1}{l!} D^{(l)}\tilde{h}(0)(v^l(z, t))) + \\ &- e^{Dh(0)t} \sum_{k=2}^{n_0} kF_k(v^{k-1}(z, t), \tilde{h}(v(z, t))) \quad \text{for } z \in B^n \text{ and } t \geq 0, \end{aligned}$$

and further, applying (23), we obtain from the above equality that

$$\begin{aligned} (26) \quad \frac{\partial u(z, t)}{\partial t} &= -e^{Dh(0)t}(\tilde{h}(v(z, t))) + \\ &- \sum_{j=2}^{n_0} \sum_{l=n_0-j+2}^{n_0} jF_j(v^{j-1}(z, t), \frac{1}{l!} D^{(l)}h(0)(v^l(z, t))) + \\ &+ e^{Dh(0)t} \sum_{k=2}^{n_0} kF_k(v^{k-1}(z, t), \tilde{h}(v(z, t))) \quad \text{for } z \in B^n \text{ and } t \geq 0. \end{aligned}$$

From properties of the function $\tilde{h}$ and Theorem 8.14.3 in [1] it follows that there exist such a constant $C_1 > 0$ and such a ball B_δ ($0 < \delta < 1$) that for $y \in B_\delta$ the following inequality holds

$$(27) \quad \|\tilde{h}(y)\| \leq C_1 \|y\|^{n_0}.$$

Let now r be a fixed number of the interval $(0, 1)$. Since inequality (4) is true then there exists a constant $C_2(r)$ such that

$$(28) \quad \|v(z, t)\| \leq e^{-m(h)t} C_2(r) \quad \text{for } z \in B_r^n \quad \text{and } t \geq 0.$$

As the mapping $F_k \in L_S^k(C^n, C^n)$ for $k = 2, \dots, n_0$, then taking (27) and (28) into consideration we obtain from (26) that there exist such $C(r) > 0$ and such $T > 0$ that for $z \in B_r^n$ and t_1, t_2 , such that $T < t_1 < t_2$, we have

$$\|u(z, t_2) - u(z, t_1)\| \leq C(r) \int_{t_1}^{t_2} e^{M(h)\tau} e^{-m(h)n_0\tau} d\tau.$$

Hence there exists a limit $\lim_{t \rightarrow \infty} e^{Dh(0)t}(v(z, t) + \sum_{k=2}^{n_0} F_k(v^k(z, t))) = f(z)$ for $z \in B^n$. Since this convergence is almost uniform then from Weierstrass theorem from [5] it follows that the function $f \in H(B^n)$. Theorem 2 implies furthermore that such defined mapping f is a solution of equation (25). It is not difficult to prove that the function f fulfills the conditions $f(0) = 0$ and $Df(0) = I$.

R e m a r k 2. Using Theorem 1 it is not difficult to prove that the sufficient conditions presented in Theorem 5 are also the necessary conditions.

R e m a r k 3. A study of the existence of the mappings F_k for $k = 2, \dots, n_0$ from Theorem 5 resolves itself into the analysis of the algebraic system of the linear equations.

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Received January 8, 1988.