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A COINCIDENCE THEOREM FOR THREE SYSTEMS OF TRANSFORMATIONS

With a view to generalizing the Banach contraction principle, Matkowski [5]-[6] proved a fixed point theorem (see Remark 1) for a system of n transformations, on a product of n metric spaces. This result has been extended and generalized by Czerwik [1]-[2], Reddy-Subrahmanyam [7] and Singh-Kulshrestha [9].

We follow the notations of Matkowski [4] and Czerwik [2]:

$$(1) \quad c_{ik}^{(0)} = \begin{cases} a_{ik} & \text{for } i \neq k, \quad i, k = 1, \dots, n, \\ 1 - a_{ik} & \text{for } i = k, \end{cases}$$

and $c_{ik}^{(t)}$ are defined recursively by

$$(2) \quad c_{ik}^{(t+1)} = \begin{cases} c_{11}^{(t)} c_{i+1,k+1}^{(t)} + c_{i+1,1}^{(t)} c_{1,k+1}^{(t)}, & \text{for } i \neq k, \\ c_{11}^{(t)} c_{i+1,k+1}^{(t)} - c_{i+1,1}^{(t)} c_{1,k+1}^{(t)}, & \text{for } i = k, \\ i, k = 1, \dots, n-t-1, \quad t = 0, 1, \dots, n-2. \end{cases}$$

The following lemma is essentially due to Matkowski [5] (see also [2], [6] and [9]).

Lemma. Let $c_{ik}^{(0)} > 0$ for $i, k = 1, \dots, n$, $n \geq 2$. The system of inequalities

$$(3) \quad \sum_{k=1}^n a_{ik} r_k < r_i, \quad i=1,2,\dots,n,$$

has a positive solution $r_1, r_2, \dots, r_n$ if and only if the following inequalities hold:

$$(4) \quad c_{ii}^{(t)} > 0, \quad i=1,2,\dots,n-1, \quad t=0,\dots,n-1.$$

Let $A_1, A_2, \dots, A_n$ be arbitrary (nonempty) sets, $A := A_1 \times A_2 \times \dots \times A_n$ and $X_1, X_2, \dots, X_n$ metric spaces (with metrics d_i , respectively).

Assume that

(5) P_i, Q_i, T_i are transformations defined on the set A with values in X_i , $i=1,2,\dots,n$;

(6) $P_i(A) \cup Q_i(A) \subset T_i(A)$, and $(T_i(A), d_i)$ are complete subspaces of X_i , $i=1,\dots,n$;

$$(7) \quad d_i(P_i(x_1, \dots, x_n), Q_i(y_1, \dots, y_n)) \leq \\ \leq \sum_{k=1}^n a_{ik} d_k(T_k(x_1, \dots, x_n), T_k(y_1, \dots, y_n)) + \\ + b[d_i(T_i(x_1, \dots, x_n), P_i(x_1, \dots, x_n)) + \\ + d_i(T_i(y_1, \dots, y_n), Q_i(y_1, \dots, y_n))]$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n)$ in A and some nonnegative number b .

The following is the main result of this paper.

Theorem. If there exist nonnegative numbers b and a_{ik} , $i, k = 1, 2, \dots, n$ such that (1)-(7), where

$$(8) \quad 0 \leq 2b < 1 - \lambda,$$

in which
$$\lambda = \max_i \left(r_i^{-1} \sum_{k=1}^n a_{ik} r_k \right),$$

then the system of equations

$$(9) \quad P_i(x_1, \dots, x_n) = Q_i(x_1, \dots, x_n) = T_i(x_1, \dots, x_n)$$

has a solution in A . Moreover, if

$$P_i(v_1, \dots, v_n) = Q_i(v_1, \dots, v_n) = T_i(v_1, \dots, v_n) := u_i$$

$$\text{and } P_i(\bar{v}_1, \dots, \bar{v}_n) = Q_i(\bar{v}_1, \dots, \bar{v}_n) = T_i(\bar{v}_1, \dots, \bar{v}_n) := \bar{u}_i$$

then $u_i = \bar{u}_i$.

Proof. First we note that, in view of the system of inequalities (3), λ defined in (8) exists and $0 < \lambda < 1$ (cf. (Czerwik [2])). From the lemma and (8), we may choose positive numbers $r_1, r_2, \dots, r_n$ such that

$$\sum_{k=1}^n a_{ik} r_k \leq \lambda r_i, \quad i=1, \dots, n.$$

Pick x_i^0 in A_i , $i=1, \dots, n$. Taking a clue from [8], we construct sequences $\{x_i^m\}$ in A_i and $\{z_i^m\}$ in $T_i(A) \subset X_i$, $i=1, \dots, n$ such that

$$P_i(x_1^{2m}, \dots, x_n^{2m}) = T_i(x_1^{2m+1}, \dots, x_n^{2m+1}) = z_i^{2m+1}$$

and

$$Q_i(x_1^{2m+1}, \dots, x_n^{2m+1}) = T_i(x_1^{2m+2}, \dots, x_n^{2m+2}) = z_i^{2m+2},$$

$m = 0, 1, 2, \dots$. We can do this since $P_i(A) \subset T_i(A)$ and $Q_i(A) \subset T_i(A)$.

We may assume that

$$d_i(z_i^0, z_i^1) \leq r_i, \quad r_i \geq 1, \quad i=1, \dots, n.$$

From (7),

$$\begin{aligned} d_i(z_i^1, z_i^2) &= d_i(P_i(x_1^0, \dots, x_n^0), Q_i(x_1^1, \dots, x_n^1)) \leq \\ &\leq \sum_{k=1}^n a_{ik} d_k(T_k(x_1^0, \dots, x_n^0), T_k(x_1^1, \dots, x_n^1)) + \end{aligned}$$

$$\begin{aligned}
& + b[d_i(T_i(x_1^0, \dots, x_n^0), P_i(x_1^0, \dots, x_n^0)) + \\
& + d_i(T_i(x_1^1, \dots, x_n^1), Q_i(x_1^1, \dots, x_n^1))] = \\
& = \sum_{k=1}^n a_{ik} d_k(z_k^0, z_k^1) + bd_i(z_i^0, z_i^1) + bd_i(z_i^1, z_i^2) \leq \\
& \leq \sum_{k=1}^n a_{ik} r_k + br_i + bd(z_i^1, z_i^2).
\end{aligned}$$

So

$$d_i(z_i^1, z_i^2) \leq qr_i$$

wherein $0 < q = (\lambda + b)/(1 - b) < 1$. Similarly

$$d_i(z_i^2, z_i^3) \leq q^2 r_i.$$

Inductively

$$d_i(z_i^m, z_i^{m+1}) \leq q^m r_i, \quad m=0, 1, 2, \dots$$

This implies that $\{z_i^m\}$ is a Cauchy sequence and has a limit in $T_i(A)$, $i=1, \dots, n$. Call it u_i , $i=1, \dots, n$.

Now let $(v_1, v_2, \dots, v_n)$ be a point in $T_i^{-1}u_i$. Thus $T_i(v_1, \dots, v_n) = u_i$. Then from (7),

$$\begin{aligned}
& d_i(P_i(v_1, \dots, v_n), Q_i(x_1^{2m+1}, \dots, x_n^{2m+1})) \leq \\
& \leq \sum_{k=1}^n a_{ik} d_k(u_k, z_k^{2m+2}) + \\
& + b[d_i(u_i, P(v_1, \dots, v_n)) + d_i(z_i^{2m+2}, z_i^{2m+3})].
\end{aligned}$$

Letting $m \rightarrow \infty$ yields

$$d_i(P_i(v_1, \dots, v_n), u_i) \leq bd_i(u_i, P(v_1, \dots, v_n)),$$

that is $P_i(v_1, \dots, v_n) = u_i$, $i=1, \dots, n$.

Similarly $Q_i(v_1, \dots, v_n) = u_i$, $i=1, \dots, n$. This proved the existence of a solution of the system (9) in A .

To prove the last part of the theorem, let $(\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n)$ in A be another solution of the system of equations (9), that is $P_i(\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n) = Q_i(\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n) = T_i(\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n) = \bar{u}_i$, say. Then, since we can assume $d_i(u_i, \bar{u}_i) \leq r_i$, $i=1, \dots, n$, we have by (7),

$$d_i(u_i, \bar{u}_i) = d_i(P_i(v_1, \dots, v_n), Q_i(\bar{v}_1, \dots, \bar{v}_n)) \leq \sum_{k=1}^n a_{ik} r_k \leq \lambda r_i.$$

Inductively

$$d_i(u_i, \bar{u}_i) \leq \lambda^m r_i.$$

Letting $m \rightarrow \infty$ gives $u_i = \bar{u}_i$. This completes the proof of the theorem.

Remarks: 1. Matkowski's fixed point theorem [5] is obtained as a corollary from the above theorem when $b = 0$, $A_i = X_i$, $P_i = Q_i$ and $T_i(x_1, x_2, \dots, x_n) = x_i$ for each x_i in X_i , $i=1, \dots, n$.

2. Kominek's coincidence theorem [4, Th.1] generalizing the coincidence theorem of Goebel [3] and Matkowski's theorem is obtained from the above theorem when $b = 0$ and $P_i = Q_i$, $i=1, \dots, n$. Indeed, Kominek's theorem is proved using Matkowski's theorem while our approach is constructive and provides a method to construct a sequence converging to the coincidence value.

3. A fixed point theorem for two systems of transformations due to Singh-Kulshrestha [9] is obtained from our theorem when $A_i = X_i$ and $T_i(x_1, \dots, x_n) = x_i$ for each x_i in X_i , $i=1, \dots, n$.

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