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ESTIMATES OF THE SOLUTION OF THE BOUNDARY-VALUE PROBLEM
WITH A COMPLEX PARAMETER FOR A CERTAIN ELLIPTIC SYSTEM

In this paper we construct a solution of the boundary-value problem with a complex parameter (having a large absolute value) for a certain elliptic system of partial differential equations of order $2s$ and obtain suitable bounds for it. These estimates play an important role in methods of contour integration for parabolic equations with time-dependent boundary conditions (see [1], [2]). For $s=1$ this problem was investigated in [2].

1. Notation and statement of the problem

Let D be a domain in the Euclidean space E^n , $n \geq 3$, bounded by a closed Lyapunov's surface S with the exponent $\kappa \in (0, 1)$.

Let us denote by

$$(1) \quad R_\delta = \{ \lambda : |\arg \lambda| \leq \frac{\pi + 2\delta}{4s}, |\lambda| \geq R > 0 \}$$

the region of complex λ -plane, R being a sufficiently large number, $\delta \in (0, \frac{\pi}{4})$ and $s \in \mathbb{N}$.

Let us consider the following boundary-value problem

$$(2) \quad A(x, \frac{\partial}{\partial x})u(x, \lambda) - \lambda^{2s}u(x, \lambda) = -\varphi(x) \quad \text{for } x \in D, \lambda \in R_\delta,$$

$$(3) \quad \lim_{D \ni x \rightarrow z \in S} B(z, \lambda^{2s}, \frac{\partial}{\partial x})u(x, \lambda) = \psi(z, \lambda) \quad \text{for } z \in S, \lambda \in R_\delta,$$

where

$$(4) \quad A(x, \frac{\partial}{\partial x}) = A^0(x, \frac{\partial}{\partial x}) + A^1(x, \frac{\partial}{\partial x}),$$

$$(5) \quad A^0(x, \frac{\partial}{\partial x}) = (-1)^s a^0(x) \sum_{i_1, \dots, i_{2s}=1}^n a_{i_1 i_2}(x) \dots a_{i_{2s-1} i_{2s}}(x) \frac{\partial^{2s}}{\partial x_{i_1} \dots \partial x_{i_{2s}}},$$

$$(6) \quad A^1(x, \frac{\partial}{\partial x}) = \sum_{k=0}^{2s-1} a_{k_1 \dots k_n}^1(x) \frac{\partial^k}{\partial x_{k_1} \dots \partial x_{k_n}},$$

$$(7) \quad B(z, \lambda^{2s}, \frac{\partial}{\partial x}) = (b^0(z) + \lambda^{2s} b^1(z)) D_{Tz}^{2s-1} + B_1(z, \lambda^{2s}, \frac{\partial}{\partial x}),$$

$$(8) \quad B_1(z, \lambda, \frac{\partial}{\partial x}) = \sum_{k=0}^{2s-2} \sum_{l=0}^1 \lambda^{2sl} b_{k_1 \dots k_n}^l(z) \frac{\partial^k}{\partial x_{k_1} \dots \partial x_{k_n}},$$

$$(9) \quad D_{Tz}^{2s-1} = \sum_{i_1, \dots, i_{2s}=1}^n a_{i_1 i_2}(x) \dots \dots a_{i_{2s-1} i_{2s}}(x) \cos(n_z, x_{i_{2s}}) \frac{\partial^{2s-1}}{\partial x_{i_1} \dots \partial x_{i_{2s-1}}}$$

with $k := k_1 + \dots + k_n$ in (6), (8).

Here, $a^0(x), a_{k_1 \dots k_n}^1(x), b^0(z), b^1(z), b_{k_1 \dots k_n}^1(z)$ denote the $N \times N$ functional matrices, $N \geq 1$; $\varphi(x), \psi(z, \lambda), u(x, \lambda)$ are N -th order vector-functions and $a_{i,j}(x), i, j = 1, \dots, n$, are real functions. In (3), the point $x \in D$ tends to $z \in S$ along a curve satisfying the condition (see [7], p.115)

$$(10)^1 \quad \lim_{\|z-z_x\| \rightarrow 0} \|z-z_x\|^\kappa \log \|x-z_x\| = 0,$$

where $\|x-z_x\| = \inf_{\xi \in S} \|x-\xi\|$ and $\kappa \in (0, 1)$.

We make the following assumptions:

(I) The matrix $a(x) = [a_{ij}(x)]_{N \times N}$ is symmetric and posi-

¹Here, and in the sequel, $\|\cdot\|$ denotes the standard norm in E^n .

tively defined at any point $x \in \bar{D} = D \cup S$.

(II) The coefficients of the operator (4) are determined in the domain $D_0 \supset \bar{D}$ and $a^0, a \in C^{2s}(D_0)$,

$$a_{k_1 \dots k_n}^1 \in C^{k_1 + \dots + k_n}(D_0), \quad 0 \leq k_1 + \dots + k_n \leq 2s-1.$$

(III) The matrices $b^0, b^1, b_{k_1 \dots k_n}^1, l = 0, 1$, are determined and continuous on the boundary S .

(IV) The vector-function $\varphi \in C^1(D)$ satisfies the condition: there exists a domain D_1 such that $\bar{D}_1 \subset D$ and $\varphi(x) = 0$ for $x \in D \setminus D_1$.

(V) The vector-function $\psi(z, \lambda)$ is continuous with respect to $z \in S$, analytic with respect to $\lambda \in R_\delta$ and $\lim_{\lambda \rightarrow \infty} \psi(z, \lambda) = 0$ uniformly with respect to $\arg \lambda$ for $\lambda \in R_\delta$.

(VI) The matrices $[b^0(z) + \lambda^{2s} b^1(z)]^{-1}$,

$$[b^0(z) + \lambda^{2s} b^1(z)]^{-1} \sum_{l=0}^1 b_{k_1 \dots k_n}^l(z) \lambda^{2sl}, \quad k = 0, 1, \dots, 2s-2,$$

are bounded for $z \in S$ and $\lambda \in R_\delta$.

$$(VII)^2 \det \left[\left(\sum_{i,j=1}^n a_{ij}(x) \alpha_i \alpha_j \right)^s a^0(x) - \lambda^{2s} I \right] \neq 0$$

for $\lambda \in R_\delta$ and any $\alpha = (\alpha_1, \dots, \alpha_n) \in E^n$.

A solution $u(x, \lambda)$ of the problem (2), (3) is inquired in class of vector-functions in E^N which have in D continuous derivatives of the $2s$ -th order with respect to x and are analytic with respect to λ for any $\lambda \in R_\delta$.

The fundamental solution of the homogeneous system

$$(11) \quad A(x, \frac{\partial}{\partial x}) u(x, \lambda) - \lambda^{2s} u(x, \lambda) = 0$$

was constructed by Mamedov in [3].

Let us assume that

² Here, and in the sequel, I denotes the unit matrix.

$$(12) \quad A^0(\xi, \frac{\partial}{\partial x}) = (-1)^s a^0(\xi) \left(\sum_{i,j=1}^n a_{ij}(\xi) \frac{\partial^2}{\partial x_i \partial x_j} \right)^s,$$

and

$$(13) \quad P_0(x, \xi, \lambda) = \\ = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-\xi, \alpha)} \times \left[\left(\sum_{i,j=1}^n a_{ij}(\xi) \alpha_i \alpha_j \right)^s a^0(\xi) - \lambda^{2s} I \right]^{-1} d\alpha.$$

Under the assumption (VII), we can conclude from the results of [3] that (13) presents a fundamental solution of the system

$$(14) \quad A^0(\xi, \frac{\partial}{\partial x}) u(x, \lambda) - \lambda^{2s} u(x, \lambda) = 0$$

and the following estimates hold for $k = 0, 1, \dots, 2s$; $\lambda \in \mathbb{R}_\delta$.

$$(15)^3 \quad \left| \frac{\partial^k P_0(x, \xi, \lambda)}{\partial x_1^k \dots \partial x_n^k} \right| \leq \frac{C \exp(-c|\lambda| \|x-\xi\|)}{\|x-\xi\|^{n-2s+k}}$$

A fundamental solution of (11) was constructed in [3] by using Levi's "parametrix" method, in the form

$$(16) \quad P(x, \xi, \lambda) = P_0(x, \xi, \lambda) + P_1(x, \xi, \lambda)$$

where

$$(16') \quad P_1(x, \xi, \lambda) = \int_D P_0(x, \eta, \lambda) h(\eta, \xi, \lambda) d\eta$$

and $h(\eta, \xi, \lambda)$ is an unknown $N \times N$ functional matrix; besides in [3], by the assumptions (II) and (VII), the following relations were shown

$$(17) \quad |h(x, \xi, \lambda)| \leq \frac{C \exp(-c|\lambda| \|x-\xi\|)}{\|x-\xi\|^{n-1}},$$

³ Here, and in the sequel, the estimates are valid for all elements of matrix on the left-hand side of (15). C and c denote some positive constants: C -sufficiently large numbers and c -sufficiently small ones.

$$(18) \quad \left| \frac{\partial^k P_1(x, \xi, \lambda)}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \right| \leq \frac{C \exp(-\varepsilon |\lambda| \|x - \xi\|)}{\|x - \xi\|^{n-1-2s+k}},$$

$$(19) \quad \left| \frac{\partial^k P(x, \xi, \lambda)}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \right| \leq \frac{C \exp(-\varepsilon |\lambda| \|x - \xi\|)}{\|x - \xi\|^{n-2s+k}}$$

for $k = 0, 1, \dots, 2s$, $\lambda \in R_\delta$ and

$$(20) \quad [A(x, \frac{\partial}{\partial x}) - \lambda^{2s} I] \int_D P(x, \xi, \lambda) \varphi(\xi) d\xi = -\varphi(x)$$

for φ satisfying the assumption (IV).

2. Auxiliary theorems

Lemma 1. If the conditions (I), (II) and (VII) are satisfied and

$$(21) \quad D_0(x, \xi, \lambda) = \left(\sum_{i,j=1}^n a_{ij}(\xi) \frac{\partial^2}{\partial x_i \partial x_j} \right)^{s-1} P_0(x, \xi, \lambda),$$

where $P_0(x, \xi, \lambda)$ is given by (13), then

$$(22) \quad D_0(x, \xi, \lambda) = D_{01}(x, \xi, \lambda) + D_{02}(x, \xi, \lambda),$$

where

$$(23)^4 \quad D_{01}(x, \xi, \lambda) = \frac{(-1)^{s-1} [\det a(\xi)]^{-\frac{1}{2}} [a^0(\xi)]^{-1}}{(n-2) |S_1| (a^{-1}(\xi)(x-\xi), x-\xi)^{\frac{n}{2}-1}},$$

$$(24) \quad \left| \frac{\partial^k D_{02}(x, \xi, \lambda)}{\partial x_{i_1}^{k_1} \partial x_{i_2}^{k_2}} \right| \leq \frac{C(\lambda)}{\|x - \xi\|^{n-3+k}}$$

for $k = k_1 + k_2 = 0, 1, 2$, $\lambda \in R_\delta$, $x \neq \xi$, and $C(\lambda)$ denoting some positive constant dependent only on λ .

⁴ The symbol $(\cdot, \cdot)$ denotes here, and in the sequel, the standard product in E^n i.e. $(x, y) = \sum_{i=1}^n x_i y_i$, where $x = (x_1 \dots x_n)$, $y = (y_1 \dots y_n)$; $|S_1| = 2(\sqrt{\pi})^n [\Gamma(n/2)]^{-1}$ is the area of the unit sphere.

Proof. On the basis of the definitions (21), (13) we get

$$D_0(x, \xi, \lambda) = \frac{(-1)^{s-1}}{(2\pi)^n} \int_{E^n} e^{i(x-\xi, \alpha)} \times \\ \times (a(\xi)\alpha, \alpha)^{s-1} [a^0(\xi) (a(\xi)\alpha, \alpha)^{s-\lambda} 2sI]^{-1} d\alpha.$$

From the assumption (I) there exists the square root of the matrix $a(\xi)$. Let $\beta = a^{1/2}(\xi)\alpha$ for $\alpha \in E^n$. Hence $d\beta = [\det a(\xi)]^{1/2} d\alpha$, $(a(\xi)\alpha, \alpha) = \|\beta\|^2$ and

$$(25) \quad D_0(x, \xi, \lambda) = \frac{(-1)^{s-1}}{(2\pi)^n (\det a(\xi))^{1/2}} \times \\ \times \int_{E^n} e^{i(x-\xi, a^{-1/2}(\xi)\beta)} \|\beta\|^{2s-2} [a^0(\xi) \|\beta\|^{2s-\lambda} 2sI]^{-1} d\beta = \\ = \frac{(-1)^{s-1}}{(2\pi)^n (\det a(\xi))^{1/2}} \int_{E^n} e^{i(a^{-1/2}(\xi)(x-\xi), \beta)} \times \\ \times \|\beta\|^{2s-2} [a^0(\xi) \|\beta\|^{2s-\lambda} 2sI]^{-1} d\beta = \frac{(-1)^{s-1} [a^0(\xi)]^{-1}}{(2\pi)^n (\det a(\xi))^{1/2}} \times \\ \times \int_{E^n} e^{i(a^{-1/2}(\xi)(x-\xi), \beta)} \|\beta\|^{-2} [I - [a^0(\xi)]^{-1} \|\beta\|^{-2s} \lambda 2sI]^{-1} d\beta.$$

Keeping in mind the identity $[I-B]^{-1} = I + [B^{-1}I]^{-1}$ for $B = [a^0(\xi)]^{-1} \|\beta\|^{-2s} \lambda 2sI$, we get

$$(26) \quad D_0(x, \xi, \lambda) = D_{01}(x, \xi, \lambda) + D_{02}(x, \xi, \lambda),$$

where

$$D_{01}(x, \xi, \lambda) = \frac{(-1)^{s-1} [a^0(\xi)]^{-1}}{(2\pi)^n (\det a(\xi))^{1/2}} \int_{E^n} e^{i(a^{-1/2}(\xi)(x-\xi), \beta)} \|\beta\|^{-2} d\beta, \\ (27) \quad D_{02}(x, \xi, \lambda) = \frac{(-1)^{s-1} \lambda 2s [a^0(\xi)]^{-1}}{(2\pi)^n (\det a(\xi))^{1/2}} \times \\ \times \int_{E^n} e^{i(a^{-1/2}(\xi)(x-\xi), \beta)} \|\beta\|^{-2} [a^0(\xi) \|\beta\|^{2s-\lambda} 2sI]^{-1} d\beta.$$

It can be shown (see [5], p.122) that

$$D_{01}(x, \xi, \lambda) = \frac{(-1)^{s-1} (\det a(\xi))^{-1/2}}{(n-2) |S_1| \|a^{-1/2}(\xi)(x-\xi)\|^{n-2}} [a^0(\xi)]^{-1},$$

and hence we get (23). To estimate the elements of the matrix

$$\frac{\partial^k D_{02}(x, \xi, \lambda)}{\partial x_{i_1}^{k_1} \partial x_{i_2}^{k_2}} \quad \text{for } k = k_1 + k_2 = 0, 1, 2, \quad i_1, i_2 = 1, \dots, n, \quad \text{denote}$$

$$(28) \quad f_k(\beta, \xi, \lambda) = \\ = (a^{-1/2}(\xi)\beta)_{i_1}^{k_1} (a^{-1/2}(\xi)\beta)_{i_2}^{k_2} \|\beta\|^{-2} [a^0(\xi) \|\beta\|^{2s-2} S_1]^{-1},$$

where $(a^{-1/2}(\xi)\beta)_{i_r}^{k_r}$ is the k_r -th power of i_r -th coordinate

of the vector $a^{-1/2}(\xi)\beta$. Let us observe that the function f_k is "practically of the type σ " (see [4], definition 1, p.33) with $\sigma = -2s-2+k$ and, hence, on the basis of the suitable theorem (see [4], theorem 10, p.50), we can prove the following inequalities

$$(29) \quad \left| \int_{E^n} e^{i(a^{-1/2}(\xi)(x-\xi), \beta)} f_k(\beta, \xi, \lambda) d\beta \right| \leq \\ \leq \begin{cases} \frac{C(\lambda)}{\|a^{-1/2}(\xi)(x-\xi)\|^{n-2s-2+k}} & \text{for } n-2s-2+k > 0, \\ -C(\lambda) \log \|a^{-1/2}(\xi)(x-\xi)\| & \text{for } n = 2s+2-k, \\ C(\lambda) & \text{for } n-2s-2+k < 0. \end{cases}$$

Since $n-2s-2+k < n-3+k$, from the above estimate we have

$$(30) \quad \left| \int_{E^n} e^{i(a^{-1/2}(\xi)(x-\xi), \beta)} f_k(\beta, \xi, \lambda) d\beta \right| \leq \frac{C(\lambda)}{\|a^{-1/2}(\xi)(x-\xi)\|^{n-3+k}}.$$

The matrix $a^{-1/2}(\xi)$ is positively defined. Then there exist positive constants C_1, C_2 such that the inequalities $C_1 \|x-\xi\| \leq \|a^{-1/2}(\xi)(x-\xi)\| \leq C_2 \|x-\xi\|$ hold.

Finally, by (30), (27) and the above inequalities, we get (24), and the proof of Lemma 1 is completed.

Lemma 2. If the assumptions (I), (II) and (VII) are satisfied, then the transversal derivative of the fundamental solution (16) of the system (11) has the following estimate

$$(31) \quad |D_{T_z}^{2s-1} P(x, \xi, \lambda)| \leq \frac{C \exp(-\varepsilon |\lambda| \|x - \xi\|)}{\|x - \xi\|^{n-1}} \{ |\cos(n_z, x - \xi)| + \|x - \xi\| \}$$

Proof. From (16) we get

$$(32) \quad D_{T_z}^{2s-1} P(x, \xi, \lambda) = D_{T_z}^{2s-1} P_0(x, \xi, \lambda) + D_{T_z}^{2s-1} P_1(x, \xi, \lambda).$$

Keeping in mind the definition (9) of operator $D_{T_z}^{2s-1}$ and the definition (21) of $D_0(x, \xi, \lambda)$, we can write

$$(33) \quad D_{T_z}^{2s-1} P_0(x, \xi, \lambda) = \sum_{i_1, \dots, i_{2s}=1}^n a_{i_1 i_2}(x) \dots a_{i_{2s-1} i_{2s}}(x) \times \\ \times \frac{\partial^{2s-1} P_0(x, \xi, \lambda)}{\partial x_{i_1} \dots \partial x_{i_{2s-1}}} \cos(n_z, x_{i_{2s}}) = \sum_{i_1, \dots, i_{2s}=1}^n [a_{i_1 i_2}(x) \dots a_{i_{2s-1} i_{2s}}(x) - \\ - a_{i_1 i_2}(\xi) \dots a_{i_{2s-1} i_{2s}}(\xi)] \frac{\partial^{2s-1} P_0(x, \xi, \lambda)}{\partial x_{i_1} \dots \partial x_{i_{2s-1}}} \cos(n_z, x_{i_{2s}}) + \\ + \sum_{i,j=1}^n a_{ij}(\xi) \cos(n_z, x_j) \frac{\partial D_0(x, \xi, \lambda)}{\partial x_i} = I_1(x, \xi, \lambda) + I_2(x, \xi, \lambda).$$

From the assumption (II) it follows that the elements of the matrix $a(x)$ satisfy the Lipschitz condition. By (15) for $k = 2s-1$, we obtain

$$(34) \quad |I_1(x, \xi, \lambda)| \leq \frac{C \exp(-\varepsilon |\lambda| \|x - \xi\|)}{\|x - \xi\|^{n-2}}.$$

To estimate $I_2(x, \xi, \lambda)$ observe that, by (25), we have

$$(35) \quad D_0(x, \xi, \lambda) = (-1)^{s-1} (2\pi)^{-n} (\det a(\xi))^{-1/2} \times \\ \times \int_{E^n} e^{i(a^{-1/2}(\xi)(x-\xi), \beta)} \|\beta\|^{2s-2} [a^0(\xi) \|\beta\|^{2s-2} I]^{-1} d\beta.$$

Put $\gamma = M\beta$ in (35), where M is the orthogonal matrix to that of rotation of the system of coordinates in which the n -

th axis coincides with the direction of the unit-vector

$$\frac{a^{-1/2}(\xi)(x-\xi)}{\|a^{-1/2}(\xi)(x-\xi)\|}.$$

$$\text{Hence } d\gamma = d\beta, \|\gamma\| = \|\beta\|, \gamma_n = \frac{(a^{-1/2}(\xi)(x-\xi), \beta)}{\|a^{-1/2}(\xi)(x-\xi)\|} \text{ and}$$

$$(36) \quad D_0(x, \xi, \lambda) = (-1)^{s-1} (2\pi)^{-n} (\det a(\xi))^{-1/2} \times \\ \times \int_{\mathbb{E}^n} e^{i\|a^{-1/2}(\xi)(x-\xi)\| \gamma_n \|\gamma\|^{2s-1}} [a^0(\xi) \|\gamma\|^{2s-\lambda} 2s_I]^{-1} d\gamma.$$

Let

$$(37) \quad \rho = \rho(x, \xi) := \|a^{-1/2}(\xi)(x-\xi)\|^2 = (a^{-1}(\xi)(x-\xi), x-\xi).$$

From (36) we obtain

$$(38) \quad D_0(x, \xi, \lambda) = \bar{D}_0(\rho(x, \xi), \xi, \lambda),$$

where

$$(39) \quad \bar{D}_0(\rho(x, \xi), \xi, \lambda) = (-1)^{s-1} (2\pi)^{-n} (\det a(\xi))^{-1/2} \times \\ \times \int_{\mathbb{E}^n} e^{i\rho^{1/2} \gamma_n \|\gamma\|^{2s-2}} [a^0(\xi) \|\gamma\|^{2s-\lambda} 2s_I]^{-1} d\gamma.$$

Differentiating (38), by (37), we have

$$(40) \quad \frac{\partial D_0}{\partial x_r} = \frac{\partial \bar{D}_0}{\partial \rho} \frac{\partial \rho}{\partial x_r} = 2 \frac{\partial \bar{D}_0}{\partial \rho} \sum_{j=1}^n a^{rj}(\xi) (x_j - \xi_j), \quad r=1, \dots, n,$$

where $a^{rj}(\xi)$ are the elements of matrix $a^{-1}(\xi)$.

Multiplying now both sides of (40) suitably by $(x_r - \xi_r)$ and summing up after r we get

$$(41) \quad \sum_{r=1}^n (x_r - \xi_r) \frac{\partial D_0}{\partial x_r} = 2 \frac{\partial \bar{D}_0}{\partial \rho} \sum_{r,j=1}^n a^{rj}(\xi) (x_j - \xi_j) (x_r - \xi_r) = 2 \frac{\partial \bar{D}_0}{\partial \rho} \rho$$

and hence

$$(42) \quad \frac{\partial \bar{D}_0}{\partial \rho} = \sum_{r=1}^n \frac{x_r - \xi_r}{2\rho} \frac{\partial D_0}{\partial x_r}.$$

By (40) and (42), we can write $I_2(x, \xi, \lambda)$ from (33) in the form

$$\begin{aligned}
(43) \quad I_2(x, \xi, \lambda) &= \sum_{r,p=1}^n a_{rp}(\xi) \cos(n_z, x_p) \frac{\partial D_0(x, \xi, \lambda)}{\partial x_r} = \\
&= \sum_{r,p=1}^n a_{rp}(\xi) \cos(n_z, x_p) \frac{\partial \bar{D}_0(\rho, \xi, \lambda)}{\partial \rho} 2 \sum_{j=1}^n a^{rj}(\xi) (x_j - \xi_j) = \\
&= 2 \frac{\partial \bar{D}_0(\rho, \xi, \lambda)}{\partial \rho} \sum_{p,j=1}^n \left[\sum_{r=1}^n a_{rp}(\xi) a^{rj}(\xi) \right] (x_j - \xi_j) \cos(n_z, x_p) = \\
&= 2 \frac{\partial \bar{D}_0(\rho, \xi, \lambda)}{\partial \rho} \sum_{j=1}^n (x_j - \xi_j) \cos(n_z, x_j) = 2 \|x - \xi\| \frac{\partial \bar{D}_0(\rho, \xi, \lambda)}{\partial \rho} \cos(n_z, x - \xi) = \\
&= \|x - \xi\| \left[\sum_{r=1}^n \frac{x_r - \xi_r}{\partial \rho} \frac{\partial D_0(x, \xi, \lambda)}{\partial x_r} \right] \cos(n_z, x - \xi).
\end{aligned}$$

The matrix $a^{-1}(\xi)$ is positively defined. Then there exist positive constants C'_1, C'_2 such that the inequalities

$$(44) \quad C'_1 \|x - \xi\|^2 \leq \rho(x, \xi) \leq C'_2 \|x - \xi\|^2$$

hold. The matrix $\frac{\partial D_0(x, \xi, \lambda)}{\partial x_r}$ (see (21)) is a combination of derivatives of order $2s-1$. Thus, by (42), (43) and (15), we get

$$(45) \quad |I_2(x, \xi, \lambda)| \leq \frac{C \exp(-\varepsilon |\lambda| \|x - \xi\|)}{\|x - \xi\|^{n-1}} |\cos(n_z, x - \xi)|.$$

From the inequality (18) for $k = 2s-1$ we have

$$(46) \quad |D_{T_z}^{2s-1} P_1(x, \xi, \lambda)| \leq \frac{C \exp(-\varepsilon |\lambda| \|x - \xi\|)}{\|x - \xi\|^{n-2}}.$$

Finally, combining (32)-(34), (45), (46), we obtain (31).

Lemma 3. Let $S = \partial D$ be a closed Lyapunov's surface with the exponent $\kappa \in (0, 1)$ and

$$(47) \quad u(x, \lambda) = \int_S P(x, \xi, \lambda) \Phi(\xi, \lambda) d\xi,$$

where $P(x, \xi, \lambda)$ is the fundamental solution of the system (11). If the function $\Phi(\xi, \lambda)$ is continuous on S for all $\lambda \in R_\delta$ and the

assumptions (I), (II), (VII) are satisfied, then the transversal derivative of the potential (47) has, for $\lambda \in R_\delta$, the following limit property

$$(48) \quad \lim_{D \ni x \rightarrow z \in S} D_{T_z}^{2s-1} u(x, \lambda) = \\ = \frac{(-1)^s}{2} [a^0(z)]^{-1} \Phi(z, \xi) + \int_S D_{T_z}^{2s-1} P(x, \xi, \lambda) \Phi(\xi, \lambda) d\xi,$$

where x tends to z along a curve satisfying the condition (10).

Proof. By (16), we can write

$$(49) \quad D_{T_z}^{2s-1} u(x, \lambda) = D_{T_z}^{2s-1} \int_S P_0(x, \xi, \lambda) \Phi(\xi, \lambda) d\xi + \\ + \int_S D_{T_z}^{2s-1} P_1(x, \xi, \lambda) \Phi(\xi, \lambda) d\xi = I_1^*(x, \lambda) + I_2^*(x, \lambda)$$

for $x \in D$, $\lambda \in R_\delta$. It follows from (46) that

$$(50) \quad \lim_{x \rightarrow z \in S} I_2^*(x, \lambda) = I_2^*(z, \lambda).$$

According to Lemma 1, we can write

$$(51) \quad I_1^*(x, \lambda) = \int_S \sum_{i_1, \dots, i_{2s}=1}^n [a_{i_1 i_2}(x) \dots a_{i_{2s-3} i_{2s-2}}(x) - \\ - a_{i_1 i_2}(\xi) \dots a_{i_{2s-3} i_{2s-2}}(\xi)] a_{i_{2s-1} i_{2s}}(x) \frac{\partial^{2s-1} P_0(x, \xi, \lambda)}{\partial x_{i_1} \dots \partial x_{i_{2s-1}}} \cos(n_z, x_{i_{2s}}) \times \\ \times \Phi(\xi, \lambda) d\xi + \int_S \sum_{i, j=1}^n a_{ij}(x) \frac{\partial D_{02}(x, \xi, \lambda)}{\partial x_i} \cos(n_z, x_j) \Phi(\xi, \lambda) d\xi + \\ + \int_S \sum_{i, j=1}^n a_{ij}(x) \frac{\partial D_{01}(x, \xi, \lambda)}{\partial x_i} \cos(n_z, x_j) \Phi(\xi, \lambda) d\xi = \\ = I_{11}^*(x, \lambda) + I_{12}^*(x, \lambda) + I_{13}^*(x, \lambda).$$

From the assumption (II) and the inequalities (15), (24) we conclude that the integrals $I_{11}^*(x, \lambda)$, $I_{12}^*(x, \lambda)$ are continuous on $\bar{D}$. The integral $I_{13}^*(x, \lambda)$ is the first-order transversal derivative of a quasi-potential (see (23)) with the density $\frac{(-1)^{s-1} [a^0(\xi)]^{-1} \Phi(\xi, \lambda)}{(\det a(\xi))^{1/2} (n-2) |S_1|}$.

Making use of the appropriate result (see [6], p. 270) we get

$$(52) \quad \lim_{D \ni x \rightarrow z \in S} I_{13}^*(x, \lambda) = \frac{(-1)^S}{2} [a^0(z)]^{-1} \Phi(z, \xi) + I_{13}^*(z, \lambda).$$

Finally, combining (49)-(52), we have the limit property (48).

3. Estimates of the solution of the boundary-value problem

We seek a solution of the problem (2), (3) in the form

$$(53) \quad u(x, \lambda) = \int_S P(x, \xi, \lambda) \Phi(\xi, \lambda) d\xi.$$

By (3) and Lemma 3, we get the integral equation

$$(54) \quad \frac{(-1)^S}{2} [b^0(z) + \lambda^{2S} b^1(z)] [a^0(z)]^{-1} \Phi(z, \xi) + \\ + \int_S B(z, \lambda^{2S}, \frac{\partial}{\partial z}) P(z, \xi, \lambda) \Phi(\xi, \lambda) d\xi = \psi(z, \lambda).$$

Under the assumption (VI), we can define the following functions

$$(55) \quad \mu(z, \lambda) = 2(-1)^S a^0(z) [b^0(z) + \lambda^{2S} b^1(z)]^{-1} \psi(z, \lambda),$$

$$(56) \quad K(z, \xi, \lambda) = 2(-1)^{S-1} a^0(z) \{ D_{T_z}^{2S-1} + \\ + [b^0(z) + \lambda^{2S} b^1(z)]^{-1} B_1(z, \lambda^{2S}, \frac{\partial}{\partial z}) \} P(z, \xi, \lambda).$$

Thus, we can rewrite (54) in the form

$$(57) \quad \Phi(z, \lambda) = \mu(z, \lambda) + \int_S K(z, \xi, \lambda) \Phi(\xi, \lambda) d\xi, \quad z \in S, \quad \lambda \in R_\delta.$$

By (II), (VI), Lemma 2 and (19), we get

$$(58) \quad |K(z, \xi, \lambda)| \leq \frac{C \exp(-c|\lambda| \|z - \xi\|)}{\|z - \xi\|^{n-1-\kappa}}$$

for $\lambda \in R_\delta$, $z, \xi \in S$, $z \neq \xi$ and $\kappa \in (0, 1)$ (cf. Lemma 3).

It can be proved by the method used in [1] (see Lemma 2, p. 385) that there is a unique solution of (57) of the form

$$(59) \quad \Phi(z, \lambda) = \mu(z, \lambda) + \int_S R(z, \xi, \lambda) \mu(\xi, \lambda) d\xi$$

and the resolvent $R(z, \xi, \lambda) = \sum_{r=1}^{\infty} K_r(z, \xi, \lambda)$ ($K_1 = K$ and K_r is the r -th iteration of the kernel K), has the following estimate

$$(60) \quad |R(z, \xi, \lambda)| \leq \frac{C \exp(-\varepsilon |\lambda| \|z - \xi\|)}{\|z - \xi\|^{n-1-\kappa}},$$

for $z, \xi \in S$, $z \neq \xi$, $\lambda \in R_\delta$ (cf. (1)). It results from (60), (55) and (VI) that the function Φ given by (59) satisfies the inequalities

$$(61) \quad |\Phi(z, \lambda)| \leq |\mu(z, \lambda)| + C_1 \sup_{\xi \in S} |\mu(\xi, \lambda)| \leq C_2 \sup_{\xi \in S} |\mu(\xi, \lambda)| \leq C \sup_{\xi \in S} |\psi(\xi, \lambda)|$$

for $z \in S$ and $\lambda \in R_\delta$. Thus, by (19) and (61), we get

$$(62) \quad \left| \frac{\partial^k u(x, \lambda)}{k_1 \partial x_1 \dots k_n \partial x_n} \right| \leq C \sup_{\xi \in S} |\psi(\xi, \lambda)| \frac{\exp(-\varepsilon |\lambda| d(x))}{d(x)^{n-2s+k}},$$

for $x \in D$; $\lambda \in R_\delta$; $k = 0, 1, \dots, 2s$, where $d(x) = \inf_{\xi \in S} \|x - \xi\|$, and

$$(63) \quad \left| \frac{\partial^k u(z, \lambda)}{k_1 \partial z_1 \dots k_n \partial z_n} \right| \leq C \sup_{\xi \in S} |\psi(\xi, \lambda)|$$

for $z \in S$, $\lambda \in R_\delta$, $k = 0, 1, \dots, 2s-2$. By Lemma 2, we have

$$(64) \quad |D_z^{2s-1} P(z, \xi, \lambda)| \leq \frac{C \exp(-\varepsilon |\lambda| \|z - \xi\|)}{\|z - \xi\|^{n-1-\kappa}}$$

for $z, \xi \in S$, $z \neq \xi$, $\lambda \in R_\delta$. Hence, we obtain

$$(65) \quad |D_z^{2s-1} u(z, \lambda)| \leq C \sup_{\xi \in S} |\psi(\xi, \lambda)| \quad \text{for } \lambda \in R_\delta.$$

Finally, we can formulate the following theorem.

Theorem 1. If the assumptions (I)-(III), (V)-(VII) are satisfied and S is the Lyapunov surface with exponent $\kappa \in (0, 1)$, then the problem (2), (3) (for $\varphi(x) \equiv 0$) has the solution given by the formula (53). This solution and its derivatives have

the estimates (62), (63), (65).

4. The Green matrix for the boundary-value problem

We seek the Green matrix for the problem (2), (3) in the form

$$(66) \quad G(x, \xi, \lambda) = P(x, \xi, \lambda) - Q(x, \xi, \lambda),$$

where $P(x, \xi, \lambda)$ is the fundamental solution of (11), and the regular part $Q(x, \xi, \lambda)$ can be represented as the potential of a single layer

$$(67) \quad Q(x, \xi, \lambda) = \int_S P(x, y, \lambda) h_1(y, \xi, \lambda) dy.$$

Thus, $Q(x, \xi, \lambda)$ is the solution of (11) and satisfies the boundary condition

$$(68) \quad \lim_{D \ni x \rightarrow z \in S} B(z, \lambda^{2s}, \frac{\partial}{\partial x}) Q(x, \xi, \lambda) = g(z, \xi, \lambda),$$

where

$$(69) \quad g(z, \xi, \lambda) = B(z, \lambda^{2s}, \frac{\partial}{\partial z}) P(z, \xi, \lambda) \quad \text{for } \xi \in D, \lambda \in R_\delta.$$

According to Lemma 3, similarly as in section 2, it can be proved that the matrix $h_1(y, \xi, \lambda)$ satisfies the integral equation

$$(70) \quad h_1(z, \xi, \lambda) = \mu_1(z, \xi, \lambda) + \int_S K(z, y, \lambda) h_1(y, \xi, \lambda) dy$$

with

$$(71) \quad \mu_1(z, \xi, \lambda) = 2(-1)^s a^0(z) [b^0(z) + \lambda^{2s} b^1(z)]^{-1} g(z, \xi, \lambda)$$

and kernel (56) satisfying (60). Hence, (70) has the unique solution

$$(72) \quad h_1(z, \xi, \lambda) = \mu_1(z, \xi, \lambda) + \int_S R(z, y, \lambda) \mu_1(y, \xi, \lambda) dy.$$

By (71), (69), (19), (31) and (VI), we get

$$(73) \quad |\mu_1(z, \xi, \lambda)| \leq \frac{C \exp(-c|\lambda| \|z - \xi\|)}{\|z - \xi\|^{n-1}} |\cos(n_z, z - \xi)|$$

for $z \neq \xi$, $z \in S$, $\xi \in \bar{D}$, $\lambda \in R_\delta$. From (73), (60), using the method analogous to that of [2] (see Lemma 3, p. 173), we get

$$(74) \quad \left| \int_S R(z, y, \lambda) \mu_1(y, \xi, \lambda) dy \right| \leq \frac{C \exp(-\varepsilon |\lambda| \|z - \xi\|)}{\|z - \xi\|^{n-1-\kappa}}$$

for $z \neq \xi$, $z \in S$, $\xi \in \bar{D}$, $\lambda \in R_\delta$. Combining (70), (73) and (74), we obtain

$$(75) \quad |h_1(z, \xi, \lambda)| \leq \frac{C \exp(-\varepsilon |\lambda| \|z - \xi\|)}{\|z - \xi\|^{n-1}} (|\cos(n_z, z - \xi)| + \|z - \xi\|^\kappa)$$

for $z \neq \xi$, $z \in S$, $\xi \in \bar{D}$, $\lambda \in R_\delta$. Hence, by the method of [2] (see Theorem 4, p.196, Lemma 3, p.176, Lemma 2, p.175) we can prove the following theorem.

Theorem 2. If the assumptions (I)-(III), (V)-(VII) are satisfied and S is the Lyapunov surface with the exponent $\kappa \in (0, 1)$, then, for all $\lambda \in R_\delta$, the problem (2), (3) (for $\psi(z, \lambda) = 0$) has the solution

$$(76) \quad u(x, \lambda) = \int_D G(x, \xi, \lambda) \phi(\xi) d\xi.$$

The regular part (67) of the Green function $G(x, \xi, \lambda)$ and its derivatives have the following estimates

$$(77) \quad \left| \frac{\partial^k Q(x, \xi, \lambda)}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \right| \leq \frac{C \exp(-\varepsilon |\lambda| \|x - \xi\|)}{d(x)^{n-2s+k}}, \quad k=0, 1, \dots, 2s,$$

for $x \neq \xi$, $x \in D_1 \subset D$, $\xi \in \bar{D}$, with $d(x) = \inf_{y \in S} \|x - y\|$,

$$(78) \quad \left| \frac{\partial^k Q(x, \xi, \lambda)}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \right| \leq \frac{C \exp(-\varepsilon |\lambda| \|x - \xi\|)}{\|x - \xi\|^{n-2s+k}}, \quad k=0, 1, \dots, 2s-2,$$

for $x \neq \xi$, $x, \xi \in \bar{D}$,

$$(79) \quad \left| \frac{\partial^k Q(x, \xi, \lambda)}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \right| \leq \frac{C \exp(-\varepsilon |\lambda| \|x - \xi\|)}{|\lambda|^{2s-1-k}}, \quad k=0, 1, \dots, 2s-2,$$

for $x \neq \xi$, $x \in D$, $\xi \in D_1$ and

$$(80) \quad |D_T^{2s-1} Q(z, \xi, \lambda)| \leq \frac{C \exp(-\varepsilon |\lambda| \|z - \xi\|)}{\|z - \xi\|^{n-1-\kappa}}$$

for $z \in S$, $\xi \in D$, $\lambda \in R_\delta$.

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