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ANOTHER CARTESIAN CLOSED CATEGORY OF PARTIAL ALGEBRAS

When investigating partial algebras from the categorical point of view, we can consider several different kinds of morphisms between them. With respect to various applications, especially to computer science, the usual homomorphisms seem to be the most important morphisms between partial algebras. Namely, the category of partial algebras of a given type with homomorphisms as morphisms has some useful properties such as amnesticity, transportability, completeness and cocompleteness, wellpoweredness and cowellpoweredness, etc. But, it in general fails to have a very important property for applications to computer science - the cartesian closedness (this category is cartesian closed iff the type of its objects is empty). Thus, it could be convenient to replace the category with some of its cartesian closed subcategories. Two such subcategories have been found in [7], two in [8] and three in [9]. In the presented note we discover another cartesian closed subcategory of the category of partial algebras of a given type and discuss it in relation to those from [7], [8] and [9]. We use a new, unified and effective way for introducing the categories considered. This way is based on the use of generalized matrices when defining the properties of partial algebras resulting in these categories. All cartesian closed categories dealt with are initially structured and we describe their function spaces.

For the basic categorical terminology used see [1]. Throughout the note, all categories are considered to be constructs, i.e., concrete categories of structured sets and structure-compatible maps. In any category, the underlying set of an object A is denoted by $|A|$. A category $\mathcal{K}$ is called *initially structured* [5] if the following three conditions are satisfied:

(1) $\mathcal{K}$ is well-fibred (i.e., $\mathcal{K}$ is fibre-small and for each set with at most one element, the corresponding fibre has exactly one element),

(2) $\mathcal{K}$ has concrete products,

(3) $\mathcal{K}$ has initial subobjects.

If A, B are objects of a given category $\mathcal{K}$, we denote by $\mathrm{Mor}_{\mathcal{K}}(A, B)$ the set of all morphisms from A to B in $\mathcal{K}$. Recall that a category $\mathcal{K}$ with finite products is said to be *cartesian closed* if for any $\mathcal{K}$-object B the functor $B \times - : \mathcal{K} \to \mathcal{K}$ has a right adjoint. It is well known ([5]) that an initially structured category $\mathcal{K}$ is cartesian closed if and only if for each pair A, B of $\mathcal{K}$-objects there is a *function space*, i.e., a $\mathcal{K}$-object A^B with $|A^B| = \mathrm{Mor}_{\mathcal{K}}(B, A)$ and such that for each $\mathcal{K}$-object C we get a bijection between $\mathrm{Mor}_{\mathcal{K}}(C \times B, A)$ and $\mathrm{Mor}_{\mathcal{K}}(C, A^B)$ when assigning f^* to each $f \in \mathrm{Mor}_{\mathcal{K}}(C \times B, A)$ where $f^*(c)(b) = f(c, b)$ whenever $c \in |C|$ and $b \in |B|$. Namely, the functor $-^B$ is then a right adjoint to $B \times -$. Function spaces in cartesian closed initially structured categories have a large collection of pleasant properties (see [5]), which makes these categories convenient for various applications.

The concepts concerning partial algebras are taken from [2]. Throughout the paper, Ω will designate an arbitrary, but fixed set, and τ will designate an arbitrary, but fixed family of sets $\tau = (K_\lambda; \lambda \in \Omega)$. The family τ will be called a *type*. By a *partial algebra* of type τ we understand a pair $\langle X, (p_\lambda; \lambda \in \Omega)\rangle$ where X is a set and p_λ is a partial K_λ-ary operation on X (i.e., a partial map $p_\lambda : X^{K_\lambda} \to X$) for each $\lambda \in \Omega$. For any $\lambda \in \Omega$ we denote by D_{p_λ} the domain of the operation p_λ, i.e., the subset of X^{K_λ} having the property that $p_\lambda(x_k; k \in K_\lambda)$ is defined iff $(x_k; k \in K_\lambda) \in D_{p_\lambda}$. Let $G = \langle X, (p_\lambda; \lambda \in \Omega)\rangle$ and $H = \langle Y, (q_\lambda; \lambda \in \Omega)\rangle$ be partial algebras of type τ. G is called a *relative subalgebra* of H if $p_\lambda(x_k; k \in K_\lambda) = x \Leftrightarrow q_\lambda(x_k; k \in K_\lambda) = x$ whenever $\lambda \in \Omega$, $(x_k; k \in K_\lambda) \in X^{K_\lambda}$ and $x \in X$. If, moreover, $D_{p_\lambda} = D_{q_\lambda} \cap X^{K_\lambda}$ for each $\lambda \in \Omega$, then G is called a *closed subalgebra* of H. By a *homomorphism* of G into H we mean any map $f : X \to Y$ such that $p_\lambda(x_k; k \in K_\lambda) = x \Rightarrow q_\lambda(f(x_k); k \in K_\lambda) = f(x)$ for each $\lambda \in \Omega$. The set of all homomorphisms from G into H will be denoted by $Hom(G, H)$. We denote by Pal_τ the category of all partial algebras of type τ with homomorphisms as morphisms. Of course, Pal_τ has concrete products (given by the usual direct products) and initial subobjects (given by relative subalgebras).

Let X, K, L be sets. By a *$K \times L$-matrix* M over X we understand any map $M : K \times L \to X$, i.e., $M = (x_{kl}; k \in K, l \in L)$ where $x_{kl} \in X$ whenever $k \in K$ and $l \in L$. We denote by M' the transposed matrix to M, i.e., the $L \times K$-matrix $M' = (x_{lk}; l \in L, k \in K)$ over X.

Let $x \in X$ be an element and $M = (x_{kl}; k \in K, l \in L)$ a $K \times L$-matrix over X. Then M is said to be *x-constant* provided that $x_{kl} = x$ for all $k \in K$

and all $l \in L$, and it is said to have *x-constant diagonal* provided that $K = L$ and $x_{kk} = x$ for all $k \in K$.

Let $M = (x_{kl}; k \in K, l \in L)$ be a $K \times L$-matrix over X and q an L-ary partial operation on X. Then M is said to be *q-operational* if $(x_{kl}; l \in L) \in D_q$ for each $k \in K$. Let p be a K-ary partial operation on X. Then M is said to be *pq-operational* provided that it is q-operational and $(q(x_{kl}; l \in L); k \in K) \in D_p$. We put $pq(M) = p(q(x_{kl}; l \in L); k \in K)$. Finally, M is said to be *diagonally p-operational* provided that $K = L$ and $(x_{kk}; k \in K) \in D_p$. We then put $\Delta_p(M) = p(x_{kk}; k \in K)$.

DEFINITION. A partial algebra $\langle X, (p_\lambda; \lambda \in \Omega)\rangle$ of type τ is called

(1) *idempotent* if for any $\lambda \in \Omega$ and any $x \in X$ we have $p_\lambda(x_k; k \in K_\lambda) = x$ whenever $x_k = x$ for all $k \in K_\lambda$ (cf. [7]),

(2) *commutative* if for any $\lambda, \mu \in \Omega$ and any $p_\lambda p_\mu$-operational $K_\lambda \times K_\mu$-matrix M over X such that M' is p_λ-operational it holds that M' is $p_\mu p_\lambda$-operational and $p_\lambda p_\mu(M) = p_\mu p_\lambda(M')$ (cf. [7]),

(3) *diagonal* if, for every $\lambda \in \Omega$, any $p_\lambda p_\lambda$-operational $K_\lambda \times K_\lambda$-matrix M over X is diagonally p_λ-operational and $p_\lambda p_\lambda(M) = \Delta_{p_\lambda}(M)$ (cf. [7]),

(4) *strongly diagonal* if, for every $\lambda \in \Omega$ and arbitrary $x \in X$, any p_λ-operational $K_\lambda \times K_\lambda$-matrix M over X is $p_\lambda p_\lambda$-operational with $p_\lambda p_\lambda(M) = x$ iff it is diagonally p_λ-operational with $\Delta_{p_\lambda}(M) = x$,

(5) *weakly diagonal* if, for every $\lambda \in \Omega$, any $p_\lambda p_\lambda$-operational $K_\lambda \times K_\lambda$-matrix M over X is diagonally p_λ-operational with $p_\lambda p_\lambda(M) = \Delta_{p_\lambda}(M)$ provided that M' is $p_\lambda p_\lambda$-operational with $p_\lambda p_\lambda(M) = p_\lambda p_\lambda(M')$,

(6) *locally diagonal* if for any $\lambda \in \Omega$, any $x \in X$ and any p_λ-operational $K_\lambda \times K_\lambda$-matrix M over X with x-constant diagonal it holds that M is $p_\lambda p_\lambda$-operational and $p_\lambda p_\lambda(M) = x$,

(7) *locally antidiagonal* if for any $\lambda \in \Omega$, any $x \in X$ and any $p_\lambda p_\lambda$-operational $K_\lambda \times K_\lambda$-matrix M over X with x-constant diagonal, from $p_\lambda p_\lambda(M) = x$ it follows that M is x-constant,

(8) *weakly locally antidiagonal* if for any $\lambda \in \Omega$, any $x \in X$ and any $p_\lambda p_\lambda$-operational $K_\lambda \times K_\lambda$-matrix M over X with x-constant diagonal and with the property that M' is $p_\lambda p_\lambda$-operational too, from $p_\lambda p_\lambda(M) = p_\lambda p_\lambda(M') = x$ it follows that M is x-constant.

REMARKS. 1) In the previous Definition, only the strong diagonality is a new property. All the other properties are equivalent to the corresponding ones from [7], [8] and [9]. More precisely, the idempotency has been introduced in [7] in the same way, and the diagonality is equivalent to the diagonality introduced in [7]. The commutative partial algebras are nothing else than the partial algebras that are said to fulfil the interchange law in [7]. The weak diagonality is equivalent to the weak diagonality from [8]. The local diagonality, local antidiagonality and weak local antidiagonality are equivalent to the same notions defined in [9].

2) Clearly, strong diagonality implies diagonality, diagonality implies weak diagonality, local antidiagonality implies weak local antidiagonality, and the conjunction of idempotency and strong diagonality implies local diagonality. For total algebras the commutativity introduced coincides with the commutativity studied in [4] and the conjunction of idempotency and diagonality coincides with the diagonality discussed in [6]. The mono-binary commutative total algebras are nothing else than the medial groupoids investigated in [3]. For mono-K-ary partial algebras the strong diagonality implies the commutativity. Of course, for total algebras the diagonality coincides with the strong diagonality.

EXAMPLES. 1. Let X be a set and ρ a binary relation on X. Let p be the binary partial operation on X given by

$D_p = \rho$, and $p(x, y) = x$ whenever $(x, y) \in D_p$

and let q be the binary partial operation on X dual to p, i.e.,

$D_q = \rho$, and $q(x, y) = y$ whenever $(x, y) \in D_q$.

Then $(X, p, q) \in \mathrm{Pal}_{(2,2)}$ and we have:

a) (X, p, q) is commutative,
b) ρ is reflexive iff (X, p, q) is idempotent,
c) ρ is transitive iff (X, p, q) is diagonal,
d) if ρ is symmetric, then (X, p, q) is locally diagonal,
e) if ρ is transitive and symmetric, then (X, p, q) is strongly diagonal,
f) ρ is a tolerance relation iff (X, p, q) is idempotent and locally diagonal,
g) ρ is antisymmetric iff (X, p, q) is weakly locally antidiagonal.

Of course, the statements a) - g) remain valid when replacing (X, p, q) with (X, p) or (X, q) respectively.

2. Let (G, p) be a partial rectangular band, i.e., $G = X \times Y$ where X, Y are sets and p is the binary partial operation on G given by $((x_1, y_1), (x_2, y_2)) \in D_p$ iff $y_1 = x_2$, and then $p((x_1, y_1), (x_2, y_2)) = (x_1, y_2)$. Clearly, $(G, p) \in \mathrm{Pal}_{(2)}$ is weakly diagonal.

3. Let X be a set and p the binary partial operation on the power-set $\mathcal{P}X$ of X given by $(A, B) \in D_p$ iff $A = B$ or A and B are nonempty with $A \cap B = \emptyset$, and then $p(A, B) = A \cap B$. Then $(\mathcal{P}X, p) \in \mathrm{Pal}_{(2)}$ is locally antidiagonal.

We denote by $IPal_\tau$, $CPal_\tau$, $DPal_\tau$, $SdPal_\tau$, $WdPal_\tau$, $LdPal_\tau$, $LaPal_\tau$ and $WlaPal_\tau$ the categories of idempotent, commutative, diagonal, strongly diagonal, weakly diagonal, locally diagonal, locally antidiagonal and weakly locally antidiagonal partial algebras of type τ, respectively - all considered to be full subcategories of Pal_τ. Further, we put $ICPal_\tau = IPal_\tau \cap CPal_\tau$, $ICDPal_\tau = IPal_\tau \cap CPal_\tau \cap DPal_\tau$, etc. Obviously, all these categories are well-fibred and closed under formation of products and initial subobjects in Pal_τ. Therefore they are initially structured. Though the following two statements give new results for the category $ISdPal_\tau$ only, they are formulated generally so as to include also the corresponding results from [7], [8] and [9].

THEOREM. *The categories $IPal_\tau$, $IDPal_\tau$, $ICDPal_\tau$, $ISdPal_\tau$, $IWdPal_\tau$, $ILdPal_\tau$, $ILaPal_\tau$ and $IWlaPal_\tau$ are cartesian closed and in each of them the function space G^H of a pair of objects $G = \langle X, (p_\lambda; \lambda \in \Omega)\rangle$, $H = \langle Y, (q_\lambda; \lambda \in \Omega)\rangle$ is defined by $G^H = \langle Hom(H,G), (r_\lambda; \lambda \in \Omega)\rangle$ where, for every $\lambda \in \Omega$, r_λ is the K_λ-ary partial operation on $Hom(H,G)$ given as follows: for any $(f_k; k \in K_\lambda) \in (Hom(H,G))^{K_\lambda}$ and any $f \in Hom(H,G)$, $r_\lambda(f_k; k \in K_\lambda) = f$ iff the implication $q_\lambda(y_k; k \in K_\lambda) = y \Rightarrow p_\lambda(f_k(y_k); k \in K_\lambda) = f(y)$ is satisfied.*

Proof. For $IPal_\tau$ and $ICDPal_\tau$ the statement follows from [7], for $IDPal_\tau$ and $IWdPal_\tau$ it follows from [8], and for $ILdPal_\tau$, $ILaPal_\tau$ and $IWlaPal_\tau$ it follows from [9]. We will prove the statement for $ISdPal_\tau$. As $ISdPal_\tau$ is a full subcategory of the cartesian closed category $IDPal_\tau$, it is sufficien to show that the class of objects of $ISdPal_\tau$ is closed under formation of function spaces in $IDPal_\tau$. So, let $G = \langle X, (p_\lambda; \lambda \in \Omega)\rangle$ and $H = \langle Y, (q_\lambda; \lambda \in \Omega)\rangle$ be objects of $ISdPal_\tau$ and let $G^H = \langle Hom(H,G), (r_\lambda; \lambda \in \Omega)\rangle$ be their function space in $IDPal_\tau$. Let $\lambda \in \Omega$ and $f \in Hom(H,G)$ be arbitrary elements and let $M = (f_{kl}; k \in K_\lambda, l \in K_\lambda)$ be an r_λ-operational matrix. If M is $r_\lambda r_\lambda$-operational with $r_\lambda r_\lambda(M) = f$, then the diagonality of G^H implies that M is diagonally r_λ-operational with $\Delta_{r_\lambda}(M) = f$. Conversely, let M be diagonally r_λ-operational with $\Delta_{r_\lambda}(M) = f$, and for each $k \in K_\lambda$ put $f_k = r_\lambda(f_{kl}; l \in K_\lambda)$. Let $(y_k; k \in K_\lambda) \in D_{q_\lambda}$, $q_\lambda(y_k; k \in K_\lambda) = y$. As H is idempotent, for any $k \in K_\lambda$ we have $q_\lambda(y_l; l \in K_\lambda) = y_k$ whenever $y_l = y_k$ for each $l \in K_\lambda$. Thus, $p_\lambda(f_{kl}; l \in K_\lambda) = f_k(y_k)$ for each $k \in K_\lambda$ and the matrix $M^* = (f_{kl}(y_k); k \in K_\lambda, l \in k_\lambda)$ is p_λ-operational. Further, $\Delta_{r_\lambda}(M) = r_\lambda(f_{kk}; k \in K_\lambda) = f$ implies $\Delta_{p_\lambda}(M^*) = p_\lambda(f_{kk}(y_k); k \in K_\lambda) = f(y)$. Hence M^* is diagonally p_λ-operational. Since G is strongly diagonal, M^* is $p_\lambda p_\lambda$-operational with $p_\lambda p_\lambda(M^*) = f(y)$. As $p_\lambda p_\lambda(M^*) = p_\lambda(p_\lambda(f_{kl}(y_k); l \in K_\lambda); k \in K_\lambda) = p_\lambda(f_k(y_k); k \in K_\lambda)$, we have $r_\lambda(f_k; k \in K_\lambda) = f$. Consequently, $r_\lambda r_\lambda(M) = r_\lambda(r_\lambda(f_{kl}; l \in K_\lambda); k \in K_\lambda) = r_\lambda(f_k; k \in K_\lambda) = f$. Hence M is $r_\lambda r_\lambda$-operational with $r_\lambda r_\lambda(M) = f$. We have shown that G^H is strongly diagonal. Therefore $ISdPal_\tau$ is cartesian closed.

CLAIM. In $IDPal_\tau$, $ICDPal_\tau$, $ISdPal_\tau$ and $IWdPal_\tau$ each function space G^H is a relative subalgebra of the direct product $G^{|H|}$. In $ICDPal_\tau$ each function space G^H is even a closed subalgebra of the direct product $G^{|H|}$.

Proof. In [8] it is shown that in $IWdPal_\tau$ each function space G^H is a relative subalgebra of the direct product $G^{|H|}$. So the same is valid also for $IDPal_\tau$ and $ICDPal_\tau$ because these categories are full subcategories of $IWdPal_\tau$ having inherited function spaces. The statement that in $ICDPal_\tau$ each function space G^H is a closed subalgebra of $G^{|H|}$ is also proved in [8].

Further initially structured and cartesian closed full subcategories of Pal_τ can be obtained as intersections of those from the Theorem.

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