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**THE STABILITY OF THE DIFFERENCE SCHEMA APPROXIMATING
CAUCHY'S PROBLEM FOR THE SECOND ORDER PARABOLIC
EQUATION WITH VARIABLE COEFFICIENTS IN L_2**

1. Introduction

In [6] a class of difference schemas was presented which allow us to approximate parabolic problem for equation of second order in L_2 . Those schemas are open and unconditionally stable. In [6] only the case with constant coefficients in space variables were considered. In the present work we will deal with the case of variable coefficients depending on the space and time parameters. We shall consider the difference schema for solving the stability problem

$$(1.1) \quad \begin{cases} \frac{\partial u}{\partial t} = a(x,t) \frac{\partial^2 u}{\partial x^2} + b(x,t) \frac{\partial u}{\partial x} + c(x,t)u + f(x,t), \\ (x,t) \in R \times (0,T), \\ U(x,0) = \varphi(x), \quad \varphi \in C(R). \end{cases}$$

We assume that the functions $a(x,t)$, $b(x,t)$, $c(x,t)$, $f(x,t)$ are elements of the space $C(R, (0,T))$ and $a(x,t) \geq a_0 > 0$. On the rectangular grid

$$R_{h\tau}^2 := \{ (x,t) : x = mh, \quad t = n\tau, \quad m, n \in \mathbb{Z}, \quad N\tau = T, \quad h, \tau > 0 \}$$

we consider the following difference schema

$$(1.2) \quad \begin{cases} \frac{2}{\tau} \left[U_{2m+1}^{n+\frac{1}{2}} - U_{2m+1}^n \right] = L_{h,2m+1}^{n+\frac{1}{2}} U_{2m+1}^n + f_{2m+1}^n, \\ \frac{2}{\tau} \left[U_{2m}^{n+\frac{1}{2}} - U_{2m}^n \right] = L_{h,2m}^{n+\frac{1}{2}} U_{2m}^{n+\frac{1}{2}} + \frac{1}{2} [f_{2m}^{n+1} + f_{2m}^n], \\ \frac{2}{\tau} \left[U_{2m}^{n+1} - U_{2m}^{n+\frac{1}{2}} \right] = L_{h,2m}^{n+\frac{1}{2}} U_{2m}^{n+\frac{1}{2}} + \frac{1}{2} [f_{2m}^{n+1} + f_{2m}^n], \\ \frac{2}{\tau} \left[U_{2m+1}^{n+1} - U_{2m+1}^{n+\frac{1}{2}} \right] = L_{h,2m+1}^{n+\frac{1}{2}} U_{2m+1}^{n+\frac{1}{2}} + f_{2m+1}^{n+1}, \\ U_m^0 = u_m, \quad m, n \in \mathbb{Z}, \quad 0 \leq n \leq N-1, \end{cases}$$

where the difference operator $((L_{h,m}^n y)_k)$ is defined on the number sequences as follows

$$(L_{h,m}^n y)_k := L_{h,m}^n y_k := a_m^n [y_{k-1} - 2y_k + y_{k+1}] + b_m^n \frac{1}{2h} [y_{n+1} - y_{n-1}] + c_m^n y_k,$$

$$\forall (m, n, k), \quad 2n \in \mathbb{Z}, \quad m, n \in \mathbb{Z},$$

and $U_m^n := u(mh, n\tau)$, $\mathbb{Z}$ - the set of integers.

2. The stability of the difference schema (1.2) for the problem (1.1)

To investigate the stability of this problem we prove the following theorem.

T h e o r e m 2.1. The sequence $\{U_m^n\}_{m,n \in \mathbb{Z}, 0 \leq n \leq N}$ satisfies the system of difference equations (1.2) if and only if this sequence satisfies the relation defined by the following difference schema

$$(2.1) \quad \begin{cases} \frac{1}{2} [U_m^{n+1} - U_m^n] = \frac{1}{2} \left[L_{h,m}^{n+\frac{1}{2}} U_m^{n+\frac{1}{2}} + f_m^{n+1} \right] + \frac{1}{2} \left[L_{h,m}^{n+\frac{1}{2}} U_m^n + f_m^n \right], \\ U_m^0 = \varphi_m, \quad m, n \in \mathbb{Z}, \quad 0 \leq n \leq N-1. \end{cases}$$

P r o o f . First we show that if the sequence $\{U_m^n\}$ satisfies (1.2), then it satisfies (2.1). Summing up the second and third equation of (1.2) and dividing by 2 we obtain

$$(2.2) \quad \frac{1}{\tau} [U_{2m}^{n+1} - U_{2m}^n] = L_{h,2m}^{n+\frac{1}{2}} U_{2m}^{n+\frac{1}{2}} + \frac{1}{2} [f_{2m}^{n+1} + f_{2m}^n],$$

and subtracting side by side we obtain

$$U_{2m}^{n+\frac{1}{2}} = \frac{1}{2} [U_{2m}^{n+1} + U_{2m}^n], \quad m \in \mathbb{Z}.$$

On both sides of the above equations we act with the operator $L_{h,2m}^{n+1/2}$ and the obtained result $L_{h,2m}^{n+1/2} U_{2m}^{n+1/2}$ we substitute to (2.2), and we obtain

$$(2.3) \quad \begin{aligned} \frac{1}{2} [U_{2m}^{n+1} - U_{2m}^n] &= \frac{1}{2} L_{h,2m}^{n+\frac{1}{2}} [U_{2m}^{n+1} + U_{2m}^n] + \frac{1}{2} [f_{2m}^{n+1} + f_{2m}^n] = \\ &= \frac{1}{2} \left[L_{h,2m}^{n+\frac{1}{2}} U_{2m}^{n+1} + f_{2m}^{n+1} \right] + \frac{1}{2} \left[L_{h,2m}^{n+\frac{1}{2}} U_{2m}^n + f_{2m}^n \right]. \end{aligned}$$

Similarly, adding first and fourth equation of the system (1.2), we obtain

$$(2.4) \quad \begin{aligned} \frac{1}{\tau} [U_{2m+1}^{n+1} - U_{2m+1}^n] &= \frac{1}{2} L_{h,2m+1}^{n+\frac{1}{2}} [U_{2m+1}^{n+1} + U_{2m+1}^n] + \frac{1}{2} [f_{2m+1}^{n+1} + f_{2m+1}^n] = \\ &= \frac{1}{2} \left[L_{h,2m+1}^{n+\frac{1}{2}} U_{2m+1}^{n+1} + f_{2m+1}^{n+1} \right] + \frac{1}{2} \left[L_{h,2m+1}^{n+\frac{1}{2}} U_{2m+1}^n + f_{2m+1}^n \right]. \end{aligned}$$

Hence, from (2.3) and (2.4) we obtain

$$\frac{1}{\tau} [U_m^{n+1} - U_m^n] = \frac{1}{2} \left[L_{hm}^{n+\frac{1}{2}} U_m^{n+1} + f_m^{n+1} \right] + \frac{1}{2} \left[f_m^n + L_{hm}^{n+\frac{1}{2}} U_m^n \right].$$

This schema together with the condition $U_m^0 = \phi_m$ for $m \in \mathbb{Z}$ gives the difference schema (2.1).

To show that the sequence $\{U_m^{m+1/2}\}$ defined by the schema (2.1) satisfies (1.2) we introduce an auxiliary sequence $\{v_m^{n+1/2}\}$ defined as follows

$$(2.5) \quad U_{2m+1}^{n+\frac{1}{2}} = U_{2n+1}^n + \frac{\tau}{2} \left[L_{h,2m+1}^{n+\frac{1}{2}} U_{2m+1}^n + f_{2m+1}^n \right],$$

$$(2.6) \quad U_{2m}^{n+\frac{1}{2}} = \frac{1}{2} (U_{2m}^{n+1} + U_{2m}^n), \text{ for } m, n \in \mathbb{Z}.$$

Using (2.6) and (2.1) we obtain

$$\begin{aligned} \frac{2}{\tau} \left[U_{2m}^{n+\frac{1}{2}} - U_{2m}^n \right] &= \frac{1}{\tau} \left[U_{2m}^{n+1} - U_{2m}^n \right] = \\ &= \frac{1}{2} \left[L_{h,2m}^{n+\frac{1}{2}} U_{2m}^n + f_{2m}^n \right] + \frac{1}{2} \left[L_{h,2m}^{n+\frac{1}{2}} U_{2m}^{n+1} + f_{2m}^{n+1} \right]. \end{aligned}$$

From the linearity of $L_{h,2m}^{n+1/2}$ and (2.6) we have

$$(2.7) \quad \frac{2}{\tau} \left[U_{2m}^{n+\frac{1}{2}} - U_{2m}^n \right] = L_{h,2m}^{n+\frac{1}{2}} U_{2m}^{n+\frac{1}{2}} + \frac{1}{2} \left[f_{2m}^{n+1} + f_{2m}^n \right].$$

Similarly

$$\begin{aligned} \frac{2}{\tau} \left[U_{2m}^{n+1} - U_{2m}^{n+\frac{1}{2}} \right] &= \frac{1}{\tau} \left[U_{2m}^{n+1} - U_{2m}^n \right] = \frac{1}{2} \left[L_{h,2m}^{n+\frac{1}{2}} U_{2m}^n + f_{2m}^n \right] + \\ &+ \frac{1}{2} \left[L_{h,2m}^{n+\frac{1}{2}} U_{2m}^{n+1} + f_{2m}^{n+1} \right] = L_{h,2m}^{n+\frac{1}{2}} U_{2m}^{n+\frac{1}{2}} + \frac{1}{2} \left[f_{2m}^{n+1} + f_{2m}^n \right], \end{aligned}$$

that is

$$(2.8) \quad \frac{2}{\tau} \left[U_{2m}^{n+1} - U_{2m}^{n+\frac{1}{2}} \right] = L_{h,2m}^{n+\frac{1}{2}} U_{2m}^{n+\frac{1}{2}} + \frac{1}{2} \left[f_{2m}^{n+1} + f_{2m}^n \right].$$

Using definitions (2.5), we obtain

$$\frac{2}{\tau} \left[U_{2m+1}^{n+1} - U_{2m+1}^{n+\frac{1}{2}} \right] = \frac{2}{\tau} \left[U_{2m+1}^{n+1} - U_{2m+1}^n \right] - L_{h,2m+1}^{n+\frac{1}{2}} U_{2m+1}^n - f_{2m+1}^n.$$

Similarly, using (2.1), we obtain

$$\frac{2}{\tau} \left[U_{2m+1}^{n+1} - U_{2m+1}^{n+\frac{1}{2}} \right] = 2 \left[\frac{1}{2} \left(L_{h,2m+1}^{n+\frac{1}{2}} U_{2m+1}^{n+1} + f_{2m+1}^{n+1} \right) + \right.$$

$$\begin{aligned}
& + \frac{1}{2} \left(L_{h,2m+1}^{n+\frac{1}{2}} U_{2m+1}^n + f_{2m+1}^n \right) \Big] - L_{h,2m+1}^{n+\frac{1}{2}} U_{2m+1}^n - f_{2m+1}^n = \\
& = L_{h,2m+1}^{n+\frac{1}{2}} U_{2m+1}^{n+1} + f_{2m+1}^{n+1},
\end{aligned}$$

that is

$$(2.9) \quad \frac{2}{\tau} \left[U_{2m+1}^{n+1} - U_{2m+1}^{n+\frac{1}{2}} \right] = L_{h,2m+1}^{n+\frac{1}{2}} U_{2m+1}^{n+1} + f_{2m+1}^{n+1}.$$

From formula (2.5) it follows that

$$\frac{2}{\tau} \left[U_{2m+1}^{n+\frac{1}{2}} - U_{2m+1}^n \right] = L_{h,2m+1}^{n+\frac{1}{2}} U_{2m+1}^n + f_{2m+1}^n.$$

Hence Theorem 2.1 has been proved.

This theorem is important, because it reduces the investigations of the stability of schema (1.2) to the stability of schema (2.1).

First, we can write the schema (2.1) in the canonical form. To this aim we denote

$$\hat{U} := U_m^{n+1}, \quad U := U_m^n, \quad U_t := \frac{U_m^{n+1} - U_m^n}{\tau}, \quad \hat{f} := f_m^{n+1},$$

$$f := f_m^n, \quad U^0 := U_m^0, \quad \varphi := \varphi_m, \quad \bar{t} = tn + \frac{1}{2}\tau, \quad \text{for } m, n \in \mathbb{Z}, \quad 0 \leq n \leq N-1.$$

Then (2.1) takes the form

$$(2.10) \quad U_t = \frac{1}{2} L(\bar{t}) (\hat{U} + U) + \frac{1}{2} (\hat{f} + f), \quad U^0 = \varphi.$$

Using the identity

$$\hat{U} = U + \tau \frac{\hat{U} - U}{\tau} = U + \tau U_t,$$

we can write (2.10) in the form

$$U_t = \frac{1}{2} L(\bar{t}) [U + \tau U_t + U] + \frac{1}{2} (\hat{f} + f), \quad U^0 = \varphi,$$

that is

$$(E - \frac{\tau}{2} L(\bar{t})) U_t - L(\bar{t}) U = f, \quad U^0 = \varphi,$$

in other words

$$(2.11) \quad BU_t + AU = f, \quad U^0 = \varphi,$$

where $B := E + \frac{\tau}{2} A$, $A := -L(\bar{t})$.

To investigate the stability of this problem, we apply Theorem 10.14 in [5]. We formulate this fact as the following theorem.

Theorem 2.2. If the coefficients of equation (1.1) and the function $f(x, t)$ are continuous with respect to x and t , and bounded in such a way that

$$\frac{\partial a(x, t)}{\partial t} \in C(R, (0, T)),$$

then there exists τ_* , depending only on the norm of coefficients, such that for $\tau \in (0, \tau_*)$ the schema (1.2) is unconditionally stable.

Proof. By Theorem 2.1, the schema (1.2) is equivalent to schema (2.1), whose unconditional stability is proved, e.g. in [5], Th. 10.14.

3. The stability of difference schemes for solving the problem of Cauchy for the parabolic equations of order 2 with many space variables

We consider the following initial problem

$$(3.1) \quad \begin{cases} \frac{\partial u}{\partial t} + Lu = f(x, t), & (x, t) \in R^M \times (0, T), \\ u(x, 0) = \varphi(x), & x \in R^M, \end{cases}$$

where L is an elliptic operator, $L \in (C^2(R^M), C(R^M))$, defined by

$$Lu := - \sum_{i,j=1}^M \frac{\partial}{\partial x_i} \left(a_{ij}(x, t) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^M b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u.$$

The difference schema approximating (3.1) will be constructed on the parallelepiped grid $R_{h\tau}^M = R_h^M \times \omega_\tau$ with $R_h^M := \{x \in R^M : x = (m_1 h_1, \dots, m_M h_M), m_1, \dots, m_M \in \mathbb{Z}\}$, $h := \max_{1 \leq i \leq M} h_i$, $h_1, \dots, h_M > h_0 > 0$, and $\omega_\tau := \{t = n\tau, n=0, \dots, N, N\tau=T\}$, where by $\partial_i, \bar{\partial}_i (i=1, \dots, M)$ we denote the difference operators

$$\partial_i v(x) = \frac{1}{h_i} [v(x + e_i h_i) - v(x)], \quad \forall x \in R^M,$$

$$\bar{\partial}_i v(x) = \frac{1}{h_i} [v(x) - v(x - e_i h_i)], \quad \forall x \in R^M,$$

for $i = \overline{1, M}$; $e_1, \dots, e_M$ denote the versors of the respective axes. Let L_h^0, L_h, L_h^1 denote the difference operators defined as follows

$$L_h^0 v(x) := -\frac{1}{2} \sum_{i,j=1}^M [\partial_i(a_{ij}(x)) \bar{\partial}_j v(x) + \bar{\partial}_i(a_{ij}(x)) \partial_j v(x)],$$

$$L_h^1 v(x) := \frac{1}{2} \sum_{i=1}^M b_i(x) [\partial_i v(x) + \bar{\partial}_i v(x)] + c(x) v(x),$$

$$L_h v(x) := L_h^0 v(x) + L_h^1 v(x).$$

Let Ω_{1h} and Ω_{2h} denote non-empty domain of R_h^M and let $R_h^M = \Omega_{1h} \cup \Omega_{2h}$, $\Omega_{1h} \cap \Omega_{2h} = \emptyset$. We determine a sequence $\{U^n(x)\}$ for $x \in R_h^M$, $n \in \mathbb{Z}$, such that $U^0(x) = \phi(x)$ for $x \in R_h^N$. The difference problem (3.1) will be approximated by the following schema

$$(3.2) \quad \begin{cases} \frac{2}{\tau} \left[U_{(x)}^{n+\frac{1}{2}} - U_{(x)}^n \right] + L_h^{n+\frac{1}{2}} U_{(x)}^n = f_{(x)}^n, & \text{if } x \in \Omega_{1h}, \\ \frac{2}{\tau} \left[U_{(x)}^{n+\frac{1}{2}} - U_{(x)}^n \right] + L_h^{n+\frac{1}{2}} U_{(x)}^{n+\frac{1}{2}} = \frac{1}{2} [f_{(x)}^{n+1} + f_{(x)}^n], & \text{if } x \in \Omega_{2h}, \\ \frac{2}{\tau} \left[U_{(x)}^{n+1} - U_{(x)}^{n+\frac{1}{2}} \right] + L_h^{n+\frac{1}{2}} U_{(x)}^{n+\frac{1}{2}} = \frac{1}{2} [f_{(x)}^{n+1} + f_{(x)}^n], & \text{if } x \in \Omega_{2h}, \\ \frac{2}{\tau} \left[U_{(x)}^{n+1} - U_{(x)}^{n+\frac{1}{2}} \right] + L_h^{n+\frac{1}{2}} U_{(x)}^{n+1} = f_{(x)}^{n+1}, & \text{if } x \in \Omega_{1h}, \\ U_{(x)}^0 = \phi(x), \end{cases}$$

where $U_{(x)}^n := u(x, n\tau)$. For the difference schema (3.2) there holds the following theorem on equivalence, analogous to Theorem 2.1.

Theorem 3.1. The sequences $\{U_{(x)}^n\}$ for $x \in R_h^M$, $n \in Z$ ($0 \leq 2n \leq N$), satisfies the system of equation (3.2) iff the sequence $\{U_{(x)}^n\}$ satisfies the following difference equation

$$(3.3) \quad \begin{cases} \frac{1}{2} [U_{(x)}^{n+1} - U_{(x)}^n] + \frac{1}{2} [L_h^{n+1/2} U_{(x)}^{n+1} + L_h^{n+1/2} U_{(x)}^n] = \frac{1}{2} [f_{(x)}^n + f_{(x)}^{n+1}], \\ \text{for } x \in R_h^M, n = 0, N-1 \\ U_{(x)}^0 = \varphi(x), \text{ for } x \in R_h^M. \end{cases}$$

The proof is similar to that of Theorem 2.1.

Theorem 3.2. If the functions $a_{1,j}$, b_1 , c , f ($i, j = 1, M$) are continuous and bounded and

$$\frac{\partial a_{1j}(x, t)}{\partial t} \in C(R, (0, T)),$$

then there exists a positive number τ_* dependent only on the norms of coefficients and such that for $\tau \in (0, \tau_*)$ the difference schema (3.2) is unconditionally stable and approximates the problem (3.1) with order $O(\tau + h^2)$.

The proof of this theorem can be found in [5].

The schema (3.2) is unconditionally stable but is not always open which is shown in the following theorem.

Theorem 3.3. The schema (3.2) with the condition $\Omega_{1h} \cap \Omega_{2h} = \emptyset$ is open iff the functions a_{ij} ($i \neq j$) are identically zero and

$$\begin{aligned} \Omega_{1h} &= \left\{ x \in R_h^M; x = (m_1 h_1, \dots, m_M h_M), \frac{1}{2} \left[\sum_{j=1}^M m_j - 1 \right] \in Z \right\}, \\ \Omega_{2h} &= \left\{ x \in R_h^M; x = (m_1 h_1, \dots, m_M h_M), \frac{1}{2} \sum_{i=1}^M m_i \in Z \right\}. \end{aligned}$$

The proof of this theorem is analogous to that of Theorem 3.3 in [6].

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