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CHARACTERIZATIONS OF POLYNOMIALS IN RIGHT INVERTIBLE OPERATORS

In this paper we introduce the so-called τ -Volterra operators and generalize the known-result of Przeworska-Rolewicz and von Trotha on Volterra characterizations of polynomials with scalar coefficients to the case of polynomials in algebraic elements in terms of τ -Volterra characterizations.

Let X be a linear space over the field of complex numbers. The set of all linear operators acting in X is denoted by $L(X)$. Write: $L_0(X) = \{A \in L(X) : \text{dom } A = X\}$. An operator $D \in L(X)$ is said to be right invertible if there exists an $R \in L_0(X)$ such that $RX \subset \text{dom } D$ and $DR = I$. Denote by $R(X)$ the set of all right invertible operators and by R_D the set of all right inverses of a $D \in R(X)$ (cf. [1]).

Recall the following results of Przeworska-Rolewicz and von Trotha [1]-[2]:

THEOREM 1 (Przeworska-Rolewicz). *Write*

$$\overline{Q}(t, s) = \sum_{k=0}^N q_k t^k s^{N-k}, \quad \overline{Q}(t) = \overline{Q}(t, 1), \quad \overline{P}(t) = t^M \overline{Q}(t)$$

where $q_0, \dots, q_{N-1} \in C$, $q_N = 1$, M is a non-negative integer. If there exists an $R \in R_D \cap V(X)$ (i.e. R is a Volterra right inverse of D), then $\overline{P}(D) \in R(X)$ and the operator

$$(1) \quad R_0 = R^{M+N} (\overline{Q}(I, R))^{-1}$$

is a Volterra right inverse of $\overline{P}(D)$.

THEOREM 2 (von Trotha). *If R_0 of the form (1) is a Volterra operator, then R is Volterra.*

We shall deal with τ -Volterra operators and generalize Theorems 1 and 2 to the case of $\overline{Q}(t, s)$ being a polynomial with algebraic operator coefficients (cf. also [3], section 6).

DEFINITION 1. An operator $R \in L_0(X)$ is said to be r -Volterra if the operator $I - \lambda R$ is right invertible for all $\lambda \in C$.

We denote by $V_r(X)$ the set of all r -Volterra operators acting in X .

LEMMA 1. Let $D \in R(X)$ and $R \in R_D \cap V_r(X)$. Then $\overline{Q}(I, R) \in R(X)$ and

$$(2) \quad \overline{R}_0 = R^{M+N}T, \quad \text{where } T \in R_{\overline{Q}(I, R)},$$

is a right inverse of $\overline{P}(D)$.

PROOF. Rewrite $\overline{Q}(I, R)$ in the form

$$\overline{Q}(I, R) = \prod_{i=1}^n (I - t_i R)^{\alpha_i}, \quad \alpha_1 + \dots + \alpha_n = N.$$

This implies that $\overline{Q}(I, R) \in R(X)$, provided that $R \in V_r(X)$. For every $T \in R_{\overline{Q}(I, R)}$, we have

$$\overline{Q}(D)R^N T = \sum_{k=0}^N q_k D^k R^N T = \sum_{k=0}^N q_k R^{N-k} T = \overline{Q}(I, R)T = I.$$

Consequently, the operator $\overline{P}(D) := D^M \overline{Q}(D)$ is also right invertible and

$$\overline{P}(D)\overline{R}_0 = D^M \overline{Q}(D)R^M R^N T = \overline{Q}(D)R^N T = I.$$

LEMMA 2. If $\overline{R}_0$ of the form (2) is a r -Volterra operator, then R is r -Volterra.

PROOF. Let $s_0, \lambda \in C (\lambda \neq 0, s_0 \neq 0)$ be fixed, $\mu = \overline{Q}(\lambda) \overline{s}_0^M$. It is easy to see that

$$I - \mu \overline{R}_0 = \overline{P}(D)R_0 - \mu \overline{R}_0 = (\overline{Q}(I, R) - \mu R^{N+M})T.$$

It follows that $\overline{Q}(I, R) - \mu R^{M+N}$ is right invertible, since $I - \mu R_0$ is right invertible.

Write: $\overline{P}_\mu(t, s_0) = \overline{Q}(t, s_0) - \mu s_0^{M+N}$.

We find

$$\begin{aligned} \overline{P}_\mu(\lambda s_0, s_0) &= \overline{Q}(\lambda s_0, s_0) - \mu s_0^{M+N} \\ &= \sum_{k=0}^N q_k \lambda^k s_0^k s_0^{N-k} - \mu s_0^{M+N} = s_0^N (\overline{Q}(\lambda) - \mu s_0^M) = 0. \end{aligned}$$

Hence

$$\overline{P}_\mu(t, s_0) = (t - \lambda s_0) \overline{Q}_\mu(t, s_0),$$

where $\overline{Q}_\mu(t, s_0)$ is a well-defined polynomial.

Since $\overline{P}_\mu(I, R) = \overline{Q}_\mu(I, R) - \mu R^{M+N}$ is right invertible, we conclude that $I - \lambda R$ is right invertible for all $\lambda \in C$, i.e. $R \in V_r(X)$.

Now we deal with the case of polynomials in algebraic operators.

Let $D \in R(X)$, $R \in R_D$ and let $A_0, \dots, A_{N-1}, A_N = I$ be mutually commutative algebraic operators. Suppose that

$$(3) \quad DA_i = A_i D \quad \text{on } \text{dom } D; \quad RA_i = A_i R \quad (i = 0, \dots, N).$$

Write

$$(4) \quad Q(t, s) = \sum_{i=0}^N A_i t^i s^{N-i}, \quad Q(t) = Q(t, 1), \quad P(t) = t^M Q(t)$$

(M is a non-negative integer).

THEOREM 1. *Let $D \in R(X)$ and let $R \in R_D \cap V_r(X)$. Then $Q(I, R)$ is right invertible and $R_0 = R^{M+N}T$ is a right inverse of $P(D)$ for every $T \in R_{Q(I, R)}$.*

Proof. Write $\hat{X}_0 = \text{lin}\{R^k; k = 0, 1, \dots\}$. Then $\hat{X}_0$ is a commutative subalgebra in $L_0(X)$ and $Q(I, R)$ is a generalized algebraic element with characteristic roots belonging to the set (cf. Theorem 3.4 in [3]):

$$(5) \quad \left\{ I + \sum_{i=1}^N t_{n-i, k} R^i, \quad (i = 1, \dots, N; k = 1, \dots, r_{N-1}) \right\},$$

where $\{t_{i,1}, \dots, t_{i,r_i}\}$ are the characteristic roots of A_i . Lemma 1 implies that every operator belonging to the set (5) is right invertible. This implies that $Q(I, R)$ is right invertible. If $T \in R_{Q(I, R)}$, then

$$Q(D)R^N T = \sum_{i=0}^N A_i D^i R^N T = \sum_{i=0}^N A_i R^{N-i} T = Q(I, R)T = I.$$

Consequently, the operator $P(D) := D^M Q(D)$ is right invertible and has a right inverse of the form $R_0 = R^{M+N}T$.

LEMMA 3. *Let $D \in R(X)$ and $R \in R_D \cap V_r(X)$. If A is an algebraic operator and $AR = RA$, then RA is a r -Volterra operator.*

Proof. Since $I + \lambda RA$ is a generalized algebraic operator over $\hat{X}_0$ for every $\lambda \in C$ with characteristic roots of the form $I + \lambda t_i R$ (where t_i ($i = 1, \dots, N$) are characteristic roots of the operator A), it follows that $I + \lambda AR$ is right invertible for all $\lambda \in C$, i.e. AR is r -Volterra operator.

A converse result to Theorem 1 is the following

THEOREM 2. Let $D \in R(X)$, $R \in R_D$ and let $A_0, \dots, A_N$ be algebraic operators satisfying the conditions (3). Let $Q(t, s)$, $Q(t)$ and $P(t)$ be defined by (4). If $Q(I, R)$ is right invertible, then

$$R_0 := R^{M+N}T \in R_{P(D)} \quad \text{for every } T \in R_{Q(I, R)}.$$

Moreover, if $A_i T = T A_i$ ($i = 1, \dots, N$) and $R_0 \in V_r(X)$, then $R \in V_r(X)$.

Proof. It is enough to check that $R \in V_r(X)$ provided that $R_0 \in V_r(X)$. Let $s_0, \beta \in C$ be fixed.

Write: $A = Q(\beta)s_0^{-M}$. Clearly, A is an algebraic operator and $AR_0 = R_0A$. Lemma 3 implies that AR_0 is a r -Volterra operator. Hence, $I - AR_0$ is right invertible.

On the other hand, $I - AR_0 = (Q(I, R) - AR^{M+N})T$. If $S \in R_{I - AR_0}$ then $I = [Q(I, R) - AR^{M+N}]TS$. It means that the operator $Q_0 := Q(I, R) - AR^{M+N}$ is right invertible. Write

$$H_A(t, s_0) = Q(t, s_0) - As_0^{M+N}.$$

Then we have

$$\begin{aligned} H_A(\beta s_0, s_0) &= Q(\beta s_0, s_0) - As_0^{M+N} \\ &= \sum_{k=0}^N q_k \beta^k s_0^k s_0^{N-k} - As_0^{M+N} = s_0^N (Q(\beta) - As_0^M) = 0. \end{aligned}$$

It implies

$$H_A(t, s_0) = (t - \beta s_0)Q_A(t, s_0).$$

Thus

$$H_A(I, R) = (I - \beta R)Q_A(I, R).$$

The operator $H_A(I, R)$ is right invertible for all $\beta \in C$, which implies that $I - \beta R$ is right invertible for all $\beta \in C$, i.e. $R \in V_r(X)$.

COROLLARY 1. Let $D \in R(X)$, $R \in R_D$, and let A be an algebraic operator commuting with R . If AR is an r -Volterra operator and A is invertible, then R is an r -Volterra operator.

COROLLARY 2. Let $D \in R(X)$, $R \in R_D \cap V_r(X)$. Suppose that A has the characteristic polynomial of the form

$$P_A(t) = \prod_{j=1}^n (t - t_j), \quad t_i \neq t_j \text{ for } i \neq j$$

and $AR = RA$.

Then every solution of the equation

$$(D - A)x = y, \quad y \in X$$

is of the form

$$(7) \quad x = \sum_{i=1}^n [T_i P_i (Ry + z) + c_i],$$

where $T_i \in R_{I-t_i R}$, $z \in \ker D$, $c_i \in \ker T_i$ and

$$P_i = \prod_{k=1, k \neq i}^n (t_i - t_k)^{-1} (A - t_k I).$$

Proof. Rewrite (6) in the form

$$(8) \quad x - RAx = Ry + z, \quad z \in \ker D.$$

Since $A = \sum_{i=1}^n t_i P_i$, (8) is equivalent to the equation

$$(9) \quad (I - \sum_{i=1}^n t_i R P_i)x = Ry + z.$$

Acting on both sides of (9) by the operator P_k , we obtain

$$(10) \quad (I - t_k R)P_k x = P_k(Ry + z), \quad (k = 1, \dots, n).$$

By our assumption $T_k \in R_{I-t_k R}$, (10) are equivalent to the following equations

$$P_k x = T_k P_k (Ry + z) + c_k,$$

where $c_k \in \ker(I - t_k R)$, $k = 1, 2, \dots, n$. Thus

$$x = \sum_{i=1}^n [T_i P_i (Ry + z) + c_i].$$

The proof is complete.

References

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