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INVERSE THEOREM IN L_p -APPROXIMATION BY MODIFIED SZÁSZ-MIRAKYAN OPERATORS

In this paper, we have studied an inverse theorem in L_p -approximation by certain linear combinations of modified Szász operators. As an important tool we use the technique of Steklov means.

1. Following Gupta et al. [3], for $f \in L_p[0, \infty)$, $p \geq 1$, the modified Szász-Mirakyan operator is defined as

$$(1.1) \quad S_n(f, t) = \int_0^\infty W(n, t, u) f(u) du,$$

where

$$W(n, t, u) = n \sum_{\nu=0}^{\infty} p_{n,\nu}(t) p_{n,\nu}(u), \quad p_{n,\nu}(t) = e^{-nt} \frac{(nt)^\nu}{\nu!}.$$

To improve the rate of convergence, we use the linear combinations $S_n(f, k, t)$ of $S_{d,j,n}(f, t)$, $j = 0, 1, 2, \dots, k$, defined by

$$S_n(f, k, t) = \sum_{j=0}^k C(j, k) S_{d,j,n}(f, t),$$

where

$$C(j, k) = \begin{cases} \sum_{i=0, i \neq j}^k \frac{d_j}{d_j - d_i} & \text{for } k \neq 0, \\ 1 & \text{for } k = 0, \end{cases}$$

and $d_0, d_1, d_2, \dots, d_k$ are $k+1$ arbitrary but fixed distinct positive integers.

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The direct results for these modified Szász–Mirakyan operators were obtained in [3]. In the present paper, we extend these results and obtain an inverse theorem in L_p -approximation by the operators (1.1).

Throughout the paper, let $0 < a_1 < a_3 < a_2 < b_2 < b_3 < b_1 < \infty$, $0 < a < b < \infty$; $I_i = [a_i, b_i]$, $i = 1, 2, 3$, and $[\beta]$ denotes the integer part of β .

For $f \in L_p[0, \infty)$, $1 \leq p \leq \infty$, and η sufficiently small, the Steklov mean of order m corresponding to f is defined as

$$f_{\eta, m}(u) = \eta^{-m} \int_{-\eta/2}^{\eta/2} \dots \int_{-\eta/2}^{\eta/2} [f(u) + (-1)^{m-1} \Delta_{u_1 + \dots + u_m}^m f(u)] du_1 du_2 \dots du_m,$$

where $u \in I_1$ and $\Delta_h^m f(u)$ is the m -th order forward difference of the function f with step length h . For properties of the Steklov means see e.g. [3], [4] and [6].

We assume f to have compact support in $[0, \infty)$. Let ψ be the characteristic function of $[a_1 - \delta, b_1 + \delta]$, $\delta > 0$. Then

$$\|S_n(f\psi, k, \cdot) - f\psi\|_{L_p(I_1)} \leq \|S_n(f, k, \cdot) - f\|_{L_p(I_1)} + \|S_n(f\psi - f, k, \cdot)\|_{L_p(I_1)}.$$

Using Jensen's inequality, we obtain

$$\|S_n(f\psi - f, k, \cdot)\|_{L_p(I_1)}^p \leq C_1 \int_{a_1}^{b_1} \int_0^\infty |f(u)|^p |\psi(u) - 1|^p W(n, t, u) du dt.$$

The presence of $(\psi - 1)$ implies that $u - t > \delta$. For u sufficiently large there exist positive constants C_2, C_3 such that $(u - t)^{2k+2}/u^{2k+2} > C_2$ for all $u > C_3$ and for $u \leq C_3$ we have $(u - t)^{2k+2} > \delta^{2k+2}$. Breaking the integration in u in two parts as $u \leq C_3$ and $u > C_3$ and proceeding as in the estimate of J_4 [3, Th. 2], we have

$$\|S_n(f\psi - f, k, \cdot)\|_{L_p(I_1)} \leq C_4 n^{-l} \|f\|_{L_p[0, \infty)},$$

where l is arbitrary but fixed.

LEMMA 1.1. *Let $g \in L_1[0, \infty)$ have a compact support. Then for $m \in \mathbb{N}$ we have*

$$\left\| n \int_0^\infty \sum_{\nu=0}^\infty p_{n, \nu}(\cdot) p_{n, \nu}(u) \left(\frac{\nu}{n} - \cdot\right)^m g(u) du \right\|_{L_1[0, \infty)} \leq C_6 n^{-m/2} \|g\|_{L_1[0, \infty)},$$

where C_6 is a constant independent of n and g .

Proof. We first use Fubini's theorem and then Hölder's inequality in

the following

$$\begin{aligned}
 I &\equiv n \int_0^\infty \int_0^\infty \sum_{\nu=0}^\infty p_{n,\nu}(t) p_{n,\nu}(u) \left| \frac{\nu}{n} - t \right|^m |g(u)| du dt \\
 &= \int_0^\infty \sum_{\nu=0}^\infty \left\{ \int_0^\infty n p_{n,\nu}(t) \left| \frac{\nu}{n} - t \right|^m dt \right\} p_{n,\nu}(u) |g(u)| du \\
 &\leq \int_0^\infty \sum_{\nu=0}^\infty \left\{ \int_0^\infty n p_{n,\nu}(t) \left| \frac{\nu}{n} - t \right|^{2m} dt \right\}^{\frac{1}{2}} p_{n,\nu}(u) |g(u)| du \\
 &= \int_0^\infty \sum_{\nu=0}^\infty \left\{ \sum_{j=0}^{2m} \binom{2m}{j} \left(\frac{\nu}{n} \right)^{2m-j} (-1)^j \prod_{i=0}^j \left(\frac{\nu}{n} + \frac{i}{n} \right) \right\}^{\frac{1}{2}} p_{n,\nu}(u) |g(u)| du,
 \end{aligned}$$

making use of the identity

$$\prod_{i=0}^j \left(\frac{\nu}{n} + \frac{i}{n} \right) = \left(\frac{\nu}{n} \right)^j + \left(\frac{\nu}{n} \right)^{j-1} \frac{1}{n} p_1(j) + \left(\frac{\nu}{n} \right)^{j-2} \frac{1}{n^2} p_2(j) + \dots,$$

where $p_k(j)$ is a polynomial of degree $2k$.

Now, as $\sum_{j=0}^{2m} \binom{2m}{j} (-1)^j j^k = 0$, $k < 2m$, one has

$$\begin{aligned}
 I &\leq \int_0^\infty \sum_{\nu=0}^\infty \left\{ \sum_{j=0}^{2m} \binom{2m}{j} \left(\frac{\nu}{n} \right)^{2m-j} (-1)^j \right. \\
 &\quad \times \left. \left[\left(\frac{\nu}{n} \right)^{j-m} \frac{1}{n^m} p_m(j) + \dots \right] \right\}^{\frac{1}{2}} p_{n,\nu}(u) |g(u)| du \\
 &= \int_0^\infty \sum_{\nu=0}^\infty \left\{ \sum_{j=0}^{2m} \binom{2m}{j} (-1)^j \left[\left(\frac{\nu}{n} \right)^m \frac{1}{n^m} p_m(j) + \dots \right] \right\}^{\frac{1}{2}} p_{n,\nu}(u) |g(u)| du \\
 &\leq C_5 \int_0^\infty \sum_{\nu=0}^\infty \left\{ \left(\frac{\nu}{n} \right)^m \frac{1}{n^m} + \dots \right\}^{\frac{1}{2}} p_{n,\nu}(u) |g(u)| du.
 \end{aligned}$$

Now, applying Hölder's inequality in the sum, we obtain

$$I \leq C_5 \int_0^\infty \left\{ \sum_{\nu=0}^\infty p_{n,\nu}(u) \left[\left(\frac{\nu}{n} \right)^m \frac{1}{n^m} + \dots \right] \right\}^{\frac{1}{2}} |g(u)| du.$$

Next, using $\sum_{\nu=0}^\infty p_{n,\nu}(u) \left(\frac{\nu}{n} \right)^j = 0(n^{-j})$, $n \rightarrow \infty$, and by compactness of $\text{supp } g$, we obtain

$$I \leq C_6 n^{-m/2} \int_0^\infty |g(u)| du = C_6 n^{-m/2} \|g\|_{L_1[0,\infty)}.$$

LEMMA 1.2. Let $g \in L_p[0, \infty)$, $p \geq 1$, have a compact support $\text{supp } g$ and $i, j \in N^0$ (the set of non-negative integers). Then, for a fixed $l > 0$ there holds

$$\left\| n \int_0^\infty \sum_{\nu=0}^\infty p_{n,\nu}(\cdot) p_{n,\nu}(u) \left(\frac{\nu}{n} - \cdot \right)^i \int_t^u (u-w)^j g(w) dw du \right\|_{L_p(I_2)} \\ \leq C_7 \{ n^{-(i+j+1)/2} \|g\|_{L_p(I_1)} + n^{-l} \|g\|_{L_p[0, \infty)} \},$$

where C_7 is a constant independent of n and g .

Proof. By the repeated use of Jensen's inequality, we have

$$\left| \sum_{\nu=0}^\infty p_{n,\nu}(t) \left(\frac{\nu}{n} - t \right)^i \int_0^\infty n p_{n,\nu}(u) \int_t^u (u-w)^j g(w) dw du \right|^p \\ \leq \sum_{\nu=0}^\infty p_{n,\nu}(t) \left| \frac{\nu}{n} - t \right|^{ip} \int_0^\infty n p_{n,\nu}(u) |u-t|^{jp+p-1} \left| \int_t^u |g(w)|^p dw \right| du \\ = \sum_{\nu=0}^\infty p_{n,\nu}(t) \left| \frac{\nu}{n} - t \right|^{ip} \int_0^\infty n \psi(u) p_{n,\nu}(u) |u-t|^{jp+p-1} \left| \int_t^u |g(w)|^p dw \right| du \\ + \sum_{\nu=0}^\infty p_{n,\nu}(t) \left| \frac{\nu}{n} - t \right|^{ip} \int_0^\infty n(1-\psi(u)) p_{n,\nu}(u) |u-t|^{jp+p-1} \left| \int_t^u |g(w)|^p dw \right| du.$$

We break the interval $[t, u]$ in the first term as

$$\bigcup_{\ell=0}^m \left([t, t + (\ell+1)n^{-\frac{1}{2}}] \cup [t - (\ell+1)n^{-\frac{1}{2}}, t] \right),$$

where $mn^{-1/2} \leq \max\{b_1 - a_2, b_2 - a_1\} < (m+1)n^{-1/2}$ similar to [3, Th. 2] and [2, Th. 3.2]. The only change is that we have to estimate

$$M = \sum_{\nu=0}^\infty p_{n,\nu}(t) \left| \frac{\nu}{n} - t \right|^{ip} \int_0^\infty n \psi(u) p_{n,\nu}(u) |u-t|^{jp+p-1} du.$$

This is done by using Hölder's inequality for an infinite sum and Jensen's inequality, and by using [5, Lemma 1] and [3, Lemma 1]. The presence of the factor $1 - \psi(u)$ yields order n^{-l} in the second term. This proves Lemma 1.2.

LEMMA 1.3 [5]. There exist polynomials $q_{i,\ell,r}(t)$ independent of n and ν such that

$$t^r \frac{d^r}{dt^r} (e^{-nt} (nt)^\nu) = \sum_{\substack{2i+\ell \leq r \\ i, \ell \geq 0}} n^i (\nu - nt)^\ell q_{i,\ell,r}(t) (e^{-nt} (nt)^\nu).$$

LEMMA 1.4. Let $g \in L_p[0, \infty)$, $p \geq 1$, and $\text{supp } g \subset I_2$; then

$$(1.2) \quad \|S_n^{(2k+2)}(g, \cdot)\|_{L_p(I_2)} \leq C_8 n^{k+1} \|g\|_{L_p(I_2)},$$

C_8 being a constant independent of n and g . If in addition g has $2k+2$ derivatives with $g^{(2k+1)} \in AC(I_2)$ and $g^{(2k+2)} \in L_p(I_2)$, then

$$(1.3) \quad \|S_n^{(2k+2)}(g, \cdot)\|_{L_p(I_2)} \leq C_9 \|g^{(2k+2)}\|_{L_p(I_2)},$$

C_9 being a constant independent of n and g .

PROOF. Since $q_{i,\ell,2k+2}(t)$ and $t^{-(2k+2)}$ are bounded on I_2 , it follows from Lemma 1.1 and Lemma 1.3 that, for $g \in L_1[0, \infty)$, we have

$$\|S_n^{(2k+2)}(g, \cdot)\|_{L_1(I_2)} \leq C_{10} n^{k+1} \|g\|_{L_1(I_2)}.$$

If $g \in L_\infty[0, \infty)$, then by Lemma 1.3 and by [5, Lemma 1] we get

$$\|S_n^{(2k+2)}(g, \cdot)\|_{L_\infty(I_2)} \leq C_{11} n^{k+1} \|g\|_{L_\infty(I_2)}.$$

Now, using Riesz–Thorin interpolation theorem, we obtain (1.2).

To prove (1.3), we use the differentiable property of g as

$$g(u) = \sum_{r=0}^{2k+1} \frac{(u-t)^r}{r!} g^{(r)}(t) + \frac{1}{(2k+1)!} \int_t^u (u-w)^{2k+1} g^{(2k+2)}(w) dw.$$

Next, using [3, Lemma 1] and Lemma 1.3, we obtain

$$\begin{aligned} S_n^{(2k+2)}(g, t) &= \frac{n}{(2k+1)! t^{2k+2}} \sum_{\nu=0}^{\infty} p_{n,\nu}(t) \left\{ \sum_{\substack{2i+\ell \leq 2k+2 \\ i, \ell \geq 0}} n^i (\nu - nt)^\ell \right. \\ &\quad \times q_{i,\ell,2k+2}(t) \int_0^{\infty} p_{n,\nu}(u) \int_t^u (u-w)^{2k+1} g^{(2k+2)}(w) dw du \Big\}. \end{aligned}$$

Now, applying Lemma 1.2 in the above relation, we get (1.3).

2. In this section we shall prove the inverse theorem.

For $f \in L_p[a, b]$, where $1 \leq p < \infty$, the integral modulus of smoothness of order m is defined as

$$\omega_m(f, \tau, p, [a, b]) = \sup_{0 < \delta \leq \tau} \|\Delta_\delta^m f(\cdot)\|_{L_p[a, b-m\delta]}.$$

THEOREM 2.1. Let $0 < \alpha < 2k+2$, $1 \leq p < \infty$ and $f \in L_p[0, \infty)$, then

$$(2.1) \quad \|S_n(f, k, \cdot) - f\|_{L_p(I_1)} = O(n^{-\alpha/2}), \quad n \rightarrow \infty,$$

implies that $\omega_{2k+2}(f, \tau, p, I_2) = O(\tau^\alpha)$, $\tau \rightarrow 0$.

PROOF. We choose a function $h \in C_0^{2k+2}$ such that $\text{supp } h \subset (x_2, y_2)$, $h(t) = 1$ on $[x_3, y_3]$ and $a_1 < x_1 < x_2 < x_3 < a_2 < b_2 < y_3 < y_2 < y_1 < b_1$.

Writing $fh = \bar{f}$, for all values of $\gamma \leq \tau$, we have

$$\Delta_\gamma^{2k+2} \bar{f}(t) = \Delta_\gamma^{2k+2} (\bar{f}(t) - S_n(\bar{f}, k, t)) + \Delta_\gamma^{2k+2} S_n(\bar{f}, k, t),$$

where Δ_γ^{2k+2} denotes $(2k+2)$ -th order forward difference.

By the repeated application of Jensen's inequality and finally Fubini's theorem in the second term, we obtain

$$\begin{aligned} \|\Delta_\gamma^{2k+2} \bar{f}\|_{L_p[x_2, y_2]} &\leq \|\Delta_\gamma^{2k+2} (\bar{f} - S_n(\bar{f}, k, \cdot))\|_{L_p[x_2, y_2]} + \\ &\quad + \gamma^{2k+2} \|S_n^{(2k+2)}(\bar{f}, k, \cdot)\|_{L_p[x_2, y_2 + (2k+2)\gamma]}. \end{aligned}$$

Now, we write $\bar{f} = (\bar{f} + \bar{f}_{\eta, 2k+2}) + \bar{f}_{\eta, 2k+2}$ in $\|S_n^{(2k+2)}(\bar{f}, k, \cdot)\|_{L_p[x_2, y_2 + (2k+2)\gamma]}$ where $\bar{f}_{\eta, 2k+2}$ is the Steklov mean of $(2k+2)$ -th order corresponding to $\bar{f}$, and then use Lemma 1.4. For sufficiently small $\eta > 0$, it follows by the properties of Steklov means that

$$\begin{aligned} \|\Delta_\gamma^{2k+2} \bar{f}\|_{L_p[x_2, y_2]} &\leq \|\Delta_\gamma^{2k+2} (\bar{f} - S_n(\bar{f}, k, \cdot))\|_{L_p[x_2, y_2]} \\ &\quad + C_{12} \gamma^{2k+2} (n^{k+1} + \eta^{-(2k+2)}) \omega_{2k+2}(\bar{f}, \eta, p, [x_2, y_2]). \end{aligned}$$

Following Berens and Lorentz [1], we can complete the proof once it is established that

$$(2.2) \quad \|\Delta_\gamma^{2k+2} (\bar{f} - S_n(\bar{f}, k, \cdot))\|_{L_p[x_2, y_2]} = O(n^{-\alpha/2}), \quad n \rightarrow \infty.$$

Thus,

$$\omega_{2k+2}(\bar{f}, \tau, p, [x_2, y_2]) = O(\tau^\alpha), \quad \tau \rightarrow 0.$$

Now, since $\bar{f}(t) = f(t)$ for $t \in [x_3, y_3]$, we get

$$\omega_{2k+2}(f, \tau, p, I_2) = O(\tau^\alpha), \quad \tau \rightarrow 0 \text{ as required.}$$

We prove (2.2) by induction on α . First we consider the case $\alpha \leq 1$. Then

$$(2.3) \quad \|S_n(fh, k, \cdot) - fh\|_{L_p[x_2, y_2]} \leq \|S_n(h(\cdot)(f(u) - f(\cdot)), k, \cdot)\|_{L_p[x_2, y_2]} \\ + \|S_n(f(u)(h(u) - h(\cdot)), k, \cdot)\|_{L_p[x_2, y_2]}.$$

Now, $h(u) - h(t) = (u - t)h'(\xi)$ for some ξ lying between u and t . Using [3, Lemma 1] and the compactness of f to estimate the second term of (2.3) and using (2.1) for the first term, we get

$$\|S_n(fh, k, \cdot) - fh\|_{L_p[x_2, y_2]} = O(n^{-\alpha/2}) + O(n^{-1/2}) = O(n^{-\alpha/2}).$$

This completes the proof of (2.2), when $\alpha \leq 1$.

Now we assume (2.2) to hold for all values of α satisfying $r - 1 < \alpha < r$ and prove that the same holds true for $r < \alpha < r + 1$. Thus, we have

$$\omega_{2k+2}(f, \tau, p, [c, d]) = O(\tau^{r-1+\beta}), \quad \tau \rightarrow 0, \quad 0 < \beta < 1,$$

for any $[c, d] \subset (a_1, b_1)$. Let ϕ denote the characteristic function of $[x_1, y_1]$.

The assumed smoothness of f implies that (see [6])

$$\begin{aligned} S_n(fh, k, t) - fh(t) &= \sum_{i=0}^{r-2} \frac{1}{i!} f^{(i)}(t) S_n((u-t)^i(h(u)-h(t)), k, t) \\ &+ \frac{1}{(r-2)!} S_n(\phi(u)(h(u)-h(t)) \int_t^u (u-w)^{r-2} (f^{(r-1)}(w) - f^{(r-1)}(t)) dw, k, t) \\ &+ S_n(F(u, t)(1-\phi(u))(h(u)-h(t)), k, t) = J_1 + J_2 + J_3, \text{ say,} \end{aligned}$$

where $F(u, t) = f(u) - \sum_{i=0}^{r-2} \frac{(u-t)^i}{i!} f^{(i)}(t)$, $u \in [0, \infty)$, $t \in [x_2, y_2]$. The direct theorem and Lemma 1 of [3] imply that

$$\|J_\nu\|_{L_p[x_2, y_2]} = O(n^{-(k+1)}), \quad n \rightarrow \infty, \quad \nu = 1, 3.$$

Next, using Lensen's inequality, mean value theorem on h and breaking $[t, u]$ as in Lemma 1.2, we have

$$\begin{aligned} &\int_{x_2}^{y_2} |S_n(\phi(u)(h(u)-h(t)) \int_t^u (u-w)^{r-2} (f^{(r-1)}(w) - f^{(r-1)}(t)) dw, k, t)|^p dt \\ &\leq C_{13} \int_{x_2}^{y_2} \int_{x_1}^{y_1} W(n, t, u) |u-t|^{rp-1} \int_t^u \phi(w) |f^{(r-1)}(w) - f^{(r-1)}(t)|^p dw du dt \\ &\leq C_{13} \sum_{\ell=1}^r \int_{x_2}^{y_2} \left\{ \int_{t+\ell n^{-1/2}}^{t+(\ell+1)n^{-1/2}} W(n, t, u) (n^2 \ell^{-4}) |u-t|^{rp+4p-1} \right. \\ &\quad \times \int_t^{t+(\ell+1)n^{-1/2}} \phi(w) |f^{(r-1)}(w) - f^{(r-1)}(t)|^p dw du \\ &\quad + \int_{t-(\ell+1)n^{-1/2}}^{t-\ell n^{-1/2}} W(n, t, u) (n^2 \ell^{-4})^p |u-t|^{rp+4p-1} \\ &\quad \times \int_{t-(\ell+1)n^{-1/2}}^t \phi(w) |f^{(r-1)}(w) - f^{(r-1)}(t)|^p dw du dt \\ &\quad \left. + \int_{x_2}^{y_2} \int_{x_2-n^{-1/2}}^{y_2+n^{-1/2}} W(n, t, u) |u-t|^{rp-1} \int_{t-n^{-1/2}}^{t+n^{-1/2}} \phi(w) |f^{(r-1)}(w) - f^{(r-1)}(t)|^p dw du dt \right\} \\ &\leq C_{14} \left\{ \sum_{\ell=1}^r (n^2 \ell^{-4})^p n^{-((r+4)p-1)/2} \int_0^{(\ell+1)n^{-1/2}} (\omega(f^{(r-1)}, w, p, [x_1, y_1]))^p dw \right. \\ &\quad \left. + n^{-(rp-1)/2} \int_0^{n^{-1/2}} (\omega(f^{(r-1)}, w, p, [x_1, y_1]))^p dw \right\}, \end{aligned}$$

on using [3, Lemma 1] and then interchanging integration in t and w . Finally, using $\omega(f^{(r-1)}, w, p, [x_1, y_1]) = O(w^\beta)$, we get

$$\|J_2\|_{L_p[x_2, y_2]} = O(n^{-(r+\beta)/2}), \quad n \rightarrow \infty.$$

Combining the estimates of J_1 , J_2 and J_3 , we obtain (2.2). The proof of (2.2) shows that

$$(2.4) \quad \omega_{2k+2}(f, \tau, p, I_2) = O(\tau^\alpha), \quad \alpha < 2k + 2.$$

The above statement implies that it is true for integer values $2, 3, \dots, 2k + 1$ also. To prove this, let $\alpha = r$, $r = 2, 3, \dots, 2k + 1$.

Then, since (2.4) is true for $\alpha \in (r, r + 1)$, we have

$$\omega_{2k+2}(f, \tau, p, I_2) = O(\tau^{r+\theta}) = O(\tau^r) \quad \text{for } 0 < \theta < 1.$$

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