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A COUNTEREXAMPLE IN THE THEORY OF MIXED
BOUNDARY VALUE PROBLEMS FOR ELLIPTIC EQUATIONS
IN NON-SMOOTH DOMAINS

0. Introduction

In the theory of mixed boundary value problems (MBVP) in domains with edges on boundary an important role is played by the formula of decomposition of the solution into regular and singular parts. Such a decomposition, which gives an explicit description of the behaviour of the solution close to the edge of boundary and is known to exist except may be for some points of the neighbourhood of the edge. In this paper we construct an example, showing that it is impossible to extend the formula across such exceptional points.

Let us consider the following problem: determine a function u satisfying

$$(0.1) \quad \begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Gamma_1, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_2, \end{cases}$$

where Ω is an open, bounded subset of $\mathbb{R}^3$ with piecewise smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_0$ where Γ_1, Γ_2 are open smooth, not intersecting submanifolds, Γ_0 is a smooth, closed curve in $\partial\Omega$ such that at every point of Γ_0 there is a well defined angle ω , $0 < \omega < 2\pi$, between Γ_1 and Γ_2 .

If we confine ourselves to the standard Sobolev space framework of this problem, then it is known that for $f \in L_2(\Omega)$ there exists $u \in H^1(\Omega)$ being a variational solution to (0.1) and that this solution cannot be regularized to $H^2(\Omega)$, as in the case of, say, Dirichlet problem in smooth domain [5]. However, it follows that such a solution is regular away from the edges thus both from the theoretical and practical points of view it is desirable to determine the behaviour of the solution close to the edges.

There are two main methods of dealing with this problem. First, we can use a weighted Sobolev space approach, where the behaviour of the solution near singular points of the boundary is described in terms of an appropriate weight [4]. Another method, which we are concerned with, is to represent the solution as a sum of a regular function and a term incorporating functions which determine all possible singularities of the solution. This procedure proved to be particularly fruitful in two-dimensional problems [5], [1]. Then Ω is assumed to be a curvilinear polygon with Γ_0 being the set of its vertices. It was proved that every variational solution u of (0.1) had the following representation in some neighbourhood of every vertex of Ω

$$(0.2) \quad u = u_{\text{reg}} + \sum_{j=1}^m c_j S_j,$$

where $u_{\text{reg}} \in H^2(\Omega)$, $c_j \in \mathbb{C}$ for $j = 1, \dots, m$, $m < \infty$, and S_j are explicitly given functions of the form $r^\alpha \cdot \sin \beta \theta$, where (r, θ) are polar coordinates centered at the vertex of Ω and α, β are parameters depending only on the geometry of the problem. By summing over all vertices of Ω it can be proved that the formula analogous to (0.2) holds globally in Ω . It is known that for some exceptional values of the measure ω of the angle of the vertex of Ω (in particular such exceptional values for (0.1) are $\pi/2$ and $3\pi/2$) the range of the operator associated with (0.1) is not closed in the space of the data [5] and subsequently such cases were excluded from considerations of the previous authors. However in [1], [2] a way of overcoming this difficulty has been found and it has been proved that (0.2) holds even if ω is exceptional, also in the case of general elliptic operators and general boundary conditions.

In n -dimensional problems situation turned out to be more difficult due to the fact that the number of singular functions, which are to be taken into account, is infinite [5]. However, in regions with edges it is possible to derive a kind of a counterpart to (0.2). To describe the situation we again confine ourselves to (0.1) which is sufficient to demonstrate main features of the problem. It follows, e.g. [6], that we have the following formulas for $H^1(\Omega)$ -solutions u of (0.1).

$$(0.3a) \quad u(x) = u_{\text{reg}}(x) \quad \text{if } 0 < \omega(t) < \frac{\pi}{2},$$

$$(0.3b) \quad u(x) = u_{\text{reg}}(x) + \eta(x) c_0(t) r^{\pi/2\omega(t)} \cdot \cos \frac{\pi\theta}{2\omega(t)}$$

$$\text{if } \frac{2}{\pi} < \omega(t) < \pi \text{ or } \pi < \omega(t) < \frac{3\pi}{2},$$

$$(0.3c) \quad u(x) = u_{\text{reg}}(x) + \eta(x) \left[c_1(t) r^{\pi/2\omega(t)} \cdot \cos \frac{\pi\theta}{2\omega(t)} + \right. \\ \left. + c_2(t) r^{3\pi/2\omega(t)} \cdot \cos \frac{3\pi\theta}{2\omega(t)} \right] \quad \text{if } \frac{3\pi}{2} < \omega(t) < 2\pi,$$

where $x := (t, r, \theta)$, $t \in \Gamma_0$, (r, θ) are polar coordinates centered at t , η is an appropriate cut-off function, u_{reg} is a regular function and c_i , $i = 1, 2, 3$, are square summable over compact subsets of intervals indicated above.

We mention that the exceptional values $\pi/2$ and $3\pi/2$ depend on the type of boundary conditions and space which we are working with [6], whereas the exceptional value π is due to the geometry of the domain. It is not difficult to show that (0.3a) (resp. (0.3b)) holds for $\omega(t) \leq \pi/2$ (resp. $\omega(t) \leq 3\pi/2$). In [3] the author proved that (0.3b) holds for $\pi/2 < \omega(t) \leq 3\pi/2$. However, the case of $\omega(t) \rightarrow \pi^+/2$ (and $3\pi^+/2$) remains unclear.

In this note we shall construct an example in which the coefficient $c_0(t)$ of the variational solution of (0.1) is not integrable over $[\pi/2, \pi/2 + \varepsilon]$, $\varepsilon > 0$, despite the fact that $f \in L_2(\Omega)$, showing thus that the universal decomposition formula is unavailable.

1. Geometrical preliminaries

Since our aim is to construct a counter-example, we confine ourselves to the simplest sufficient situation (which the general one is usually reduced to). Therefore we shall work in a special domain called "generalized dihedral angle" [3].

Let $\Theta(\vec{a}, \vec{b})$ denote the measure of the angle between given vectors $\vec{a}$ and $\vec{b}$, we also denote $\vec{x}_3(x) := (x_1, x_2, 0)$ where $x = (x_1, x_2, x_3)$. Then we can define a domain

$$(1.1) \quad \Omega := \{x \in \mathbb{R}^3; 0 < \Theta(\vec{e}_1, \vec{x}_3(x)) < \omega(x_3)\}$$

where $\vec{e}_1 = (1, 0, 0)$ and $\omega(\cdot)$ is a scalar function of one variable having an appropriate degree of smoothness.

In other words Ω is such a domain that its intersection with the plane $\Pi(t) := \{x \in \mathbb{R}^3; x_3 = t = \text{const}\}$ is a rectilinear angle of measure $\omega(x_3)$. Having compared this definition with the notation of the Introduction, we see that $\Gamma_2 = \{x \in \mathbb{R}^3; x_2 = 0\}$, $\Gamma_1 = \{x \in \mathbb{R}^3; \Theta(\vec{x}_3(x), \vec{e}_1) = \omega(x_3)\}$, $\Gamma_0 = \{x \in \mathbb{R}^3; x_1 = x_2 = 0\}$.

Such a domain has an advantage of providing a possibility to apply two-dimensional results for rectilinear angles in the set defined by

$$(1.2) \quad \Omega(t) = \Omega \cap \Pi(t).$$

Similarly we define

$$(1.3) \quad \Gamma_i(t) := \Gamma_i \cap \Pi(t) \quad i = 0, 1, 2.$$

2. Two-dimensional problem with parameter

In this section we consider the following problem: determine $u(\cdot, t)$ such that

$$(2.1) \quad \begin{cases} \Delta_2 u = u_{x_1 x_2} + u_{x_2 x_2} = f & \text{in } \Omega(t), \\ u = 0 & \text{on } \Gamma_1(t), \\ u_{x_2} = 0 & \text{on } \Gamma_2(t), \end{cases}$$

where $f = f(\cdot, t) \in L_2(\Omega(t))$ for every $t \in \mathbb{R}$ and u_{x_i} denotes partial derivative with respect to x_i . This is a family of two dimensional problems in rectilinear angles with measures $\omega(t)$. We assume that $\omega(0) = \pi/2$ and $\omega(t) > \pi/2$ for $t > 0$. It is known [1] that the variational solution $u(\cdot, t)$ of (2.1) has for $t \geq 0$ a representation:

$$(2.2) \quad u(r, \theta, t) = u_{\text{reg}}(r, \theta, t) + \eta(r)c(t)r^{\pi/2\omega(t)} \cdot \cos \frac{\pi\theta}{2\omega(t)},$$

where (r, θ) are polar coordinates centered at $(0, 0)$, $u_{\text{reg}}(\cdot, t)$ is an H^2 -function for almost every t , η is a cut-off function, equal to one for $r < e^{-1}$ and satisfying the boundary conditions of (2.1) (such a construction is possible, see [5]). It follows [6] that in such a simple case the function c can be determined by the formula

$$(2.3) \quad c(t) = \frac{2}{\pi} \left[\int_{\Omega(t)} \eta(r)f(r, \theta, t)r^{-\pi/2\omega(t)+1} \cdot \sin \frac{\pi\theta}{2\omega(t)} d\theta dr \right. \\ \left. + \int_{\Omega(t)} u(r, \theta, t)\Delta_2 \left[\eta r^{-\pi/2\omega(t)} \cdot \sin \frac{\pi\theta}{2\omega(t)} \right] r dr d\theta \right].$$

LEMMA 2.1. *There exist a monotonic function ω defined on $[0, t_\varepsilon]$, with $\omega(t_\varepsilon) := \frac{\pi}{2} + \varepsilon$ for some $\varepsilon > 0$ and a function $f(\cdot, t) \in L_2(\Omega(t))$, $t \in [0, t_\varepsilon]$, such that*

$$(2.4) \quad \int_0^{t_\varepsilon} |c(t)|^2 dt = +\infty.$$

Proof. Let ω be an arbitrary function satisfying assumptions of the lemma. First of all let us notice that, due to the fact that $\eta \equiv 1$ close to zero and $r^{-\pi/2\omega(t)} \cdot \sin(\pi\theta/2\omega(t))$ is a harmonic function of (r, θ) , the contribution to (2.4) of the second integral I_2 in (2.3) is always finite. Thus, we shall focus

ourselves on the first integral I_1 in (2.3). Let us consider

$$(2.5) \quad f(r, \theta) = \begin{cases} r^{-1}(-\log r)^q & \text{for } 0 \leq r \leq e^{-1}, \\ 0 & \text{for } r > e^{-1}. \end{cases}$$

We have

$$(2.6) \quad \int_{\Omega(t)} |f|^2 dx = \frac{\omega(t)}{2} \int_0^{e^{-1}} r^{-1}(-\log r)^{-2q} dr < +\infty,$$

if $q > 1/2$. On the other hand, putting $\alpha(t) := 1 - \pi/2\omega(t)$, we have

$$I_1 = \frac{4\omega(t)}{\pi^2} \int_0^{e^{-1}} r^{\alpha(t)-1} \cdot (-\ln r)^{-q} dr = \frac{4\omega(t)\alpha^{q-1}(t)}{\pi^2} \int_{\alpha(t)}^{\infty} s^{-q} e^{-s} ds.$$

Now, $g(t) := \int_{\alpha(t)}^{\infty} s^{-q} e^{-s} ds$ is a continuous function of t and $0 < g(0) < \infty$ provided $q < 1$. Therefore, c belongs to $L_2([0, t_\epsilon])$ iff α^{q-1} does. However, let $\omega(t) = t^2 + \pi/2$; then $\alpha^{q-1} = (t^2/\pi + 2t^2)^{q-1}$ does not belong to $L_2([0, t_\epsilon])$ for $q < 3/4$ and thus (2.6) and (2.4) hold for $1/2 < q < 3/4$. ■

3. The three-dimensional problem

To reach our aim we have to show that there exists a function $F \in L_2(\Omega)$ (see Section 1) such that $u(\cdot, t)$ is a H^1 -solution of a three dimensional problem. It would be sufficient to show that u_t and u_{tt} belong to $L_2(\Omega)$. Indeed, then we would define $F(r, \theta, t) = \vartheta(t)f(r) + u_{tt}$, where f is defined by (2.5) and ϑ is a cut-off function equal to one in a neighbourhood of 0. We shall proceed along this lines, however the problem turns out to be more complicated.

Thanks to the simple form of (2.1), we can determine $u(\cdot, t)$ explicitly by means of the Mellin transform. We define (cf. [7])

$$(3.1) \quad M(v(r))(s) := \hat{v}(s) := \int_0^{\infty} r^{s-1} v(r) dr, \quad s = s_r + is_i \in \mathbb{C}.$$

The Mellin transform is closely related to the complex Fourier transform, thus we shall refer the reader to e.g. [7]. We state only the counterpart of Plancherel's identity which is the main tool of our investigations. Let $L_2(A, \rho)$ denote the set of (classes of) functions which are square-integrable over A with the weight ρ and denote $P(s_r) := \{s; s = s_r + i\sigma, \sigma \in \mathbb{R}\}$. Then we have

$$\hat{v} \in L_2(P(s_r)) \text{ iff } v \in L_2([0, \infty], r^{2s_r-1})$$

with

$$(3.2) \quad \int_{s_r - i\infty}^{s_r + i\infty} |\widehat{v}(s)|^2 ds = 2\pi \int_0^\infty |v(r)|^2 r^{2s_r - 1} dr.$$

THEOREM 3.1. *Let $J_\varepsilon := \{t; \pi/\varepsilon \leq \omega(t) \leq \pi/2 + \varepsilon\}$ for some $\varepsilon > 0$ and $s_r \in E := \mathbb{R} \setminus (]-\infty, -1] \cup [\pi/(\pi + 2\varepsilon), 1] \cup \mathbb{Z})$, where $\mathbb{Z}$ denotes the set of integers. If u is the solution of (2.1) then we have*

$$(3.3) \quad u_t, u_{tt}, ru_t, (ru_r)_t, (ru_r)_{tt} \in L_2(\Omega, r^{2s_r - 2}).$$

Proof. Let us consider the right-hand side of the equation in (2.1) given by $\vartheta(t)f(r)$ where ϑ is a $C_0^\infty(\mathbb{R})$ function equal to 0 in the neighbourhood of 0. Changing (2.1) into (r, θ) variables, denoting $f_1(r) = r^2 f(r)$ and applying the Mellin transform, we obtain the following problem

$$(3.4) \quad \begin{cases} \widehat{u}_\phi + s^2 u = \vartheta \cdot \widehat{f}_1, \\ \widehat{u}(0) = 0, \\ \widehat{u}_\phi(\omega) = 0. \end{cases}$$

Let us denote $G(s, \phi, t) = \frac{\sin \frac{s\phi}{2} \sin \frac{s(\phi - 2\omega)}{2}}{s^2 \cos(s\omega)}$, $\omega = \omega(t)$. After simple calculations we get

$$(3.5) \quad \widehat{u}(s, \phi, t) = \vartheta(t) \widehat{f}_1(s) G(s, \phi, s).$$

By (3.2), $\widehat{f}_1 \in L_2(P(s_r))$ for every $s_r > -1$ and in order to obtain $u \in L_2(\Omega)$ it is necessary to have $\widehat{u}(\phi, s, t) \in L_2(P(1))$. However, if $\omega(t) = \pi/2$, then $\cos(s\omega) = 0$ for $s_r = 1$ and $s_r = 0$, thus G is unbounded and so in general is not square-integrable over $P(1)$. Let us consider $t \in J_\varepsilon$, then $G(s, \phi, t)$ is bounded on every line $P(s_r)$ if $s_r \in E$, hence by (3.2) $u \in L_2(\Omega, r^{2s_r - 2})$. Since $\widehat{ru}_r = s\widehat{u}$ and both $G_t(s, \phi, t)$ and $G_{tt}(s, \phi, t)$ are equal to $O(e^{\pm s_i(\phi - 2\omega(t))})$ as $s_i \rightarrow \pm\infty$ with $0 \leq \phi \leq \omega(t)$, we obtain the remaining relations of (3.3). ■

However, this is insufficient for our purposes, since if we take $s_r < 1$, then we have better behaviour at 0 and worse at infinity than required in $L_2(\Omega)$ and otherwise, if $s_r > 1$. To overcome this difficulty we modify u to have a bounded support. In such a case weighted estimates for $s_r < 1$ are sufficient for $L_2(\Omega)$. Let us consider $\chi \in C_0^\infty([0, \infty[)$ such that $\chi(r) \equiv 1$ for $r < e^{-1}$.

LEMMA 3.2. *Under assumptions of Theorem 3.1, for $v := \chi u$ we have*

$$(3.6) \quad v, v_t, v_{tt} \in L_2(\Omega).$$

Proof. The function $v = \chi u$ where u is the solution of (2.1) satisfies

$$(3.7) \quad \begin{cases} \Delta_2 v = \chi \vartheta f = \nabla \chi \nabla u - u \nabla \chi = \chi \vartheta f + g = F, \\ v(r, 0) = 0, \\ v_\phi(r, 0) = 0, \end{cases}$$

where, by (3.3), $F, F_t, F_{tt} \in L_2(\Omega)$, since $|\nabla \chi| = \Delta \chi \equiv 0$ for $r < e^{-1}$. As before, we reduce (3.7) to (3.4). However, the right-hand side of the equation depends now also on ϕ , so we have to make use of the Green function $\mathcal{G}$ of the problem (3.4) which gives $\hat{v}$ in the form (cf. [7])

$$(3.8) \quad \begin{aligned} \hat{v}(s, \phi, t) &= s^{-2} \cos^{-1}(s\omega) \left\{ \cos[s(\phi - \omega)] \int_0^\phi \hat{F}(s, \theta, t) \sin(s\theta) d\theta \right. \\ &\quad \left. + \sin(s\phi) \int_\phi^\omega \hat{F}(s, \theta, \omega) \cos[s(\theta - \omega)] d\theta \right\} \\ &= s^{-2} \cos^{-1}(s\omega) \int_0^\omega \mathcal{G}(\phi, \theta, s, \omega) \hat{F}(s, \theta, t) d\theta. \end{aligned}$$

Now we have for some $s_r \in E$

$$(3.9) \quad \begin{aligned} &\int_{s_r - i\infty}^{s_r + i\infty} |\hat{v}(s, \phi, \omega)|^2 ds \\ &= \int_{s_r - i\infty}^{s_r + i\infty} |s^{-2} \cos^{-1}(s\omega) \int_0^\omega \mathcal{G}(\phi, \theta, s, \omega) \hat{F}(\theta, s, \omega) d\theta|^2 ds \\ &\leq \int_{s_r - i\infty}^{s_r + i\infty} \left(s^{-4} \cos^{-2}(s\omega) \int_0^\omega |\mathcal{G}(\phi, \theta, s, \omega)|^2 d\theta \cdot \int_0^\omega |\hat{F}(\theta, s, \omega)|^2 d\theta \right) ds. \end{aligned}$$

Since

$$\int_{-\infty}^{+\infty} \int_{s_r - i\infty}^{s_r + i\infty} \int_0^\omega |\hat{F}(\theta, s, \omega)|^2 d\theta ds dt < +\infty \quad (\omega = \omega(t)),$$

by (3.6) and (3.2), it is enough to prove the boundedness of

$$\begin{aligned} I := |s^{-4} \cos^{-2}(s\omega)| &\left(|\cos[s(\phi - \omega)]|^2 \int_0^\phi |\sin(s\theta)|^2 d\theta \right. \\ &\quad \left. + |\sin(s\phi)|^2 \int_\phi^\omega |\cos[s(\theta - \omega)]|^2 d\theta \right). \end{aligned}$$

However, short calculations show that

$$(3.10) \quad I = O(s_i^{-4} \exp[\pm 2s_i(\phi - \omega)]) \quad \text{as } s_i \rightarrow \mp\infty,$$

so, since $0 \leq \phi \leq \omega$, we obtain $v \in L_2(\Omega, r^{2s_r-2})$ which implies $v \in L_2(\Omega)$ thanks to the fact that $v \equiv 0$ in some neighbourhood of 0.

Further we have

$$\begin{aligned} \hat{v}_t(s, \phi, t) = & -2s^{-1}\omega_t \sin(s\omega) \cos^{-2}(s\omega) \int_0^\omega \mathcal{G}(\phi, \theta, s, \omega) \hat{F}(\theta, s, \omega) d\theta \\ & + \omega_t s^{-2} \cos^{-1}(s\omega) \left(\mathcal{G}(\phi, \omega, s, \omega) \hat{F}(\omega, s, \omega) \right. \\ & + \int_0^\omega \mathcal{G}_\omega(\phi, \theta, s, \omega) \hat{F}(\theta, s, \omega) d\theta \\ & \left. + \int_0^\omega \mathcal{G}(\phi, \theta, s, \omega) \hat{F}_\omega(\theta, s, \omega) d\theta \right) = I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Now, we get analogously to (3.10) for $s_i \rightarrow \pm\infty$

$$\begin{aligned} |I_1|^2 & \leq O(s_i^{-1} \exp(\pm 2s_i \omega)) \int_0^\omega |\mathcal{G}(\phi, \theta, s, \omega)|^2 d\theta \cdot \int_0^\omega |\hat{F}(\theta, s, \omega)|^2 d\theta \\ (3.11) \quad & \leq \text{const} \int_0^\omega |\hat{F}(\theta, s, \omega)|^2 d\theta. \end{aligned}$$

Next, we have $\mathcal{G}(\phi, \omega, s, \omega) = \sin(s\phi)$, $\hat{F}(\omega, s, \omega)$ is a trace of function $\hat{F}$ on the surface $\{\theta = \omega\}$, thus at least square-integrable by (3.6), since, by (3.3), $\eta u \in H^2(\Omega)$ for any function η vanishing in some neighbourhood of 0. Therefore $I_2 \in L_2(P(s_r))$ for any $s_r \in E$.

Further

$$\mathcal{G}_\omega(\phi, \theta, s, \omega) = s \begin{cases} \sin[s(\phi - \omega)] \sin(s\theta) & 0 \leq \theta \leq \phi \\ \sin(s\phi) \sin[s(\theta - \omega)] & \phi \leq \theta \leq \omega \end{cases}$$

thus I_3 can be estimated as I in (3.9). Similar estimates hold also for I_4 by (3.6). Thus, $v_t \in L_2(\Omega, r^{2s_r-2})$ for $s_r \in E$ and, having support bounded away from infinity, $v_t \in L_2(\Omega)$.

Analogous arguments can be carried out for v_{tt} , however we would have then 22 components of the second derivative to be estimated. Therefore we omit calculations and state that also $v_{tt} \in L_2(\Omega)$. ■

We can state final theorem of this paper.

THEOREM 3.3. *Let us consider three-dimensional problem (0.1) with Ω given by (1.1). There exists such $f \in L_2(\Omega)$ that the coefficient c given by (2.3) of the asymptotic expansion (2.2) of the variational solution to (0.1) has a singularity at $\omega = \pi/2$ which is not square-integrable.*

Proof. Similar considerations as in Lemma 3.2. show that both rv_r and v_ϕ belongs to $L_2(\Omega, r^{2s_r-2})$ for every $s_r \in E$ (since v_ϕ contains components of the same type as v_ω and $r\widehat{v_r} = s\widehat{v}$). Thus, if we select $s_r < 1/2$ from E , then we will have both v_r and $r^{-1}v_\phi$ belonging to $L_2(\Omega)$, since the support of v is bounded. Therefore $v \in H^1(\Omega)$ and, since $v_{tt} \in L_2(\Omega)$, we see that v is a variational solution of the three-dimensional problem (0.1) with right-hand side given by $F = \chi\vartheta f - \nabla\chi\nabla u - u\Delta\chi + v_{tt} \in L_2(\Omega)$, where the additional term satisfies $\nabla\chi\nabla u - \chi\vartheta f \equiv 0$ for $r < e^{-1}$ and hence it has no influence on the behaviour of the coefficient c in (2.3). ■

Remark. Similar considerations can be carried out in Sobolev spaces W_2^p and in weighted Sobolev spaces, showing thus that the appearance of such exceptional points is in some sense universal, though the particular value of ω which such a singularity appears at depends on the space we are working with, see e.g. [4], [5], [6].

References

- [1] J. Banasiak, G. F. Roach, *On mixed boundary value problems of Dirichlet-oblique derivative problems in plane domains with piecewise-differentiable boundary*, J. Diff. Eqaut., 79, (1989), 111–131.
- [2] J. Banasiak, *On corner singularities of solutions to mixed boundary value problems for second order elliptic and parabolic equations*, Proc. R. Soc. Lond., A 433 (1991), 209–217.
- [3] J. Banasiak, *On asymptotics of solutions of mixed boundary value problems of second order in domains with vanishing edges*, SIAM J. Math. An. (23) 5 (1992), 1117–1124.
- [4] M. Dauge, *Elliptic boundary problems on corner domains*, L.N.M. 1341, Springer Verlag 1988.
- [5] P. Grisvard, *Elliptic boundary value problems in non-smooth domains*, Pitman 1985.
- [6] A. Kufner, A. M. Sändig, *Some applications of weighted Sobolev spaces*, Teubner Texte zur Mathematik, Leipzig 1987.
- [7] E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Fair Lawn, New York, 1948.
- [8] V. A. Vladimirov, *Uraunenia matematičeskoj fiziki*, Nauka, Moskva 1976 (in Russian).

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