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A GENERALIZATION OF p -VALENT CLASSES RELATED TO CONVEX FUNCTIONS

Abstract. In this paper we shall study some properties of the class $C(b, p, A, B)$, where $C(b, p, A, B)$ is defined as follows :

$$p + \frac{1}{b} \left(1 + \frac{zf''(z)}{f'(z)} - p \right) \prec p \frac{1 + Az}{1 + Bz}, z \in U,$$

where $\prec$ denote subordination, $b \neq 0$ complex, $p \in \mathbb{N}$ and $-1 \leq B < A \leq 1$.

Introduction

Let $A_p(p \in \mathbb{N})$ denote the class of functions of the form

$$(1.1) \quad f(z) = z^p + \sum_{k=p+1}^{\infty} (p \in \mathbb{N})$$

which are analytic and p -valent in the unit disc $U = \{z : |z| < 1\}$.

For $|\lambda| < \frac{\pi}{2}$ and p a fixed positive integer, let $S_p^\lambda(\alpha)$ denote the class of functions $f(z) \in A_p$ which satisfy the inequality

$$(1.2) \quad \operatorname{Re} \left\{ e^{i\lambda} \frac{zf'(z)}{f'(z)} \right\} > \alpha \cos \lambda \quad (z \in U),$$

for some $\alpha(0 \leq \alpha < p)$. Following Patil and Thakare [9], we say that the functions belonging to the class $S_p^\lambda(\alpha)$ are p -valent λ -spiral-like functions of order α .

Furthermore, for $|\lambda| < \frac{\pi}{2}$ and p a fixed positive integer, let $C_p^\lambda(\alpha)$ denote the class of functions $f(z) \in A_p$ which satisfy the inequality

$$(1.3) \quad \operatorname{Re} \left\{ e^{i\lambda} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \alpha \cos \lambda, \quad z \in U,$$

for some $\alpha (0 \leq \alpha < p)$. We say that the functions belonging to the class $C_p^\lambda(\alpha)$ are p -valent λ -Robertson functions of order α .

It follows from (1.2) and (1.3) that

$$(1.4) \quad f(z) \in C_p^\lambda(\alpha) \quad \text{if and only if} \quad \frac{zf'(z)}{p} \in S_p^\lambda(\alpha).$$

DEFINITION 1 [1]. A function $f(z) \in A_p$ is said to be p -valent convex function of order b ($b \neq 0$ complex), that is $f(z) \in C(b, p)$ if and only if

$$(1.5) \quad \operatorname{Re} \left\{ p + \frac{1}{b} \left(1 + \frac{zf''(z)}{f'(z)} - p \right) \right\} > 0, \quad z \in U.$$

For $b \neq 0$ complex, $p \in N$, and $-1 \leq B < A < 1$, let $C(b, p, A, B)$ denote the class of functions of A_p which satisfy the condition

$$(1.6) \quad p + \frac{1}{b} \left(1 + \frac{zf''(z)}{f'(z)} - p \right) \prec \frac{1 + Az}{1 + Bz}, \quad z \in U.$$

In other words $f(z) \in C(b, p, A, B)$ if and only if there exists a function $w(z) \in \Omega$ such that

$$(1.7) \quad \frac{zf''(z)}{f'(z)} = \frac{(p-1) + [(A-B)pb + (p-1)B]w(z)}{1 + Bw(z)}.$$

And the above condition is equivalent to

$$(1.8) \quad \left| \frac{\left(1 + \frac{zf''(z)}{f'(z)} - p \right)}{pb(A-B) - B \left(1 + \frac{zf''(z)}{f'(z)} - p \right)} \right| < 1,$$

holds true for some $b \neq 0$ complex, $p \in N$, $-1 \leq B < A \leq 1$, and for all $z \in U$.

It is noticed that, by giving specific values to b, p, A and B , we obtain the following important subclasses studied by various authors in earlier works:

- (i) $C(1, 1, 1, -1) = C$ is the well known class of convex functions;
- (ii) $C((1-\alpha), 1, 1, -1) = C(\alpha)$ (Robertson [11], and Pinchuk [10]);
- (iii) $C((1-\alpha) \cos \lambda e^{-1\lambda}, 1, 1, -1) = C^\lambda(\alpha)$ (Chichra [2] and Sizuk [13]);
- (iv) $C(b, 1, 1, -1) = C(b)$ (Wiatrowski [14], Nasr and Aouf [6], [7]);
- (v) $C(1, 1, A, B) = C^*(A, B)$ (Mazur [5] and Silverman and Siliva [12]);
- (vi) $C(\cos \lambda e^{-1\lambda}, 1, A, B) = C^\lambda(A, B)$, where $C^\lambda(A, B)$ represents the class of functions $f(z) \in S$ satisfying the condition

$$(1.9) \quad 1 + \frac{zf''(z)}{f'(z)} = \frac{1 + [B + (A-B)e^{-1\lambda} \cos \lambda]w(z)}{1 + Bw(z)},$$

where $w(z) \in \Omega$, $|\lambda| < \frac{\pi}{2}$, and $-1 \leq B < A \leq 1$;

(vii) $C(1, p, 1, -1) = C(p)$, is the class of p -valent convex functions considered by Goodman [4];

(viii) $C(b, p, 1, -1) = C(b, p)$ (Aouf [1]);

(ix) $C(b, 1, A, B) = C(b, A, B)$; where $C(b, A, B)$ represents the class of functions $f(z) \in S$ satisfying the condition

$$(1.10) \quad 1 + \frac{1}{b} \frac{zf''(z)}{f'(z)} = \frac{1 + Aw(z)}{1 + Bw(z)},$$

where $w(z) \in \Omega$, $b \neq 0$ complex, and $-1 \leq B < A \leq 1$.

In this paper we determine representation formulas, sharp coefficient estimates for the class $C(b, p, A, B)$ and we maximize $|a_{p+2} - \mu a_{p+1}^2|$ over the class $C(b, p, A, B)$. Also, we get sufficient condition for a function to be in the class $C(b, p, A, B)$, and sharp radius of convexity for the class $C(b, p, A, B)$.

2. Representation formulas for the class $C(b, p, A, B)$

We shall require the following lemmas in our investigation:

LEMMA 1. Let the function $w(z)$ defined by

$$(2.1) \quad w(z) = \sum_{k=1}^{\infty} c_k z^k$$

be in the class Ω . Then

$$(2.2) \quad |c_1| \leq 1$$

and

$$(2.3) \quad |c_2| \leq 1 - |c_1|^2.$$

LEMMA 2. Let the function $f(z)$ be defined by (1.1). Then $f(z) \in C(b, p, A, B)$, $p \in \mathbb{N}$ if and only if it can be represented as follows

$$(2.4) \quad f'(z) = pz^{p-1}[h'(z)]^p (z \in U),$$

for some $h(z) \in C(b, A, B)$.

Proof. By direct computation, we find from representation (2.4) that

$$(2.5) \quad p + \frac{1}{b} \left(1 + \frac{zf''(z)}{f'(z)} - p \right) = p \left(1 + \frac{1}{b} \frac{zh''(z)}{h'(z)} \right).$$

Now the assertion (2.4) follows from (2.5) in view of definition (2.4).

LEMMA 3. Let the function $f(z)$ be defined by (1.1). Then $f(z) \in C(b, p, A, B)$, $p \in \mathbb{N}$ if and only if it can be represented as follows

$$(2.6) \quad f'(z) = pz^{p-1}[g'(z)]^{pb} \quad (z \in U),$$

for some $g(z) \in C(1, 1, A, B) \equiv C^*(A, B)$.

Proof. By direct computation, we find from representation (2.6) that

$$(2.7) \quad p + \frac{1}{b} \left(1 + \frac{zf''(z)}{f'(z)} - p \right) = p \left(1 + \frac{zg''(z)}{g'(z)} \right).$$

Now the assertion (2.6) follows from (2.7) in view of Definition 1.

3. Coefficient estimates for the class $C(b, p, A, B)$

We need in our investigation this lemma:

LEMMA 4. Let the function $w(z)$ defined by (2.1) be in the class Ω . Then

$$(2.8) \quad |c_2 - \mu c_1^2| \leq \max\{1, |\mu|\},$$

for any complex number μ . Equality in (2.8) may be attained with the functions $w(z) = z^2$ and $w(z) = z$ for $|\mu| < 1$ and $|\mu| \geq 1$, respectively.

THEOREM 1. If a function $f(z)$ defined by (1.1) is in the class $C(b, p, A, B)$ and if μ is any complex number, then

$$(3.1) \quad |a_{p+2} - \mu a_{p+1}^2| \leq \frac{(A-B)p^2|b|}{2(p+2)} \max \left\{ 1, \left| [(A-B)pb - B] - \frac{2(A-B)(p+2)p^2b}{(p+1)^2} \mu \right| \right\}.$$

The result (3.1) is sharp for each μ .

Proof. Since $f(z) \in C(b, p, A, B)$, then from (1.7), we have

$$(3.2) \quad w(z) = \frac{(p-1)f'(z) - zf''(z)}{Bzf''(z) - [(p-1)B + (A-B)pb]f'(z)},$$

and applying the definition (1.1), it can be shown that

$$(3.3) \quad w(z) = \frac{(p+1)}{(A-B)p^2b} a_{p+1}z + \frac{1}{(A-B)p^2b} \left[2(p+2)a_{p+2} - \frac{(p+1)^2[(A-B)pb - B]}{(A-B)p^2b} a_{p+1}^2 \right] z^2 + \dots$$

Now compare the coefficients of z and z^2 on both sides of (3.3), using the definition (2.1). We thus obtain

$$(3.4) \quad c_1 = \frac{(p+1)}{(A-B)p^2b} a_{p+1},$$

and

$$(3.5) \quad c_2 = \frac{2(p+2)}{(A-B)p^2b} a_{p+2} - \frac{(p+1)^2[(A-B)pb - B]}{(A-B)^2p^4b^2} a_{p+1}^2.$$

Consequently, we have

$$(3.6) \quad a_{p+1} = \frac{(A-B)p^2b}{(p+1)} c_1,$$

and

$$(3.7) \quad a_{p+2} = \frac{(A-B)p^2b}{2(p+2)}c_2 + \frac{(p+1)^2[(A-B)pb-B]}{2(A-B)(p+2)p^2b}a_{p+1}^2.$$

Using (2.8), (3.6) and (3.7), we readily obtain (3.1).

Finally, the assertion (3.1) of Theorem 1 is sharp in view of the fact that the assertion (2.8) of Lemma 4 is sharp.

COROLLARY 1. *If the function $f(z)$ defined by (1.1) is in $C(b, p, A, B)$, then*

$$(3.8) \quad |a_{p+1}| \leq \frac{(A-B)p^2|b|}{(p+1)},$$

and

$$(3.9) \quad |a_{p+2}| \leq \frac{(A-B)p^2|b|}{2(p+2)} \max\{1, |(A-B)pb-B|\}.$$

The bounds in (3.8) and (3.9) are attained by the function $f^*(z)$ defined by

$$(3.10) \quad f^{*'}(z) = pz^{p-1}(1+Bz)^{\frac{p(A-B)b}{B}} (B \neq 0).$$

Proof. The assertions (3.8) and (3.9) of Corollary 1 follow directly from (3.6) and (3.1), respectively.

We shall require the following Lemma in our investigation:

LEMMA 5. *If $h(z) = z + \sum_{k=2}^{\infty} b_k z^k \in C(b, A, B)$, then*

$$(3.11) \quad |b_2| \leq \frac{(A-B)|b|}{2},$$

and

$$(3.12) \quad |b_3| \leq \frac{(A-B)|b|}{6} \max\{1, |(A-B)b-B|\}.$$

The bounds in (3.11) and (3.12) are attained by the function $h^*(z)$ defined by

$$(3.13) \quad h^{*'}(z) = (1+Bz)^{\frac{(A-B)b}{B}} (B \neq 0).$$

The bounds on the modulus of the second and third coefficients for functions in the class $C(b, p, A, B)$ are attained by another method as follows:

THEOREM 2. *Let the function $f(z)$ defined by (1.1) be in $C(b, p, A, B)$, then*

$$(3.14) \quad |a_{p+1}| \leq \frac{(A-B)p^2|b|}{(p+1)},$$

and

$$(3.15) \quad |a_{p+2}| \leq \frac{(A-B)p^2|b|}{2(p+2)} \{ \max\{1, |(A-B)b - B|\} + (A-B)(p-1)|b| \}.$$

These results are sharp with equality for $f^*(z)$ defined by (3.10).

Proof. By Lemma 2, there exists an $h(z) = z + \sum_{k=2}^{\infty} b_k z^k \in C(b, A, B)$ such that

$$(3.16) \quad f'(z) = pz^{p-1} + \sum_{k=p+1}^{\infty} ka_k z^{k-1} = pz^{p-1} \left[1 + \sum_{k=z}^{\infty} kb_k z^{k-1} \right]^{p-\alpha}.$$

Expanding the right hand side of (3.16), we obtain

$$(3.17) \quad f'(z) = pz^{p-1} + 2p^2 b_2 z^p + p(3pb_3 + 2p(p-1)b_2^2)z^{p+1} + \dots$$

Equating coefficients from (2.3.16) and (2.3.17), we have

$$(3.18) \quad (p+1)a_{p+1} = 2p^2 b_2,$$

$$(3.19) \quad (p+2)a_{p+2} = p(3pb_3 + 2p(p-1)b_2^2).$$

Thus, the result follows from Lemma 5.

Remarks on Corollary 1 and Theorem 2. Comparing the results in Corollary 1 and Theorem 2 we see that:

(1) When $\max\{1, |(A-B)b - B|\}$ in Theorem 2 is $|(A-B)b - B|$, then Corollary 1 is a better result.

(2) When $\max\{1, |(A-B)b - B|\}$ in Theorem 2 is 1, then Theorem 2 is a better result.

LEMMA 6. If p and m are positive integers, $b \neq 0$ complex and $-1 \leq B < A \leq 1$, then

$$(3.20) \quad \prod_{j=0}^{m-1} \frac{|(A-B)pb - Bj|^2}{(j+1)^2} \\ = \frac{1}{m^2} \left\{ (A-B)^2 p^2 |b|^2 \right. \\ + \sum_{k=1}^{m-1} [(A-B)^2 p^2 |b|^2 - 2k B p (A-B) \operatorname{Re}\{b\} + k^2 (B^2 - 1)] \\ \times \left. \prod_{j=0}^{k-1} \frac{|(A-B)pb - Bj|^2}{(j+1)^2} \right\}.$$

Proof. For $m = 1$, the assertion (3.20) is obvious. Thus, in order to prove Lemma 6 by induction on m , suppose that the result (3.20) holds true for

$m = q - 1$ ($q = 1, 2, 3, \dots$). Then, observing that

$$\begin{aligned}
 (3.21) \quad & \frac{1}{q^2} \left\{ (A - B)^2 p^2 |b|^2 + \sum_{k=1}^{m-1} \left[(A - B)^2 p^2 |b|^2 - 2k B p (A - B) \operatorname{Re}\{b\} \right. \right. \\
 & \quad \left. \left. + k^2 (B^2 - 1) \right] \prod_{j=0}^{k-1} \frac{|(A - B)pb - Bj|^2}{(j + 1)^2} \right\} \\
 &= \frac{1}{q^2} \left\{ (A - B)^2 p^2 |b|^2 + \sum_{k=1}^{q-2} \left[(A - B)^2 p^2 |b|^2 - 2k B p (A - B) \operatorname{Re}\{b\} \right. \right. \\
 & \quad \left. \left. + k^2 (B^2 - 1) \right] \prod_{j=0}^{k-1} \frac{|(A - B)pb - Bj|^2}{(j + 1)^2} \right\} \\
 & \quad + \left[(A - B)^2 p^2 |b|^2 + (q - 1)^2 (B^2 - 1) - 2(q - 1) B p (A - B) \operatorname{Re}\{b\} \right] \times \\
 & \quad \times \prod_{j=0}^{q-2} \frac{|(A - B)pb - Bj|^2}{(j + 1)^2} \\
 &= \left(\frac{q - 1}{q} \right)^2 \prod_{j=0}^{q-2} \frac{|(A - B)pb - Bj|^2}{(j + 1)^2} \\
 & \quad + \frac{[(A - B)^2 p^2 |b|^2 - 2(q - 1) B p (A - B) \operatorname{Re}\{b\} + (q - 1)^2 (B^2 - 1)]}{q^2} \\
 & \quad \times \prod_{j=0}^{q-2} \frac{|(A - B)pb - Bj|^2}{(j + 1)^2} \\
 &= \prod_{j=0}^{q-2} \frac{|(A - B)pb - Bj|^2}{(j + 1)^2} \cdot \frac{|(A - B)pb - (q - 1)B|^2}{q^2} \\
 &= \prod_{j=0}^{q-1} \frac{|(A - B)pb - Bj|^2}{(j + 1)^2},
 \end{aligned}$$

we conclude that the result (3.20) is valid also for $m = q$. This evidently completes the proof of Lemma 6.

THEOREM 3. *Let the function $f(z)$ defined by (1.1) be in $C(b, p, A, B)$. If*

$$(3.22) \quad (A - B)^2 p^2 |b|^2 + k^2 (B^2 - 1) > 2k B p (A - B) \operatorname{Re}\{b\},$$

then

$$(3.23) \quad |a_n| \leq \frac{p}{n} \cdot \prod_{k=0}^{n-p-1} \frac{|(A-B)pb - Bk|}{(k+1)}.$$

The bounds in (3.23) are sharp for all admissible b , p , A and B , and for each integer $n \geq p+1$.

Proof. Since $f(z) \in C(b, p, A, B)$, (1.7) gives

$$(3.24) \quad \{Bzf''(z) - [B(p-1) + (A-B)pb]f'(z)\}w(z) \\ = (p-1)f'(z) - zf''(z), \quad w \in \Omega.$$

Rewriting (3.24) in the form

$$(3.25) \quad \left\{ B[p(p-1)z^{p-1} + \sum_{k=1}^{\infty} (p+k)(p+k-1)a_{p+k}z^{p+k-1}] - \right. \\ \left. - [B(p-1) + (A-B)pb] \cdot [pz^{p-1} + \sum_{k=1}^{\infty} (p+k)a_{p+k}z^{p+k-1}] \right\} w(z) \\ = (p-1) \left[pz^{p-1} + \sum_{k=1}^{\infty} (p+k)a_{p+k}z^{p+k-1} \right] \\ - \left[p(p-1)z^{p-1} + \sum_{k=1}^{\infty} (p+k)(p+k-1)a_{p+k}z^{p+k-1} \right]$$

or, equivalently,

$$(3.26) \quad \left\{ (A-B)p^2b + \sum_{k=1}^{\infty} (-Bk + (A-B)pb)(p+k)a_{p+k}z^k \right\} w(z) \\ = - \sum_{k=1}^{\infty} k(p+k)a_{p+k}z^k,$$

we find that

$$(3.27) \quad \sum_{k=0}^{\infty} \left\{ [-Bk + (A-B)pb](p+k)a_{p+k}z^k \right\} w(z) = - \sum_{k=0}^{\infty} k(p+k)a_{p+k}z^k,$$

where $a_p = 1$ and $w(z)$ is given, as before, by (2.1).

Equating the coefficients of z^m on both sides of (3.27), we obtain

$$(3.28) \quad \sum_{k=0}^{m-1} [(-Bk + (A-B)pb)a_{p+k}c_{m-k}] = -m(p+m)a_{p+m},$$

which shows that a_{p+m} on the right-hand side depends only on the coefficients

$$a_p, a_{p+1}, \dots, a_{p+m-1}.$$

Occurring on the left-hand side. Hence we can write

$$(3.29) \quad \sum_{k=0}^{m-1} \{[-Bk + (A-B)pb](p+k)\} a_{p+k} z^k \} w(z) \\ = \sum_{k=0}^m k(p+k) a_{p+k} z^k + \sum_{k=m+1}^{\infty} A_k z^k,$$

for $m = 1, 2, 3, \dots$, and for a proper choice of A_k ($k \geq 0$).

Let $z = re^{i\theta}$, $0 < r < 1$, $0 \leq \theta \leq 2\pi$. Then

$$(3.30) \quad \sum_{k=0}^{m-1} |[-Bk + (A-B)pb](p+k)|^2 |a_{p+k}|^2 r^{2k} \\ = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^{m-1} \{[-Bk + (A-B)pb](p+k)\} a_{p+k} r^k e^{ik\theta} \right|^2 d\theta \\ \geq \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^{m-1} \{[-Bk + (A-B)pb](p+k)\} a_{p+k} r^k e^{ik\theta} \right|^2 |w(re^{i\theta})|^2 d\theta \\ \geq \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^m k(p+k) r^k e^{ik\theta} + \sum_{k=m+1}^{\infty} A_k r^k e^{ik\theta} \right|^2 d\theta \\ \geq \sum_{k=0}^m k^2 |a_{p+k}|^2 r^{2k} + \sum_{k=m+1}^{\infty} |A_k|^2 r^{2k} \geq \sum_{k=0}^m k^2 |a_{p+k}|^2 r^{2k}.$$

Letting $r \rightarrow 1$ in (3.30), we obtain

$$(3.31) \quad \sum_{k=0}^{m-1} [|(-Bk + (A-B)pb)(p+k)|^2 - k^2(p+k)^2] |a_{p+k}|^2 \\ \geq m^2(p+m)^2 |a_{p+m}|^2.$$

Setting $m = n - p$ in (3.31), we are led finally to the inequality

$$(3.32) \quad n^2(n-p)^2 |a_n|^2 \leq (A-B)^2 p^4 |b|^2 \\ + \sum_{k=1}^{n-p-1} [|(-Bk + (A-B)pb)(p+k)|^2 - k^2(p+k)^2] |a_{p+k}|^2 \\ (n = p+1, p+2, p+3, \dots).$$

Then, for $n = p + 1$, (3.3) immediately yields

$$(3.33) \quad |a_{p+1}| \leq \frac{(A-B)p^2|b|}{(p+1)},$$

which proves (3.24) for $n = p + 1$.

To establish (3.23) for $n > p + 1$, we use an induction argument.

Fix n , $n \geq p + 2$ and suppose (3.23) holds for $k = 1, 2, \dots, n - (p + 1)$. Then

$$(3.34) \quad |a_n|^2 \leq \frac{p^2}{n^2(n-p)^2} \left\{ (A-B)^2 p^2 |b|^2 + \sum_{k=1}^{n-(p+1)} [(A-B)^2 p^2 |b|^2 - 2kBp(A-B) \operatorname{Re}\{b\} + k^2(B^2 - 1)] \times \prod_{j=0}^{k-1} \frac{|(A-B)pb - Bj|^2}{(j+1)^2} \right\}.$$

Thus from (3.32), (3.34) and Lemma 2.3.1 with $m = n - p$, we obtain

$$|a_n|^2 \leq \frac{p^2}{n^2} \cdot \prod_{j=0}^{n-p-1} \frac{|(A-B)pb - Bj|^2}{(j+1)^2}.$$

This completes the proof of (3.23). This proof is based on a technique found in Clunie [3].

4. A sufficient condition for the class $C(b, p, A, B)$

We now establish a sufficient condition for a function to be in the class $C(b, p, A, B)$.

THEOREM 4. *Let the function $f(z)$ defined by (1.1) be analytic in U . Then $f(z) \in C(b, p, A, B)$ if, for $b \neq 0$ complex, $p \in N$, and $1 \leq B < A \leq 1$,*

$$(4.1) \quad \sum_{n=p+1}^{\infty} \frac{n}{p} \{(n-p)B\} |a_n| \leq p(A-B)|b|.$$

Proof. Let $|z| = r < 1$. Noting that

$$(4.2) \quad |zf''(z) - (p-1)f'(z)| < \sum_{n=p+1}^{\infty} n(n-p)|a_n|r,$$

and

$$(4.3) \quad |pb(A-B)f'(z) - B[zf''(z) - (p-1)f'(z)]| \\ > \{p^2|b|(A-B) - \sum_{n=p+1}^{\infty} n|pb(A-B) - (n-p)B|a_n|\}r,$$

we see that

$$(4.4) \quad |zf''(z) - (p-1)f'(z)| - |pb(A-B)f'(z) - B[zf''(z) - (p-1)f'(z)]| \\ < \left\{ \sum_{n=p+1}^{\infty} n\{(n-p) + |pb(A-B) - (n-p)B||a_n| - p^2(A-B)|b|\} \right\}r.$$

The right-hand side of (4.4) is non-positive by (4.1), so that $f(z) \in C(b, p, A, B)$ by Definition 1.

5. Radius of convexity for the class $C(b, p, A, B)$

THEOREM 5. *The sharp radius of convexity of the class $C(b, p, A, B)$ is given by*

$$(5.1) \quad r_c = \begin{cases} 2\{(A-B)|b|\sqrt{|b|^2(A-B)^2 + 4B[B + (A-B)\operatorname{Re}\{b\}]} \}^{-1} & (B \neq 0), \\ \{A|b|\}^{-1} & (B = 0). \end{cases}$$

(5.1) is real and finite only when $B \neq 0$ and such that

$$(5.2) \quad |b|^2(A-B)^2 + 4B[B + (A-B)\operatorname{Re}\{b\}] > 0.$$

The bounds in (5.1) are sharp for the function $f^*(z)$ given by

$$(5.3) \quad f^{*'}(z) = \begin{cases} pz^{p-1}(1+Bz)^{\frac{pb(A-B)}{B}} & (B \neq 0), \\ pz^{p-1}\exp\{pbAz\} & (B = 0). \end{cases}$$

Proof. Since $f(z) \in C(b, p, A, B)$, the condition (1.9) in conjunction with the Schwarz lemma [8] implies that

$$\left| \frac{zf''(z)}{f'(z)} - \zeta \right| < R,$$

where

$$\zeta = \frac{(p-1) - B[(p-1)B + pb(A-B)]r^2}{1 - B^2r^2}$$

and

$$R = \frac{p|b|(A-B)r}{1 - B^2r^2} \quad (|z| = r).$$

Hence we have

$$(5.4) \quad \frac{p - p|b|(A - B)r - B[pB + p\operatorname{Re}\{b\}(A - B)]r^2}{1 - B^2r^2} \\ \leq \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \leq \frac{p + p|b|(A - B)r - B[pB + p\operatorname{Re}\{b\}(A - B)]r^2}{1 - B^2r^2}.$$

From the first part of the inequalities in (5.4), it follows readily that

$$(5.5) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \quad \text{for } |z| < r_c,$$

where r_c is given by (5.1), provided that the condition (5.2) is satisfied. To show that the result (5.1) is sharp, we let $f^{*'}(z)$ be given (5.3), and put

$$(5.6) \quad \zeta = \begin{cases} -\left(\frac{Br^2 + \sqrt{\frac{b}{b}}r}{1 + B\sqrt{\frac{b}{b}}r} \right) & B \neq 0, \\ -\left(\sqrt{\frac{b}{b}}r \right) & B = 0. \end{cases}$$

We thus obtain

$$(5.7) \quad 1 + \frac{\zeta f^{*''}(\zeta)}{f^{*'}(\zeta)} = \begin{cases} \frac{p - p(A - B)|b|r - pB[B + (A - B)b]r^2}{1 - B^2r^2}, & B \neq 0 \\ p(1 - A|b|r), & B = 0 \end{cases}$$

which obviously has a zero real part when r is given by (5.1). This completes the proof of Theorem 5.

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Received July 12, 1999.