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THE THREE-PARABOLIC PROBLEM FOR THE HALF-UNBOUNDED DOMAIN WITH CURVILINEAR BOUNDARY CONDITIONS OF LAURICELLA TYPE

1. Formulation of the problem

The subject of the paper is the construction of a solution of the equation

$$(1) \quad P^3 u(x, t) = f(x, t), \quad P^3 = P(P^2), \quad P^2 = P(P), \quad P = D_x^2 - D_t$$

in the domain

$$D = \{(x, t) : x > p(t), t \in (0, T), T < \infty\}.$$

satisfying the initial conditions

$$(2) \quad D_t^i u(x, 0) = f_i(x), \quad i = 0, 1, 2 \quad \text{for } x > p(0)$$

and the boundary conditions

$$(3) \quad D_x^i u(p(t), t) = h_{i+1}(t), \quad i = 0, 1, 2 \quad \text{for } t \in (0, T),$$

$$(4) \quad D_x^i u(\infty, t) = 0, \quad i = 0, 1, 2 \quad \text{for } t \in (0, T),$$

where $f, f_j, j = 0, 1, 2, h_k, k = 1, 2, 3$, are given functions.

2. Motivation of the considered problem

In [10] and [2] similar problems for the equation $P^3 u = f$ for the time-spatial curvilinear trapezium and for n -dimensional half-space are treated.

In [1] and [11] similar problems for bicaloric equation are given.

In the monograph [5] similar problem for the equation $Pu = f$ with initial and boundary conditions was solved. In [12] the similar problem for the biparabolic equation with limit conditions of Lauricella type was treated.

In the monograph [11], vol. II, p. 235 the biparabolic Cauchy problem was solved.

In [3] and [4] the similar biparabolic problems for the curvilinear trapezium and for the strip are treated respectively. In [15] the nonlinear biparabolic problem was solved. In [6] and [7] the three-parabolic problem for three-dimensional cylinder with boundary conditions of Riquier type and for the strip with boundary conditions of Lauricella type was solved respectively.

In [8] and [9] the polyparabolic problems with boundary conditions of Lauricella type for the quart-time-plane and for the strip was treated respectively.

3. Some definitions

In the sequel $C, C_i, i \in \mathbb{N}$, denote positive constants.

DEFINITION 1. Denote by (K_1) the class of all functions $u : D \rightarrow \mathbb{R}$, such that $u(x, t) \in C^{6,3}(D) \cap C^{2,2}(D \cup S)$ and $D_x^i u, D_t^i u \in B(D)$, $i = 0, 1, 2$, where $S = \{(x, t) : x = p(t), t \in (0, T)\}$, and $B(D)$ denote a class of all functions u bounded in D , such that $D_x^i u(\infty, t) = 0$, $i = 0, 1, 2$, $t \in (0, T]$.

DEFINITION 2. Denote by (K_2) the class of all functions $p : [0, T] \rightarrow \mathbb{R}$, such that $p \in C^2([0, T])$, $D_t p(t) > 0$ for $t \in [0, T]$.

DEFINITION 3. Denote by (K_3) the class of all functions $f : D_1 \rightarrow \mathbb{R}$, $D_1 = \{(x, 0) : x > p(0)\}$, such that $f \in C^6(D_1) \cap C^2(\overline{D}_1)$ and $D_x^i f(p(0)) = D_x^i f(\infty) = 0$, $i = 0, 1, 2$.

DEFINITION 4. Denote by (K_4) the class of all functions $h : [0, T] \rightarrow \mathbb{R}$, such that $h \in C^3([0, T])$ and $D_t^i h(0) = 0$, $i = 0, 1, 2$.

DEFINITION 5. Denote by (K_5) the class of all functions $F : D \rightarrow \mathbb{R}$, such that $F \in C^{2,1}(D)$ and $\text{comp. supp } F \subset D$.

4. Reduction of the initial conditions to the homogeneous ones

Let us consider the function

$$w(x, t) = u(x, t) - r(x, t)$$

where

$$r(x, t) = f_0(x) + t f_1(x) + \frac{1}{2} t^2 f_2(x).$$

LEMMA 1. If $p \in (K_2)$, $f_i \in (K_3)$, $i = 0, 1, 2$, $h_{i+1} \in (K_4)$, $i = 0, 1, 2$ and $u \in (K_1)$ is a solution of the problem (1)–(4), then $w \in (K_1)$ and satisfies the conditions

$$(1a) \quad P^3 w(x, t) = F(x, t), \quad F(x, t) = f(x, t) - P^3 r(x, t), \quad (x, t) \in D,$$

$$(2a) \quad D_t^i w(x, 0) = 0, \quad i = 0, 1, 2, \quad x > p(0),$$

$$(3a) \quad D_x^i w(p(t), t) = H_{i+1}(t), \quad H_{i+1}(t) = h_{i+1}(t) - D_x^i r(p(t), t), \\ i = 0, 1, 2,$$

$$(4a) \quad D_x^i w(\infty, t) = 0, \quad i = 0, 1, 2, \quad t \in (0, T).$$

Conversely. If the function $w \in (K_1)$ and satisfies (1a)–(4a), then the function $u = (w + r) \in (K_1)$ and is a solution of the problem (1)–(4).

We omit the simple proof.

5. Potentials w_i

Let us consider the potentials

$$w_1(x, t) = A \int_0^t q_1(s)(t-s)^{3/2} \exp(B(t, s)(x-p(s))^2) ds,$$

$$w_2(x, t) = A \int_0^t q_2(s)(t-s)^{1/2}(x-p(s)) \exp(B(t, s)(x-p(s))^2) ds,$$

$$w_3(x, t) = A \int_0^t q_3(s)(t-s)^{-1/2}(x-p(s))^2 \exp(B(t, s)(x-p(s))^2) ds,$$

$$w_4(x, t) = A \int_0^t \int_{p(s)}^\infty F(y, s)(t-s)^{3/2} \exp(B(t, s)(x-y)^2) dy ds,$$

$$w_4^1(t) = w_4(p(t), t) = \\ = A \int_0^t \int_{p(s)}^\infty F(y, s)(t-s)^{3/2} \exp(B(t, s)(p(t)-y)^2) dy ds,$$

$$D_x w_4(p(t), t) = w_4^{1,1}(t) = \\ = \frac{1}{2} A \int_0^t \int_{p(s)}^\infty F(y, s)(t-s)^{1/2}(p(t)-y) \exp(B(p(t), y)^2) dy ds,$$

$$D_x^2 w_4(p(t), t) = w_4^{2,1}(t) = -2A \int_0^t \int_{p(s)}^\infty F(y, s)(t-s)^{3/2} B(t, s) \times \\ \times (1 - 2B(t, s)(p(t)-y)^2) \exp(B(t, s)(p(t)-y)^2) dy ds,$$

$$D_x^2 w_4(p(t), t) = w_4^{2,2}(t) = 2A \int_0^t \int_{p(s)}^\infty F(y, s)(t-s)^{3/2} B(t, s) \times \\ \times (1 - 2B(t, s)(p(t)-y)^2) \exp(B(t, s)(p(t)-y)^2) dy ds,$$

where $A = (\pi)^{-1}$, $B(t, s) = (-4(t-s))^{-1}$, $q_i(s)$, $i = 1, 2, 3$, are unknown functions continuous for $s \in [0, T]$.

6. Properties of the potential w_4

LEMMA 2. If $f_i \in (K_3)$, $h_j \in (K_4)$, $i = 0, 1, 2$, $j = 1, 2, 3$, $F \in (K_5)$ and if $F(p(s), s) = F(\infty, s) = 0$, $D_y F(p(s), s) = D_y F(\infty, s)$, $D_s F(y, 0) = 0$, then:

- 1⁰ $P^3 w_4(x, t) = F(x, t)$ for $(x, t) \in D$,
- 2⁰ $D_t^i w_4(x, 0) = 0$, $i = 0, 1, 2$, $x \in (p(0), \infty)$,
- 3⁰ there exist the boundary functions w_4^1 , continuous for $t \in [0, T]$.
- 4⁰ $D_x^i w_4(\infty, t) = 0$, $i = 0, 1, 2$, $t \in [0, T]$.

PROOF. Ad 1⁰. Let us consider the integrals

$$I_4^j(x, t) = A \int_0^t \int_{p(s)}^\infty F(y, s)(t-s)^{3/2} D_x^j \exp(B(t, s)(x-y)^2) dy ds, \quad j = 0, 1, 2$$

and let

$$\begin{aligned} \bar{F}(y, s) &= F(y, s) \quad \text{for } (y, s) \in D \quad \text{and} \\ \bar{F}(y, s) &= 0 \quad \text{for } (y, s) \in ((-\infty, \infty) \times [0, T]) \setminus D. \end{aligned}$$

For $j = 0$ we have

$$|I_4^0(x, t)| \leq A \left(\sup_{\bar{D}(T)} |F| \right) \int_0^t (t-s)^{3/2} ds = C_0 t^{5/2} = C_0 T^{5/2} \leq C_0 T^{5/2}.$$

Consequently the integral $I_4^0(x, t)$ is locally uniformly convergent for $x \in [p(s), \infty)$, $s \in (0, T]$ and is continuous for $x = p(t)$, $t \in [0, T]$. i.e.

$$I_4^0(p(t), t) = w_4^1(t) \in C([0, T]).$$

For $j = 1$ we have

$$\begin{aligned} I_4^1(x, t) &= -A \int_0^t \int_{p(s)}^\infty F(y, s)(t-s)^{3/2} (D_y \exp(B(t, s)(x-y)^2)) dy ds = \\ &= -A \int_0^t \bar{F}(y, s)(t-s)^{3/2} \exp(B(t, s)(x-y)^2) \Big|_{y=-\infty}^{y=\infty} ds + \\ &\quad + A \int_0^t \int_{-\infty}^\infty (D_y \bar{F}(y, s))(t-s)^{3/2} \exp(B(t, s)(x-y)^2) dy ds = \\ &= A \int_0^t \int_{-\infty}^\infty (D_y \bar{F}(y, s))(t-s)^{3/2} \exp(B(t, s)(x-y)^2) dy ds. \end{aligned}$$

Consequently $I_4^1(x, t)$ is locally uniformly convergent for $x = p(t)$ and $D_x w_4(p(t), t) = w_4^1(t)$, are continuous for $t \in (0, T]$.

For $j = 2$, we have

$$I_4^2(x, t) = A \int_0^t \int_{p(s)}^\infty (D_y^2 F(y, s))(t-s)^{3/2} \exp(B(t, s)(x-y)^2) dy ds.$$

Similarly as for $I_4^1(x, t)$ the integral $I_4^2(x, t)$ is locally uniformly convergent for $x \in [p(s), \infty)$, $s \in (0, T]$ and is continuous for $x = p(t)$, i.e.

$$D_x^2 w_4(p(t), t) = w_3^2(t) \in C([0, T]),$$

For $w_4(x, t) = I_4^0(x, t)$, we have

$$\begin{aligned} PI_4^0(x, t) &= D_x^2 I_4^0(x, t) - D_t I_4^0(x, t) = \\ &= A \int_0^t \int_{-\infty}^\infty \{ \bar{F}(y, s)(t-s)^{3/2} D_x^2 \exp(B(t, s)(x-y)^2) - \\ &\quad - \bar{F}(y, s) D_t [(t-s)^{3/2} \exp(B(t, s)(x-y)^2)] \} dy ds = \\ &= A \int_0^t \int_{-\infty}^\infty \bar{F}(y, s) [D_x^2 [(x-s)^{3/2} \exp(B(t, s)(x-y)^2)] - \\ &\quad - D_t [(t-s)^{3/2} \exp(B(t, s)(x-y)^2)]] dy ds = \\ &= A \int_0^t \int_{p(s)}^\infty F(y, s) [D_x^2 [(t-s)^2 (t-s)^{-1/2} \exp(B(t, s)(x-y)^2)] - \\ &\quad - D_t [(t-s)^2 (t-s)^{-1/2} \exp(B(t, s)(x-y)^2)]] dy ds = \\ &= A \int_0^t \int_{p(s)}^\infty F(y, s) \{ (t-s)^2 D_x^2 [(t-s)^{-1/2} \exp(B(t, s)(x-y)^2)] - \\ &\quad - (t-s)^2 D_t [(t-s)^{-1/2} \exp(B(t, s)(x-y)^2)] - \\ &\quad - 2(t-s) [(t-s)^{-1/2} \exp(B(t, s)(x-y)^2)] \} dy ds = \\ &= -2A \int_0^t \int_{p(s)}^\infty F(y, s)(t-s)v(x, y, t, s) dy ds, \end{aligned}$$

where

$$v(x, y, t, s) = (t-s)^{-1/2} \exp(B(t, s)(x-y)^2).$$

Similarly

$$P^2 I_4^0(x, t) = -2A \int_0^t \int_{p(s)}^\infty F(y, s)v(x, y, t, s) dy ds$$

and by Poisson theorem [11], vol. I, p. 522, we obtain

$$P^3 I_4^0(x, t) = F(x, t) \quad \text{for } (x, t) \in D.$$

Ad 2⁰ We have the estimations

$$(5) \quad \begin{cases} |w_4(x, t)| \leq C \int_0^t (t-s)^{3/2} ds = C_1 t^{5/2} \leq C_1 T^{5/2}, \\ |D_t w_4(x, t)| \leq C_2 \int_0^t (t-s)^{1/2} ds = C_3 t^{3/2} \leq C_3 T^{3/2}, \\ |D_t^2 w_4(x, t)| \leq C_4 \int_0^t (t-s)^{-1/2} ds = C_5 t^{1/2} \leq C_5 T^{1/2}. \end{cases}$$

By (5) we obtain the assertions 2⁰ and 3⁰.

Ad 4⁰. Let $M = \text{supp } F = \{(x, t) : p(t) + a_1 < x < a_2, a_2 < \infty\}$. For $j = 0$, we have

$$\begin{aligned} |w_3(x, t)| &= \\ &= \left| A \int_0^t \int_{p(s)+a_1}^{a_2} F(y, s) (t-s)^{1/2} \exp\left(\frac{-(x-y)^2}{4(t-s)}\right) dy ds \right| \leq \\ &\leq \left| C \int_0^t \int_{p(s)+a_1}^{a_2} (t-s)^{1/2} (x-y)(x-y)^{-1} \exp\left(\frac{-(x-y)^2}{4(t-s)}\right) dy ds \right| \leq \\ &\leq \left| C \int_0^t \int_{p(s)+a_1}^{a_2} \frac{1}{x-y} \frac{(x-y)}{(t-s)^{1/2}} \exp\left(\frac{-(x-y)^2}{4(t-s)}\right) dy ds \right| \leq \\ &\leq \left| C \int_0^t \int_{p(s)+a_1}^{a_2} \frac{1}{x-y} dy ds \right| \leq \left| C \int_0^t \int_{p(s)+a_1}^{a_2} \frac{1}{x-a_2} dy ds \right| = \\ &= Ct \frac{1}{x-a_2} (a_2 - p(s) - a_1) \leq CT \frac{1}{x-a_2} \rightarrow 0 \quad \text{as } x \rightarrow \infty. \end{aligned}$$

7. Properties of the potentials w_i for $i = 1, 2, 3$

LEMMA 3. If $q_i(s) \in C([0, T])$, $q_i(0) = 0$, $i = 1, 2, 3$, then:

1⁰ $P^3 w_i(x, t) = 0$ for $(x, t) \in D$, $i = 1, 2, 3$,

2⁰ $D_t^k w_i(x, 0) = 0$ for $x \in (p(0), \infty)$, $k = 0, 1, 2$, $i = 1, 2, 3$,

3⁰ $D_x^k w_i(p(t), t) \in C([0, T])$, $k = 0, 1, 2$, $i = 1, 2, 3$,

4⁰ $D_w^k w_i(\infty, t) = 0$, $t \in [0, T]$, $k = 0, 1, 2$, $i = 1, 2, 3$.

Proof. Ad 1⁰. For the function w_1 we have

$$Pw_1(x, t) = D_x^2 w_1(x, t) - D_t w_1(x, t),$$

$$\begin{aligned}
D_x^2 w_1(x, t) &= \\
&= A \int_0^t q_1(s)(t-s)^{3/2} D_x^2 \exp(B(t, s)(x-p(s))^2) ds = \\
&= A \int_0^t q_1(s)(t-s)^2 D_x^2 [(t-s)^{-1/2} \exp(B(t, s)(x-p(s))^2)] ds = I_{11}(x, t), \\
D_t w_1(x, t) &= \\
&= \lim_{s \rightarrow t} q_1(s)(t-s)^{3/2} \exp(B(t, s)(x-p(s))^2) + I_{11}(x, t) = I_{11}(x, t),
\end{aligned}$$

where

$$\begin{aligned}
I_{11}(x, t) &= \\
&= A \int_0^t q_1(s) D_t [(t-s)^2 (t-s)^{-1/2} \exp(B(t, s)(x-p(s))^2)] ds = \\
&= A \int_0^t q_1(s) \{ [(t-s)^2 (t-s)^{-1/2} \exp(B(t, s)(x-p(s))^2)] + \\
&\quad + 2(t-s) [(t-s)^{-1/2} \exp(B(t, s)(x-p(s))^2)] \} ds. \\
Pw_1(x, t) &= \\
&= A \int_0^t q_1(s) [(t-s)^2 [D_x^2 V(x, t, s) - D_t V(x, t, s)] + \\
&\quad + 2(t-s)V(x, t, s)] ds = 2 \int_0^t q_1(s)(t-s)V(x, t, s) ds.
\end{aligned}$$

$$\begin{aligned}
P^2 w_1(x, t) &= \\
&= 2 \int_0^t q_1(s) [(t-s) D_x^2 V(x, t, s) - (t-s) D_t^2 V(x, t, s) + V(x, t, s)] ds = \\
&= 2 \int_0^t q_1(s) V(x, t, s) ds,
\end{aligned}$$

$$\text{where } V(x, t, s) = (t-s)^{-1/2} \exp(B(t, s)(x-p(s))^2).$$

By [10], vol. I, p. 484, we obtain $P^3 w_1(x, t) = 2 \int_0^t q_1(s) P_{x,t} V(x, t, s) = 0$. For the function w_2 , we have

$$\begin{aligned}
w_2(x, t) &= A \int_0^t q_2(s)(t-s)^2 [((x-p(s))(t-s)^{-3/2}) \times \\
&\quad \times \exp(B(t, s)(x-p(s))^2)] ds,
\end{aligned}$$

$$\begin{aligned}
Pw_2(x, t) &= A \int_0^t q_2(s)(t-s)^2 [D_x^2 W(x, t, s)] ds - \\
&\quad - A \int_0^t q_2(s)[(t-s)^2 D_t W(x, t, s) + 2(t-s)W(x, t, s)] ds \\
&= 2A \int_0^t q_2(s)(t-s)W(x, t, s) ds,
\end{aligned}$$

where

$$\begin{aligned}
W(x, t, s) &= ((x - p(s))/(t-s)^{3/2}) \exp(B(t, s)(x - p(s))^2), \\
P^2 w_2(x, t) &= 2A \int_0^t q_2(s)(t-s)D_x^2 W(x, t, s) ds - \\
&\quad - 2A \int_0^t q_2(s)(t-s)D_t W(x, t, s) ds - \\
&\quad - 2A \int_0^t q_2(s)W(x, t, s) ds - 2A \int_0^t q_2(s)W(x, t, s) ds, \\
P^3 w_2(x, t) &= -2A \int_0^t q_2(s)D_x^2 W(x, t, s) ds + 2A \int_0^t q_2(s)D_t W(x, t, s) ds = \\
&= 2A \int_0^t q_2(s)[-D_x^2 W(x, t, s) + D_t W(x, t, s)] ds = 0.
\end{aligned}$$

For the function w_3 , we have

$$\begin{aligned}
w_3(x, t) &= A \int_0^t q_3(s)(x - p(s))^2 V(x, t, s) ds, \\
Pw_3(x, t) &= A \int_0^t q_3(s)[(x - p(s))^2 D_x^2 V(x, t, s) + 4(x - p(s))D_x V(x, t, s) + \\
&\quad + 2V(x, t, s)] - A \int_0^t q_3(s)(x - p(s))D_x V(x, t, s) ds = \\
&= A \int_0^t q_3(s)4(x - p(s))D_x V(x, t, s) ds + 2A \int_0^t q_3(s)V(x, t, s) ds,
\end{aligned}$$

$$\begin{aligned}
P^2 w_3(x, t) &= P(Pw_3(x, t)) = AP \int_0^t q_3(s) V(x, t, s) ds + \\
&+ A \int_0^t q_3(s) [8D_x^2 V(x, t, s) + 4(x - p(s))D_x^3 V(x, t, s) - \\
&- 4(x - p(s))D_x^2 D_t V(x, t, s)] ds = 8A \int_0^t D_x^2 V(x, t, s) ds, \\
P^3 w_1(x, t) &= 8A \int_0^t P_{xt} D_x^2 V(x, t, s) ds = \\
&= 8A \int_0^t D_x^2 P_{xt} V(p_1(s), t, s) ds = 0.
\end{aligned}$$

Ad 2⁰. Obviously $w_1(x, 0) = 0$, because

$$\begin{aligned}
|w_1(x, t)| &\leq C_6 \int_0^t (t-s)^{3/2} ds \leq -(t-s)^{5/2} \Big|_0^t \leq C_7^{5/2} \leq C_7 T^{5/2}, \\
|D_t w_1(x, t)| &\leq C_8 \int_0^t (t-s)^{-1/2} ds \leq C_9 t^{1/2} \leq C_9 T^{1/2}, \\
|D_t w_1(x, t)| &\leq A \int_0^t q_1(s) [C_{10}(t-s)^{-1/2} \exp(B(x-p(s))^2) + \\
&+ C_{11}(t-s)^{1/2} (x-p(s))^2 (t-s)^{-2} \exp(B(t,s)(x-p(s))^2) + \\
&+ C_{12}(t-s)^{3/2} (x-p(s))^2 (t-s)^{-3} \exp(B(t,s)(x-p(s))^2) + \\
&+ C_{13}(x-p(s))^4 (t-s)^{-5/2} \exp(B(t,s)(x-p(s))^2)] ds.
\end{aligned}$$

We have the estimations

$$\begin{aligned}
&\left| A \int_0^t q_1(s) C_{10}(t-s)^{-1/2} \exp(B(t,s)(x-p(s))^2) \right| \leq \\
&\leq C_{14} \int_0^t (t-s)^{-1/2} ds \leq C_{15} t^{1/2} \leq C_{15} T^{1/2}, \\
&\left| C_{11} A \int_0^t q_1(s) (x-p(s))^2 (t-s)^{1/2} (t-s)^{-2} \exp(B(t,s)(x-p(s))^2) ds \right| \leq \\
&\leq C_{16} \int_0^t \frac{(x-p(s))^2}{(t-s)} (t-s)^{-1/2} \exp(B(t,s)(x-p(s))^2) ds \leq
\end{aligned}$$

$$\leq C_{17} \int_0^t (t-s)^{-1/2} ds \leq C_{18} t^{1/2} \leq C_{18} T^{1/2}.$$

Similarly

$$\left| C_{12} A \int_0^t q_1(s) (t-s)^{3/2} (x-p(s))^2 (t-s)^{-3} \exp(B(t,s)(x-p(s))^2) ds \right| \leq \\ \leq C_{19} t^{1/2} \leq C_{19} T^{1/2}$$

and

$$\left| C_{13} A \int_0^t q_1(s) (x-p(s))^4 (t-s)^{-5/2} \exp(B(t,s)(x-p(s))^2) ds \right| \leq \\ \leq C_{20} t^{1/2} \leq C_{20} T^{1/2}.$$

Similarly we obtain assertion 2^0 for w_i , $i = 2, 3$.

Ad 3^0 . Analogously as for 2^0 , in the proof of assertion 1^0 , we obtain the estimations

$$(6) \quad |D_x^k w_i(x, t)| \leq C_{21} t^a \leq C_{21} T^a, \quad k = 0, 1, 2, \quad i = 1, 2, \dots, 6,$$

where a being a positive number.

By (6), we obtain the assertion 3^0 .

Ad 4^0 . Let $I_{ij}(x, t) = \int_0^t q_i(s) (t-s)^{(3/2)-i} D_x^j (x-p(s))^{i-1} \exp(B(t,s) \times (x-p(s))^2) ds$, $i = 1, 2, 3$, $j = 0, 1, 2$.

Dividing and multiplying the kernels of the integrals I_{ij} by the function

$$(x-p(s))^j (t-s)^{-j/2},$$

we obtain

$$|D_x^j w_i(x, t)| \leq C \int_0^t (t-s)^{1-\frac{n}{2}-\frac{j}{2}} \sum_{n=0}^j C_n (x-p(s))^{j-n+1} \times \\ \times \exp\left(\frac{1}{2} B(t,s)(x-p(s))^2\right) ds \leq \\ \leq C (x-p(s))^{-1} \sum_{n=0}^j (x-p(s))^{-n} \int_0^t (t-s)^{n-i-\frac{n}{2}+\frac{j}{2}-1} \times \\ \times \frac{(x-p(s))}{(t-s)^{j/2}} \exp\left(\frac{1}{2} B(t,s)(x-p(s))^2\right) ds \leq \\ \leq \left[C \sum_{n=0}^j (x-p(s))^{-(n+1)} \right] \int_0^t (t-s)^{n+i-(n/2)+(j/2)+j-1} \leq$$

$$\begin{aligned}
&\leq \left[C \sum_{n=0}^j (x - p(s))^{-(n+1)} \right] t^{n-i-(n/2)+j-1} \leq \\
&\leq \left[C \sum_{n=0}^j (x - p(s))^{-(n+1)} \right] T^{n-i-(n/2)+(j/2)+j-1} \rightarrow 0 \quad \text{as } x \rightarrow \infty.
\end{aligned}$$

8. Integral equations compatible with the boundary conditions

We suppose that the function

$$w(x, t) = \sum_{i=1}^4 w_i(x, t)$$

is a solution (1a)–(4a) problem. By (3a)–(4a), we obtain the system of the integral equations for the functions q_i , $i = 1, 2, \dots, 6$, of the form:

$$(I_1a) \quad \sum_{i=1}^6 I_{1,i}(t) = H^1(t),$$

$$(I_2a) \quad \sum_{i=1}^3 I_{2,i}(t) = H^2(t),$$

$$(I_3a) \quad \sum_{i=1}^3 I_{3,i}(t) = H^3(t),$$

where

$$\begin{aligned}
I_{k,i}(t) &= D_x^{k-1} w_i(p(t), t), \quad k = 1, 2, 3, \quad i = 1, 2, 3, \\
H^k(t) &= H_k(t) - w_4^k(p(t), t), \quad k = 1, 2, 3.
\end{aligned}$$

9. Abel transformation of the system $(I_i a)$, $i = 1, 2, 3$

Applying Abel transformation [14], Vol. I, p. 14, to the system $(I_i a)$, $i = 1, 2, 3$, we obtain the following

LEMMA 4. *If $q_i \in C([0, T])$, $H^i \in C^{m-i}([0, T])$, $D_t^{m-i} H^i(0) = 0$, $i = 1, 2, 3$, then the system $(I_i a)$, $i = 1, 2, 3$ is equivalent to following system of Volterra integral equations of the second kind:*

$$\begin{aligned}
(I_1b) \quad &A_{11}(t)q_1(t) + A_{12}(t)q_2(t) + A_{13}(t)q_3(t) + \\
&+ \sum_{j=1}^3 \int_0^t D_t S_{1,j}(t, s_1) q_j(s_1) ds_1 = \overline{H}^1(t),
\end{aligned}$$

$$(I_2b) \quad A_{22}(t)q_2(t) + A_{23}(t)q_3(t) + \sum_{j=1}^3 \int_0^t D_t S_{2,j}(t, s_1) q_j(s_1) ds_1 = \overline{H}^2(t),$$

$$(I_3b) \quad A_{33}(t)q_3(t) + \sum_{j=1}^3 \int_0^t D_t S_{3,j}(t, s_1) q_j(s_1) ds_1 = \overline{H}^3(t),$$

where

$$A_{11}(t) = \frac{3}{4} A \beta \left(\frac{1}{2}, \frac{1}{2} \right) (1 + (p'(t))^2),$$

$$A_{12}(t) = \frac{1}{4} A \beta \left(\frac{1}{2}, \frac{1}{2} \right) (p'(t))^2,$$

$$A_{13}(t) = A \beta \left(\frac{1}{2}, \frac{1}{2} \right) \left(-4p'(t)p''(t) + \frac{7}{4}(p'(t))^2 \right),$$

$$A_{21}(t) = 0,$$

$$A_{22}(t) = \frac{1}{2} A \beta \left(\frac{1}{2}, \frac{1}{2} \right),$$

$$A_{23}(t) = \frac{1}{3} A (-p'(t)),$$

$$A_{33}(t) = -\frac{2}{3} \beta \left(\frac{1}{2}, \frac{1}{2} \right),$$

$$\overline{H}^k(t) = D_t H^k(t), \quad k = 1, 2, 3.$$

$$S_{k,j}(t, s_1) = \int_0^1 D_t^{3-i} [(t-s)^{(-1/2)+3-j} (p(t) - p(s))^{j-1} \times \\ \times \exp(B(t, s)(p(t) - p(s))^2)] \Big|_{\substack{s:=s_1 \\ t:=s_1+(t-s_1)u}} du,$$

Proof. For $I_{1,1}$, we have

$$I_{1,1} = w_1(p(t), t) = A \int_0^t q_1(s) (t-s)^{3/2} \exp(B(t, s)(p(t) - p(s))^2) ds.$$

Differentiating $I_{1,1}(t)$, we obtain

$$\begin{aligned} D_t^2 I_{1,1}(t) = \\ = A \int_0^t (t-s)^{-1/2} \left\{ \frac{3}{4} + \left[-\frac{1}{4} [-(t-s)^{-2} (p(t) - p(s))^2 + \right. \right. \\ \left. \left. + 2(t-s)^{-1} (p(t) - p(s)) p'(t) \right] + (t-s) \left[-\frac{1}{4} [2(t-s)^{-3} (p(t) - p(s))^2 - \right. \right. \\ \left. \left. - 2(t-s)^{-2} (p(t) - p(s)) p'(t) - \right. \right. \\ \left. \left. - 2(t-s)^{-2} (p(t) - p(s)) p'(t) - 2(t-s)^{-1} (p'(t))^2 + 2(t-s)^{-1} (p(t) - p(s)) p''(t) + \right. \right. \end{aligned}$$

$$+\frac{1}{4}(t-s)^{-2}(p(t)-p(s))^2+2(t-s)^{-1}(p(t)-p(s))p'(t)]^2\Bigg\}\times$$

$$\times \exp(B(t,s)(p(t)-p(s))^2)q_1(s)ds.$$

Applying to the $D_t^2 I_{1,1}(t)$ Abel transformation [14], Vol. I, p. 16, we obtain

$$A \int_0^t (t-s)^{-1/2} \int_0^s (s-s_1)^{-1/2} \left\{ \frac{3}{4} + \left[-\frac{1}{4}(-(s-s_1)^{-2}(p(s)-p(s_1))^2 + \right. \right.$$

$$+2(s-s_1)^{-1}(p(s)-p(s_1))p'(s)) \Bigg] +$$

$$+(s-s_1) \left[-\frac{1}{4}[2(s-s_1)^{-3}(p(s)-p(s_1))^2 - 2(s-s_1)^{-2}(p(s)-p(s_1))p'(s) - \right.$$

$$-2(s-s_1)^{-2}(p(s)-(p(s_1))p'(s) - 2(s-s_1)^{-1}(p'(s))^2 +$$

$$+2(s-s_1)^{-1}(p(s)-p(s_1))p''(s) +$$

$$\left. \left. +\frac{1}{4}(s-s_1)^{-2}(p(s)-p(s_1))^2 + 2(s-s_1)^{-1}(p(s)-p(s_1))p'(s) \right]^2 \right\} \times$$

$$\times \exp(B(s,s_1)(p(s)-p(s_1))^2)q_1(s_1)ds_1.$$

Interchanging the order of the integration in the last formula and applying the change variable of the integral variable

$$s = s_1 + (t-s_1)u, \quad ds = (t-s_1)du, \quad u \in (0,1),$$

we obtain

$$A \int_0^t \left\{ \int_0^1 \frac{1}{(1-u)^{1/2}u^{1/2}} \left\{ \frac{3}{4} + \left[\frac{3}{4} + \left[-\frac{1}{4}((t-s_1)u)^{-2}(p(s_1+(t-s_1)u)-p(s_1))^2 + \right. \right. \right.$$

$$+2((t-s_1)u)^{-1}(p(s_1+(t-s_1)u)-p(s_1))p'(s_1+(t-s_1)u) +$$

$$+((t-s_1)u) \left[-\frac{1}{4}[2((t-s_1)u)^{-3}(p(s_1+(t-s_1)u)-p(s_1))^2 - \right.$$

$$-2((t-s_1)u)^{-2}(p(s_1+(t-s_1)u)-p(s_1))p'(s_1+(t-s_1)u) -$$

$$-2(t-s_1)u)^{-2}(p(s_1+(t-s_1)u)-p(s_1))p'(s_1+(t-s_1)u) +$$

$$+2((t-s_1)u)^{-1}(p'(s_1+(t-s_1)u))^2 +$$

$$+2((t-s_1)u)^{-1}p(s_1+(t-s_1)u-p(s_1))p''(s_1+(t-s_1)u) +$$

$$\left. \left. +\left[\frac{1}{4}((t-s_1)u)^{-2}(p(s_1+(t-s_1)u)-p(s_1))^2 + \right. \right. \right.$$

$$\left. \left. +2((t-s_1)u)^{-1}(p(s_1+(t-s_1)u)-p(s_1))p'(s_1+(t-s_1)u) \right]^2 \right\} \Bigg\} \times$$

$$\times \exp((-4(t-s_1)u)^{-1})p(s_1+(t-s_1)u-p(s_1))^2 du \Big\} q_1(s_1) ds_1.$$

Applying to the last formula the mean value theorem and differentiating with respect to t , we obtain

$$\begin{aligned} q_1(t) \int_0^t \frac{1}{(1-u)^{1/2}u^{1/2}} \Big\{ \frac{3}{4} \Big[-\frac{1}{4}(t-s_1)^{-2}u^{-2}(t-s_1)^2u^2p'(s_1+\theta(t-s_1))^2 \Big] + \\ + 2(t-s_1)^{-1}u^{-1}(t-s_1)up'(s_1+\theta(t-s_1))p'(s_1+(t-s_1)u) + \\ + (t-s_1)u[(t-s_1)^{-3}u^{-3}((t-s_1)^2u^2(p'(s_1+\theta(t-s_1)))^2) - \\ - 2(t-s_1)^{-2}u^{-2}(t-s_1)up'(s_1+\theta(t-s_1))p'(s_1+(t-s_1)u) + \\ + 2(t-s_1)^{-1}u^{-1}(p'(s_1+(t-s_1)u))^2 + \\ + 2(t-s_1)^{-1}u^{-1}(t-s_1)up'(s_1+\theta(t-s_1))p''(s_1+(t-s_1)u)] + \\ + \Big[\frac{1}{4}(t-s_1)^{-2}u^{-2}((t-s_1)up'(s_1+\theta(t-s_1)))^2 + \\ + 2(t-s_1)^{-1}u^{-1}((t-s_1)up'(s_1+\theta(t-s_1)))p'(s_1+(t-s_1)u) \Big]^2 \Big\} \times \\ \times \exp((-4(t-s_1)u)^{-1})((t-s_1)up'(s_1+\theta(t-s_1)))^2 \Big|_{s_1=t} du + \\ + \int_0^t D_t S_{1,1}(t, s_1) q_1(s_1) ds_1, \end{aligned}$$

where

$$S_{1,1}(t, s_1) = \int_0^1 D_t^2 [(t-s)^{3/2} \exp(B(t, s)(p(t)-p(s))^2)] \Big|_{\substack{s=s_1 \\ t=s_1+(t-s_1)u}} du.$$

By last formula we obtain

$$q_1(t) \frac{3}{4} \beta\left(\frac{1}{2}, \frac{1}{2}\right) (1 + (p'(t))^2) + \int_0^1 D_t S_{1,1}(t, s_1) q_1(s_1) ds_1,$$

where β denote beta Euler function.

Denoting $A_{11}(t) = \frac{3}{4} \beta(\frac{1}{2}, \frac{1}{2}) (1 + (p'(t))^2)$, we obtain

$$A_{11}(t) q_1(t) + \int_0^t D_t S_{1,1}(t, s_1) q_1(s_1) ds_1.$$

Similarly for $I_{2,i}(t) = w_i(p(t), t)$, $i = 2, 3$, we obtain respectively

$$A_{12}(t) q_2(t) + \int_0^t D_t S_{1,2}(t, s_1) q_2(s_1) ds_1,$$

where

$$A_{1,2}(t) = \frac{1}{2}A\beta\left(\frac{1}{2}, \frac{1}{2}\right)(p'(t))^2,$$

$$S_{1,2}(t, s_1) = \int_0^1 D_t^2[(t-s)^{1/2} \exp(B(t, s)(p(t) - p(s))^2)] \Big|_{\substack{s:=s_1 \\ t:=s_1+(t-s_1)u}} du$$

and

$$A_{13}(t)q_3(t) + \int_0^t D_t S_{1,3}(t, s_1) ds_1,$$

where

$$A_{1,3}(t) = A\beta\left(\frac{1}{2}, \frac{1}{2}\right)\left(-4p'(t)p''(t) - \frac{7}{4}(p'(t))^2\right),$$

$$S_{1,3}(t, s_1) = \int_0^1 D_t^2[(t-s)^{-1/2} \exp(B(t, s)p(t)(p(t) - p(s))^2)] \Big|_{\substack{s:=s_1 \\ t:=s_1+(t-s_1)u}} du.$$

Differentiating $H^1(t)$ 2-times, substituting t by s , s by s_1 , multiplying by $(t-s)^{-1/2}$ and integrating in the interval $[0, t]$, we obtain

$$\int_0^t \frac{1}{(t-s_1)^{1/2}} D_{s_1}^2 H^1(s_1) ds_1$$

Integrating the last formula by parts, we obtain

$$2(t-s)^{1/2} D_{s_1}^2 H^1(s_1) \Big|_0^t - 2 \int_0^t (t-s_1)^{1/2} D_{s_1}^3 H^1(s_1) ds_1.$$

Differentiating the last formula with respect to t , we obtain

$$-2 \int_0^t (t-s_1)^{1/2} D_{s_1}^3 H^1(s_1) ds_1$$

because by assumptions is known that

$$D_s^2 H^1(0) = 0.$$

Denote by

$$\overline{H}^1(t) = -2 \int_0^t (t-s_1)^{1/2} D_{s_1}^3 H^1(s_1) ds_1.$$

Consequently by $(I_1 a)$ we obtain the equivalent equation $(I_1 b)$ in the following form

$$A_{11}(t)q_1(t) + A_{12}(t)q_2(t) + A_{13}(t)q_3(t) + \sum_{j=1}^3 \int_0^t D_t S_{1,j}(t, s_1) q_j(s_1) ds_1 = \overline{H}^1(t).$$

Similarly differentiating the equations $(I_j a)$, $j = 2, 3$, $3 - j$ -times, substituting t by s and s by s_1 , multiplying both sides by $(t - s)^{-1/2}$ and integrating in the interval $[0, t]$, interchanging the order of the integration, applying the change variable $s = s_1 + (t - s_1)u$, $u \in (0, 1)$ in the integrals, applying the mean value theorem $p(s_1 + (t - s_1)u) - p(s_1) = (t - s_1)up'(s_1 + \theta(t - s_1))$, $\theta \in (0, 1)$ and differentiating with respect to t , finally we obtain the equivalent equations $(I_j b)$, $j = 2, 3$.

Thus the proof of Lemma 4 is finished.

10. Solution of the system $(I_i b)$, $i = 1, 2, 3$

LEMMA 5. *If the assumptions of the Lemma 4 are satisfied, then the solution of the system $(I_i b)$, $i = 1, 2, 3$, is of form $(I_i c)$*

$$(I_i c) \quad q_i(t) = F^i(t) + \sum_{j=1}^3 \int_0^t q_j(s) N_{ij}(t, s) ds, \quad i = 1, 2, 3,$$

where

$$F^1(t) = (A_{11}(t))^{-1}[\overline{H^1(t)} - A_{12}(t)F^2(t) - A_{13}(t)F^3(t)],$$

$$F^2(t) = (A_{22}(t))^{-1}[\overline{H^2(t)} - A_{23}(t)F^3(t)],$$

$$F^3(t) = (A_{33}(t))^{-1}\overline{H^3(t)},$$

$$N_{1j}(t, s) = (A_{11}(t))^{-1}[A_{12}(t)N_{2j}(t, s) + A_{13}(t)N_{3j}(t, s) + D_t S_{1j}(t, s)],$$

$$j = 1, 2, 3,$$

$$N_{2j}(t, s) = (A_{22}(t))^{-1}[N_{3j}(t, s) - D_t S_{2j}(t, s)], \quad j = 1, 2, 3,$$

$$N_{3j}(t, s) = (A_{33}(t))^{-1}D_t S_{3j}(t, s), \quad j = 1, 2, 3.$$

Proof. Applying the method of the successive elimination of the unknown functions q_i , $i = 1, 2, 3$, we shall determine successively the densities q_i .

By the equation $(I_3 b)$, we obtain

$$q_3(t) = F^3(t) + \sum_{j=1}^3 \int_0^t q_j(s) N_{3j}(t, s) ds,$$

where $F^3(t)$ and $N_{3j}(t, s)$ are defined below.

By the equation $(I_2 b)$, we obtain

$$q_2(t) = F^2(t) + \sum_{j=1}^3 \int_0^t q_j(s) N_{2j}(t, s) ds,$$

where $F^2(t)$ and $N_{2j}(t, s)$ are defined below.

By the equation (I₁b), we obtain

$$q_1(t) = F^1(t) + \sum_{j=1}^3 \int_0^t q_j(s) N_{1j}(t, s) ds,$$

where $F^1(t)$ and $N_{1j}(t, s)$ are defined below.

11. Solution of the system (I_ic), $i = 1, 2, 3$

Let us consider the system with parameter λ compatible to the system (I_ic) of the form

$$(I_i\lambda) \quad q_i(t) = F^i(t) + \lambda \sum_{j=1}^3 \int_0^t q_j(s) N_{ij}(t, s) ds, \quad i = 1, 2, 3.$$

Let

$$N_{ij}^1(t, s) = \sum_{k=1}^3 \int_s^t N_{ik}(t, s_1) N_{kj}(s_1, s) ds_1,$$

$$N_{ij}^{n+1}(t, s) = \sum_{k=1}^3 \int_s^t N_{ik}(t, s_1) N_{kj}^n(s_1, s) ds_1, \quad n = 1, 2, \dots, \quad i, j = 1, 2, 3,$$

$$R_{ij}(t, s, \lambda) = N_{ij}(t, s) + \lambda N_{ij}^1(t, s) + \dots + \lambda^n N_{ij}^n(t, s) + \dots, \\ n = 1, 2, \dots, \quad i, j = 1, 2, 3.$$

By [14], Vol. I, p. 4 or by [5], p. 97, we obtain

LEMMA 6. *If the functions $F^i \in C^{3-j}([0, T])$ and $D_t^{2-j} H^i(0) = 0$, where $i = 1, 2, 3$ and $j = 0$ for $k = 1$, $j = 1$ for $k = 2$, $j = 2$ for $k = 3$, then the functions*

$$q_i(t, \lambda) = F^i(t) + \sum_{j=1}^3 \int_0^t R_{ij}(t, s, \lambda) F^j(s) ds, \quad i = 1, 2, 3$$

are the solutions of class $C([0, T])$ for every $\lambda \in \mathbb{R}$ of the system (I_iλ) and the functions

$$q_i(t) = q_i(t, 1), \quad i = 1, 2, 3$$

are the unique solutions of class $C([0, T])$ of the system (I_ic).

12. Fundamental theorem

By Lemmas 1–6, we obtain the following result.

THEOREM. *If the assumptions of Lemmas 1–6 are satisfied, then the function*

$$w(x, t) = \sum_{i=1}^4 w_i(x, t)$$

is a solution of the problem (1a)–(4a) and the function

$$u(x, t) = w(x, t) + r(x, t)$$

is a solution of the (1)–(4) problem.

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