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ON TENSOR PRODUCTS IN CONCRETE CATEGORIES

Abstract. In this paper we consider the notion of tensor product in a concrete category, in the sense of [5]. For such a tensor product, which we refer as a *concrete tensor product* we study some important properties: commutativity, associativity, epifunctoriality and zero object. We also consider examples and some special properties of tensor products and of concrete categories with tensor products for: arbitrary topological spaces, compact spaces, left modules, right H–comodules and left H–modules, for H a Hopf algebra.

1. Introduction

Several authors have dealt with certain types of tensor products in a category. Firstly, the notion of closed category was considered by S. Eilenberg & G. M. Kelly [4]. A tensor product is a symmetric monoidal structure extendable to a structure of closed category. For example, if X and Y are two topological spaces, then taking on $X \times Y$ the initial topology with respect to the class of all maps $f : X \times Y \longrightarrow Z$, $Z \in \mathrm{Top}^0$, such that $f(a, -) : Y \longrightarrow Z$ and $f(-, b) : X \longrightarrow Z$ are continuous maps for each $a \in X$, $b \in Y$, a tensor product $X \otimes Y$ is obtained. We notice that this topology is somehow dual of the product topology.

Tensor products in categories were considered after that by G. M. Kelly [6]. G. M. Kelly studies the associativity and left and right identities. Interesting results have been obtained by J. Činčura. In [1], it is proven that in the category Top of topological spaces and in the category of T_0–spaces, there exists only one tensor product (up to isomorphism). In [2], it is established that the category of pseudoradial spaces admits at most two tensor products and that the category of Hausdorff pseudoradial spaces admits exactly two tensor products. B. A. Davey and G. Davis in [3] consider the tensor product in a variety k as a free k-algebra. These authors prove a lot of good properties for their tensor product when k is an entropic variety.

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Recently, D. Jagiełło in [5] has defined a new tensor product in a concrete category and has given sufficient conditions (Theorem 2 of [5]) for a concrete category to have arbitrary tensor products. As application of this theorem, it is proven that the category Comp of compact topological spaces has tensor products.

In this Note, we consider the notion of tensor product according to [5]. For this there are studied some important properties: commutativity, associativity, epifunctoriality and zero object. There are also considered examples and special properties of tensor products and of concrete categories with tensor products for: arbitrary topological spaces, compact spaces, left modules, right H-comodules and left H-modules, for H a Hopf algebra.

2. Concrete tensor products

We remind that a *concrete category* $\mathcal{U}$ is a pair $(\mathcal{U}, U)$, with $\mathcal{U}$ a category and $U : \mathcal{U} \longrightarrow \text{Ens}$ a faithful covariant functor.

Firstly, we recall according to [5] the notion of concrete tensor product, some notations and results that are used in this paper.

Given the sets $M_1, \ldots, M_n$, and $m_j \in M_j, j = 1, \ldots, i-1, i+1, \ldots, n$, let $\pi^{M_i}_{m_1,\ldots,m_{i-1},m_{i+1},\ldots,m_n} : M_i \longrightarrow M_1 \times \cdots \times M_n$ be the function of sets defined by

(1) $\pi^{M_i}_{m_1,\ldots,m_{i-1},m_{i+1},\ldots,m_n}(x_i) = (m_1, \ldots, m_{i-1}, x_i, m_{i+1}, \ldots, m_n)$, for $x_i \in M_i$.

Let $(\mathcal{U}, U)$ be a concrete category and $\mathcal{U}^0$ the class of objects of $\mathcal{U}$.

Definition 1. [5]. Let $A_1, \ldots, A_n$, $B \in \mathcal{U}^0$. A function of sets

$$\varphi : U(A_1) \times \cdots \times U(A_n) \longrightarrow U(B)$$

is called an *n–morphism* for the concrete category $(\mathcal{U}, U)$ if for any $i = 1, \ldots, n$ and any $a_j \in U(A_j)$, $j = 1, \ldots, i-1, i+1, \ldots, n$, there exists a morphism in $\mathcal{U}$, $\gamma^{A_i}_{a_1,\ldots a_{i-1},a_{i+1},\ldots,a_n} : A_i \to B$ such that $\varphi\pi^{U(A_i)}_{a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n} = U(\gamma^{A_i}_{a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n})$.

Definition 2 [5]. Let $(A_1, \ldots, A_n)$ be a sequence of objects of the category $\mathcal{U}$. By a *tensor product* (*concrete tensor product*) of this sequence we call a pair (T, τ) with $T \in \mathcal{U}^0$ and $\tau : U(A_1) \times \cdots \times U(A_n) \longrightarrow U(T)$ an n–morphism for $(\mathcal{U}, U)$, having the following universality property: for any object $X \in \mathcal{U}^0$ and any n-morphism $\varphi : U(A_1) \times \cdots \times U(A_n) \longrightarrow U(X)$ there exists exactly one morphism $\zeta : T \longrightarrow X$ such that $U(\zeta)\tau = \varphi$.

The concrete tensor product, if it exists, is unique up to an isomorphism, as it follows from [5, Th.1].

If (T, τ) is a concrete tensor product of the sequence $(A_1, \dots, A_n)$, we shall sometimes denote $T = A_1 \otimes \cdots \otimes A_n$.

The concrete tensor product is a covariant multifunctor.

The first property, that we prove and that is mentioned also in [5] is commutativity.

PROPOSITION 1. *Let $(\mathcal{U}, U)$ be a concrete category with concrete tensor products. Then, for any $A_1, A_2 \in \mathcal{U}^0$, we have: $A_1 \otimes A_2 \simeq A_2 \otimes A_1$.*

Proof. Let $U(A_1) \times U(A_2) \xrightarrow{\tau_1} U(A_1 \otimes A_2)$ be the 2-morphism from the definition of tensor product and similarly, $U(A_2) \times U(A_1) \xrightarrow{\tau_2} U(A_2 \otimes A_1)$.

Denote $\varphi : U(A_1) \times U(A_2) \longrightarrow U(A_2 \otimes A_1)$ the function defined by $(x, y) \longmapsto \tau_2(y, x)$. It is easy to check that φ is a 2-morphism. Similarly, we define $\psi : U(A_2) \times U(A_1) \longrightarrow U(A_1 \otimes A_2)$ by $(y, x) \longmapsto \tau_1(x, y)$, which is a 2-morphism. It results that there exist exactly one morphism $f : A_1 \otimes A_2 \longrightarrow A_2 \otimes A_1$ and one morphism $g : A_2 \otimes A_1 \longrightarrow A_1 \otimes A_2$ such that $U(f)\tau_1 = \varphi$ and $U(g)\tau_2 = \psi$. It follows that $U(g)U(f)\tau_1 = \tau_1 \iff U(gf)\tau_1 = \tau_1$. From the universality property, it results that $gf = 1_{A_1 \otimes A_2}$. Analogously, $fg = 1_{A_2 \otimes A_1}$.

We shall now give more properties of concrete tensor product.

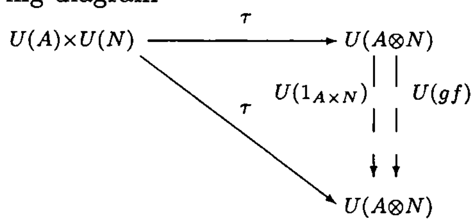
THEOREM 1. *Let $(\mathcal{U}, U)$ be a concrete category that has a zero object N and concrete tensor products. Then, for any object $A \in \mathcal{U}^0$, we have: $A \otimes N \simeq N$.*

Proof. Because N is a zero object, it results that there exist exactly one morphism $f : A \otimes N \longrightarrow N$ and one morphism $g : N \longrightarrow A \otimes N$. It follows that $fg : N \longrightarrow N$ is a morphism $\Longrightarrow fg = 1_N$.

On the other hand, for any $x \in U(A)$, let $\pi_x^{U(N)} : U(N) \longrightarrow U(A) \times U(N)$ defined by $y \longmapsto (x, y)$. As $\tau : U(A) \times U(N) \longrightarrow U(A \otimes N)$ is a 2-morphism, it results that it exists $g^x : N \longrightarrow A \otimes N$, morphism in $(\mathcal{U}, U)$, such that $\tau\pi_x^{U(N)} = U(g^x)$. But, N is an initial object. It follows that $g^x = g$, for any $x \in U(A) \Longrightarrow \forall(x \in U(A)),\ U(g) = \tau\pi_x^{U(N)}$.

Let $\varphi : U(A) \times U(N) \to U(N)$ be the function of sets defined by $(x, y) \mapsto y$. We have: $U(g)\varphi = \tau\pi_x^{U(N)}\varphi$ and $(\tau\pi_x^{U(N)}\varphi)(x_0, y_0) = (\tau\pi_{x_0}^{U(N)}\varphi)(x_0, y_0) = \tau\pi_{x_0}^{U(N)}(y_0) = \tau(x_0, y_0)$, for any $(x_0, y_0) \in U(A) \times U(N)$. Thus, $\tau\pi_x^{U(N)}\varphi = \tau$ $\Longrightarrow U(g)\varphi = \tau \Longrightarrow U(f)\tau = U(f)U(g)\varphi = U(fg)\varphi = U(1_N)\varphi = \varphi$. We obtain that $U(gf)\tau = U(g)U(f)\tau = U(g)\varphi = \tau$.

From the following diagram



it results that $gf = 1_{A\otimes N}$. We have obtained that $A\otimes N \simeq N$.

DEFINITION 3. Let $(\mathcal{U}, U)$ be a concrete category with concrete tensor products. We say that the 2-tensor product in $(\mathcal{U}, U)$ is *special* if for any $A_1, A_2, A_3 \in \mathcal{U}^0$, with

$$\tau : U(A_1)\times U(A_2) \to U(A_1\otimes A_2) \text{ and } \tau' : U(A_2)\times U(A_3)\to U(A_2\otimes A_3),$$

and any functions of sets

$$\varphi : U(A_1\otimes A_2)\times U(A_3) \longrightarrow U(B),\ \varphi' : U(A_1)\times U(A_2\otimes A_3)\longrightarrow U(B),$$

we have the following implications:

(i) If $\forall(z\in U(A_3))\ \exists(\gamma_z^{A_1\otimes A_2} : A_1\otimes A_2 \longrightarrow B)$ such that $\varphi\pi_z{}^{U(A_1\otimes A_2)} = U(\gamma_z^{A_1\otimes A_2})$ and $\varphi\circ(\tau\times 1_{U(A_3)})$ is a 3-morphism then φ is a 2-morphism.

(ii) If $\forall(x\in U(A_1))\ \exists(\gamma_x^{A_2\otimes A_3} : A_2\otimes A_3 \longrightarrow B)$ such that $\varphi'\pi_x^{U(A_2\otimes A_3)} = U(\gamma_x^{A_2\otimes A_3})$ and $\varphi'\circ(1_{U(A_1)}\times\tau')$ is a 3-morphism then φ' is a 2-morphism.

THEOREM 2. *Let $(\mathcal{U}, U)$ be a concrete category with 2-tensor products such that the 2-tensor product is special. Then $(A_1\otimes A_2)\otimes A_3 \simeq A_1\otimes(A_2\otimes A_3)$, for any $A_1, A_2, A_3 \in \mathcal{U}^0$.*

Proof. Let $U(A_2)\times U(A_3) \xrightarrow{\tau_1} U(A_2\otimes A_3)$, $U(A_1)\times U(A_2\otimes A_3) \xrightarrow{\tau_2} U(A_1\otimes(A_2\otimes A_3))$, $U(A_1)\times U(A_2)\xrightarrow{\tau_3} U(A_1\otimes A_2)$ and $U(A_1\otimes A_2)\times U(A_3)\xrightarrow{\tau_4} U((A_1\otimes A_2)\otimes A_3)$ be the 2-morphisms from the definition of tensor products. Let $z\in U(A_3)$. Let $u_z : U(A_1)\times U(A_2)\longrightarrow U(A_1\otimes(A_2\otimes A_3))$ be the function defined by $(x,y)\longmapsto \tau_2(x,\tau_1(y,z))$. We check that u_z is a 2-morphism. For any $\pi_{a_2}^{U(A_1)} : U(A_1)\longrightarrow U(A_1)\times U(A_2)$, $a_1\longmapsto(a_1,a_2)$, $(u_z\pi_{a_2}^{U(A_1)})(a_1) = \tau_2(a_1,\tau_1(a_2,z)) = U(\gamma^{A_1}_{\tau_1(a_2,z)})(a_1)$ from the fact that τ_2 is a 2-morphism. Also, because τ_1 and τ_2 are 2-morphisms, for any $\pi_{a_1}^{U(A_2)} : U(A_2)\longrightarrow U(A_1)\times U(A_2)$, $a_2\longmapsto(a_1,a_2)$, $(u_z\pi_{a_1}^{U(A_2)})(a_2) = \tau_2(a_1,\tau_1(a_2,z)) = U(\alpha_{a_1})(\tau_1(a_2,z)) = U(\alpha_{a_1})U(\beta_z)(a_2) = U(\alpha_{a_1}\beta_z)(a_2) = U(\gamma_{a_1}^{A_2})(a_2)$.

We have shown that u_z is a 2-morphism. From the following diagram, it exists exactly one morphism $v_z : A_1\otimes A_2\longrightarrow A_1\otimes(A_2\otimes A_3)$ such that $U(v_z)\tau_3 = u_z$.

(2)
$$\begin{array}{ccc} U(A_1)\times U(A_2) & \xrightarrow{\tau_3} & U(A_1\otimes A_2) \\ & \searrow^{u_z} & \downarrow U(v_z) \\ & & U(A_1\otimes(A_2\otimes A_3)) \end{array}$$

We define $U(A_1 \otimes A_2) \times U(A_3) \xrightarrow{\varphi} U(A_1 \otimes (A_2 \otimes A_3))$ by $\varphi(\omega, z) = U(v_z)(\omega)$. Check that φ is a 2-morphism. Because the 2-tensor product is special, it is sufficient to prove that $\varphi \circ (\tau_3 \times 1_{U(A_3)})$ is a 3-morphism. We have that

$$(\varphi \circ (\tau_3 \times 1_{U(A_3)}))(x, y, z) = \varphi(\tau_3(x, y), z) = U(v_z)(\tau_3(x, y)) = u_z(x, y),$$

for any $(x, y, z) \in U(A_1) \times U(A_2) \times U(A_3)$.

For any $\pi^{U(A_1)}_{(a_2,a_3)} : U(A_1) \longrightarrow U(A_1) \times U(A_2) \times U(A_3)$, $a_1 \longmapsto (a_1, a_2, a_3)$, we have:

$$(\varphi \circ (\tau_3 \times 1_{U(A_3)}) \circ \pi^{U(A_1)}_{(a_2,a_3)})(a_1) = u_{a_3}(a_1, a_2) = U(\gamma^{A_1}_{(a_2,a_3)})(a_1)$$

as u_{a_3} is a 2-morphism.

Similarly, $\varphi(\tau_3 \times 1_{U(A_3)})\pi^{U(A_2)}_{(a_1,a_3)} = U(\gamma^{A_2}_{(a_1,a_3)})$.

Finally, for any $\pi^{U(A_3)}_{(a_1,a_2)}$,

$$\begin{aligned} &u_{a_3}(a_1, a_2) = \tau_2(a_1, \tau_1(a_2, a_3)) = U(\alpha^1_{a_1})(\tau_1(a_2, a_3)) = \\ &= U(\alpha^1_{a_1})U(\beta^1_{a_2})(a_3) = U(\alpha^1_{a_1}\beta^1_{a_2})(a_3) \\ &\Longrightarrow \varphi(\tau_3 \times 1_{U(A_3)})\pi^{U(A_3)}_{(a_1,a_2)} = U(\alpha^1_{a_1}\beta^1_{a_2}). \end{aligned}$$

We have obtained that φ is a 2-morphism.

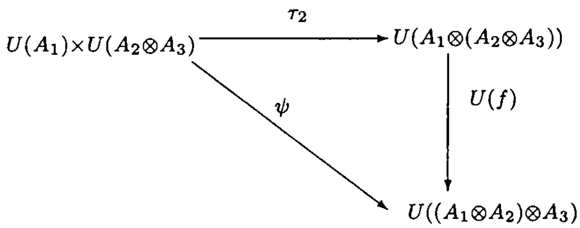
Let $g : (A_1 \otimes A_2) \otimes A_3 \longrightarrow A_1 \otimes (A_2 \otimes A_3)$ be the morphism in $\mathcal{U}$ uniquely defined such that the following diagram is commutative:

(3)
$$\begin{array}{ccc} U(A_1\otimes A_2)\times U(A_3) & \xrightarrow{\tau_4} & U((A_1\otimes A_2)\otimes A_3) \\ & \searrow^{\varphi} & \downarrow U(g) \\ & & U(A_1\otimes(A_2\otimes A_3)) \end{array}$$

In the same way, consider for any $x \in U(A_1)$, $\delta_x : U(A_2) \times U(A_3) \longrightarrow U((A_1 \otimes A_2) \otimes A_3)$, $(y, z) \longmapsto \tau_4(\tau_3(x, y), z)$, that is a 2-morphism. It follows that there exists exactly one morphism $\mu_x : A_2 \otimes A_3 \longrightarrow (A_1 \otimes A_2) \otimes A_3$ such that $U(\mu_x)\tau_1 = \delta_x$. Now, let $\psi : U(A_1) \times U(A_2 \otimes A_3) \longrightarrow U((A_1 \otimes A_2) \otimes A_3)$,

$(x, \alpha) \longmapsto U(\mu_x)(\alpha)$. ψ is a 2-morphism because $\psi \circ (1_{U(A_1)} \times \tau_1)$ is a 3-morphism. It follows that there exists only one morphism $f : A_1 \otimes (A_2 \otimes A_3) \longrightarrow (A_1 \otimes A_2) \otimes A_3$ such that the following diagram is commutative:

(4)



It remains to prove that $fg = 1_{(A_1 \otimes A_2) \otimes A_3}$ and $gf = 1_{A_1 \otimes (A_2 \otimes A_3)}$. We have: $U(f)(\tau_2(x, \tau_1(y, z))) = \delta_x(y, z) = \tau_4(\tau_3(x, y), z)$ and $U(g)(\tau_4(\tau_3(x, y), z) = \tau_2(x, \tau_1(y, z))$. It follows that

$$U(gf)(\tau_2(x, \tau_1(y, z)) = \tau_2(x, \tau_1(y, z)),$$

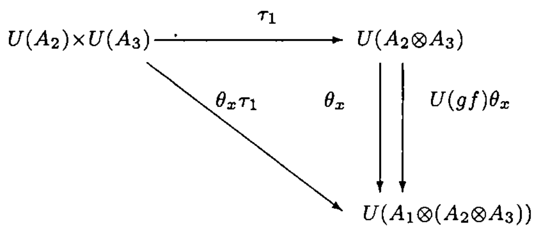
for any $(x, y, z) \in U(A_1) \times U(A_2) \times U(A_3)$.

Let $x \in U(A_1)$ and $\theta_x : U(A_2 \otimes A_3) \longrightarrow U(A_1 \otimes (A_2 \otimes A_3))$ be defined by $\alpha \longmapsto \tau_2(x, \alpha)$. It results that

$$(\theta_x \tau_1)(y, z) = \tau_2(x, \tau_1(y, z)) \Longrightarrow \forall(x \in U(A_1)),\ U(gf)\theta_x \tau_1 = \theta_x \tau_1.$$

But $\theta_x \tau_1$ is a 2-morphism. From the diagram

(5)



and the fact that $\theta_x = U(\widetilde{\alpha}_x)$, where $\widetilde{\alpha}_x : A_2 \otimes A_3 \longrightarrow A_1 \otimes (A_2 \otimes A_3)$ morphism in $(\mathcal{U}, U)$, it results that

$$\widetilde{\alpha}_x = gf\widetilde{\alpha}_x \Longrightarrow \theta_x = U(gf)\theta_x \Longrightarrow \tau_2(x, \alpha) = U(gf)(\tau_2(x, \alpha)),$$

for any $x \in U(A_1)$ and $\alpha \in U(A_2 \otimes A_3)$. It results that $\tau_2 = U(gf)\tau_2$ and from the universality property, it follows that $gf = 1_{A_1 \otimes (A_2 \otimes A_3)}$. Similarly, $fg = 1_{(A_1 \otimes A_2) \otimes A_3}$.

Definition 4. We say that a 2-tensor product $\tau : U(A_1) \times U(A_2) \longrightarrow U(A_1 \otimes A_2)$ in a concrete category $(\mathcal{U}, U)$ is *dense* if for every morphism $f : A \longrightarrow A_1 \otimes A_2$ for which $\operatorname{Im} \tau \subseteq \operatorname{Im} U(f)$, it follows that $U(f)$ is surjective.

Theorem 3. *Let $(\mathcal{U}, U)$ be a concrete category with 2-tensor products such that U preserves epimorphisms. Then, if $\tau : U(A_1) \times U(A_2) \longrightarrow U(A_1 \otimes A_2)$*

is dense and $f : A_1' \longrightarrow A_1$ *is an epimorphism, it results that the morphism*

$$f \otimes 1_{A_2} : A_1' \otimes A_2 \longrightarrow A_1 \otimes A_2$$

is also an epimorphism.

Proof. We recall that $f \otimes 1_{A_2}$ is defined in this way: let us consider $\varphi : U(A_1') \times U(A_2) \to U(A_1 \otimes A_2)$ defined by $\varphi = \tau \circ (U(f) \times 1_{U(A_2)})$. Then φ is a 2-morphism: $\varphi(a_1', a_2) = \tau(U(f)(a_1'), a_2) = U(\gamma_{a_2})(U(f)(a_1')) = U(\gamma_{a_2} f)(a_1')$ and $\varphi(a_1', a_2) = U(\gamma_{U(f)(a_1')})(a_2)$ because τ is a 2-morphism. It follows that there exists only one morphism $\bar{f} : A_1' \otimes A_2 \longrightarrow A_1 \otimes A_2$ such that $U(\bar{f})\tau' = \varphi$, where $\tau' : U(A_1') \times U(A_2) \longrightarrow U(A_1' \otimes A_2)$ is from the definition of tensor product. We denote $\bar{f}$ by $f \otimes 1_{A_2}$. As f is an epimorphism, it results that $U(f)$ is surjective. This implies that $U(f) \times 1_{U(A_2)}$ is surjective. We have that $U(f \otimes 1_{U(A_2)})\tau' = \tau(U(f) \times 1_{A_2})$. It results that $\operatorname{Im} \tau \subseteq \operatorname{Im} U(f \otimes 1_{A_2})$ and because τ is dense, it follows that $U(f \otimes 1_{A_2})$ is surjective. As U is faithful, $f \otimes 1_{A_2}$ is an epimorphism.

3. Examples

In this section, U is the forgetful functor.

a) *Tensor products of sets*

PROPOSITION 2. *For any* $A_1, \dots, A_n \in \mathrm{Ens}^0$, $A_1 \times \cdots \times A_n = A_1 \otimes \cdots \otimes A_n$.

Proof. It is obvious because every function of sets $\varphi : A_1 \times \cdots \times A_n \longrightarrow B$ is an n–morphism.

b) *Tensor products of topological spaces.* It can be shown that for two arbitrary topological spaces X and Y, the pair $(X \otimes Y, 1_{X \otimes Y})$, where $X \otimes Y$ is obtained like in the introduction, is a concrete tensor product in Top. We shall refer to this as the canonical tensor product in Top. We shall explain this tensor product in the situation when X is a discrete topological space.

PROPOSITION 3. *Let* X *be a discrete topological space and* Y *an arbitrary topological space.*

If $\bigvee_X Y$ *is the topological sum* $\bigcup_{x \in X} \{(x, y) \mid y \in Y\}$ *and* $\tau : X \times Y \longrightarrow \bigvee_X Y$ *is the map given by* $\tau(x, y) = (x, y)$, *then* $(\bigvee_X Y, \tau) = X \otimes Y$ *in* (Top, U).

Particularly, if $X = \{*\}$ *is a singleton, then* $\{*\} \otimes Y = Y$.

Proof. First we prove that τ is a 2-morphism. The function $Y \longrightarrow \bigvee_X Y$ defined by $y \longmapsto (x, y)$, where $x \in X$ is fixed, is continuous because $\{(x, y) \mid y \in Y\}$ is a subspace of the sum $\bigvee_X Y$ homeomorphic with Y. Also, because X is discrete, the following function $X \longrightarrow \bigvee_X Y$, defined by $x \longmapsto (x, y)$, for $y \in Y$ fixed, is continuous.

Let $\varphi : X \times Y \longrightarrow Z$ be a 2–morphism in (Top, U). It follows that the functions $X \longrightarrow Z$ defined by $x \longmapsto \varphi(x,y)$ and $Y \longrightarrow Z$ given by $y \longmapsto \varphi(x,y)$ are continuous. Define $\gamma : \underset{X}{\vee} Y \longrightarrow Z$ by $\gamma(x,y) = \varphi(x,y)$. This is continuous because $Y \longrightarrow \underset{X}{\vee} Y \longrightarrow Z$, $y \longmapsto (x,y) \longmapsto \varphi(x,y)$, $x \in X$ fixed, is continuous. We have: $(U(\gamma)\tau)(x,y) = U(\gamma)(x,y) = \varphi(x,y)$, so $U(\gamma)\tau = \varphi$. Uniqueness: let $\gamma' : \underset{X}{\vee} Y \longrightarrow Z$ with

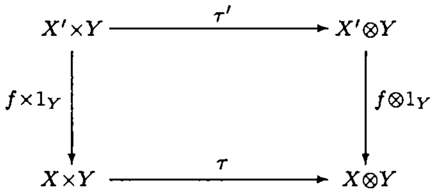
$$U(\gamma')\tau = \varphi \Longrightarrow U(\gamma')(x,y) = U(\gamma)(x,y) \Longrightarrow U(\gamma') = U(\gamma) \Longrightarrow \gamma' = \gamma.$$

REMARK 1. For the canonical tensor product $(X_1 \otimes X_2, \tau)$ in Top, τ is surjective and then we can apply Theorem 2.

PROPOSITION 4. *Let $X, Y, X' \in \mathrm{Comp}^0$, where* Comp *is the category of compact Hausdorff spaces. Assume that $\tau(X \times Y)$ is dense in a tensor product $X \otimes Y$ (cf.* [5]*, in Comp there exist tensor products). Then, if $f : X' \longrightarrow X$ is a continuous surjection, the map $f \otimes 1_Y : X' \otimes Y \longrightarrow X \otimes Y$ is surjective.*

Proof. The following diagram is commutative:

(6)



We have that, for any $x \in X$, there exists $x' \in X'$ such that $f(x') = x$. It follows that $\tau(x,y) = \tau(f(x'),y) = \tau(f \times 1_Y)(x',y) = (f \otimes 1_Y)(\tau'(x',y))$. We obtain that $\tau(X \times Y) \subset \mathrm{Im}(f \otimes 1_Y)$.

Now let $z \in X \otimes Y$. Because $\tau(X \times Y)$ is dense in $X \otimes Y$, it follows that there exists a generalized sequence $(z_i)_{i \in I} \subset \tau(X \times Y)$ such that $z_i \longrightarrow z$. It results that there exists $(\zeta_i)_{i \in I}$ such that $z_i = (f \otimes 1_Y)(\zeta_i)$. Because $X' \otimes Y$ is a compact, $(\zeta_i)_{i \in I}$ has a generalized subsequence $(\zeta_{i_j})_{j \in J}$ such that $\zeta_{i_j} \longrightarrow \zeta$ in $X' \otimes Y$. As $f \otimes 1_Y$ is continuous, it follows that $z_{i_j} \longrightarrow (f \otimes 1_Y)(\zeta)$. We obtain that $z = (f \otimes 1_Y)(\zeta)$ as $X \otimes Y$ is separate.

c) *Left A–modules, H–comodules and H–modules.*

Another example of concrete tensor product can be given in the category of H–comodules, where H is a Hopf algebra.

PROPOSITION 5. *Let V, W be k–linear spaces and H a Hopf algebra with e_H the unity. V and W are right H–comodules with the maps $\rho_V : V \longrightarrow V \otimes_k H$, $v \longmapsto v \otimes e_H$ and respectively, $\rho_W : W \longrightarrow W \otimes_k H$, $w \longmapsto w \otimes e_H$. Then $V \otimes_k W$ as H–comodule is the concrete tensor product of V and W in the category of right H–comodules, $\mathcal{M}^H$.*

Proof. $V \otimes_k W$ becomes a right H–comodule via $\rho_{V \otimes_k W} : V \otimes_k W \longrightarrow V \otimes_k W \otimes_k H$, $v \otimes w \longmapsto v \otimes w \otimes e_H$. Let $\tau : V \times W \longrightarrow V \otimes_k W$ from the definition of tensor product in the category of k–linear spaces. Let us verify that τ is a 2-morphism in $\mathcal{M}^H$. Let $v \in V$ be fixed and $f : W \longrightarrow V \otimes_k W$, $f(w) = v \otimes w$. f is a morphism in $\mathcal{M}^H$. Indeed, f is a k–linear map and the following diagram is commutative:

(7)
$$\begin{array}{ccc} W & \xrightarrow{\quad \rho_W \quad} & W \otimes_k H \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes_k 1_H} \\ V \otimes_k W & \xrightarrow{\quad \rho_{V \otimes_k W} \quad} & V \otimes_k W \otimes_k H \end{array}$$

Similarly, $g : V \longrightarrow V \otimes_k W$, $v \longmapsto v \otimes w$ is a morphism in $\mathcal{M}^H$. Let $M \in (\mathcal{M}^H)^0$ and $\varphi : V \times W \longrightarrow M$ be a 2–morphism in $\mathcal{M}^H$. It follows that φ is a bilinear map over k. From the universality property for k–linear spaces, there exists exactly one k–linear map $\psi : V \otimes_k W \longrightarrow M$ such that $\psi\tau = \varphi$. We check that ψ is a morphism in $\mathcal{M}^H$. We have: $(\psi \otimes_k 1_H)(\rho_{V \otimes_k W}(v \otimes w)) = (\psi \otimes_k 1_H)(v \otimes w \otimes e_H) = \psi(v \otimes w) \otimes e_H = \varphi(v, w) \otimes e_H = \rho_M(\varphi(v, w)) = \rho_M(\psi(v \otimes w))$ because φ is a 2–morphism in $\mathcal{M}^H$. Thus, the following diagram commutes:

(8)
$$\begin{array}{ccc} V \otimes_k W & \xrightarrow{\quad \rho_{V \otimes_k W} \quad} & V \otimes_k W \otimes_k H \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \psi \otimes_k 1_H} \\ M & \xrightarrow{\quad \rho_M \quad} & M \otimes_k H \end{array}$$

The uniqueness is obvious.

REMARK 2. In this situation, the concrete tensor product coincides with the tensor product of right H–comodules met in [7].

REMARK 3. Proposition 5 is valid also for H only a bialgebra.

We shall now give examples of categories with concrete tensor products. D. Jagiełło gives in his paper sufficient conditions for a concrete category to have concrete tensor products (cf. Theorem 2 from [5]). We shall use this for other examples.

PROPOSITION 6. *Let A be a ring with unity $1 \neq 0$. Then ${}_A\mathcal{M}$, the category of left A–modules, has concrete tensor products.*

Proof. We verify the following conditions (cf. Theorem 2 from [5]).

(i) ${}_A\mathcal{M}$ is complete with respect to inputs because it has products and equalizers.

(ii) ${}_A\mathcal{M}$ is complete with respect to coequalizers. Every pair of morphisms $M \overset{f}{\underset{g}{\rightrightarrows}} N$ has a coequalizer $(N/\mathrm{Im}(f-g); p)$.

(iii) $U : {}_A\mathcal{M} \longrightarrow \mathrm{Ens}$, the forgetful functor, has a left adjoint $L : \mathrm{Ens} \longrightarrow {}_A\mathcal{M}$, the free functor:

$$\begin{array}{ccc} X & \longmapsto & L(X) \\ f\downarrow & & \downarrow L(f) \\ Y & \longmapsto & L(Y) \end{array}$$

where $L(X)$ is the free left A–module with basis X.

(iv) U preserves epimorphisms because epimorphisms in ${}_A\mathcal{M}$ are surjective morphisms.

(v) Every injection is an embedding (a morphism $\xi : M \longrightarrow N$ is called an embedding if for any function of sets $f : P \longrightarrow M$ such that ξf is a morphism, it follows that f is a morphism).

(vi) For each $M \in ({}_A\mathcal{M})^0$, the class of all the structures of left A–modules on M is a set. Indeed, it is known that for any structure on a group M corresponds a ring morphism $A \longrightarrow \mathrm{End}(M)$ and conversely. Also, the class of all the group structures on the set M is a set.

REMARK 4. By following the proof of Theorem 2 from [5], it results that for any $M_1, \ldots, M_n \in ({}_A\mathcal{M})^0$, $M_1 \otimes \cdots \otimes M_n = L(M_1 \times \cdots \times M_n)/N$, where N is the submodule of $L(M_1 \times \cdots \times M_n)$ generated by the elements:

$$(a_1, \ldots, a_i + \tilde{a}_i, \ldots, a_n) - (a_1, \ldots, a_i, \ldots, a_n) - (a_1, \ldots, \tilde{a}_i, \ldots, a_n),$$
$$(a_1, \ldots, \lambda a_i, \ldots, a_n) - \lambda(a_1, \ldots, a_i, \ldots, a_n).$$

REMARK 5. The associativity holds because the 2-tensor product is special.

REMARK 6. Because of Remark 4, every 2–tensor product is dense.

REMARK 7. By a similar proof like that of Proposition 6, it follows that the category of groups Gr has concrete tensor products.

Moreover, a canonical tensor product can be obtained. If G and H are arbitrary groups, one considers the free group generated by $G \times H$, $Gp\{G \times H\}$, and the normal subgroup N generated by the elements:

$$(g_1g_2, h)(g_2, h)^{-1}(g_1, h)^{-1} \text{ and } (g, h_1h_2)(g, h_2)^{-1}(g, h_1)^{-1}$$

for any $g, g_1, g_2 \in G$ and $h, h_1, h_2 \in H$. Then $Gp\{G \times H\}/N$ verifies the universality property for the groups G and H.

By taking $G = \{e\}$ a trivial group, we have $G \otimes H = G$, in concordance with Theorem 1.

PROPOSITION 7. *Let V, W be k-linear spaces and H a Hopf algebra. V and W are left H-modules with the structures: $h \cdot v = \varepsilon(h)v$, respectively $h \cdot w = \varepsilon(h)w$, for any $h \in H$, $v \in V$, $w \in W$. Then $V \otimes_k W$, as left H-module, is the concrete tensor product of V and W in ${}_H\mathcal{M}$.*

Proof. $V \otimes_k W$ becomes an H-module via $h \cdot (v \otimes w) = \Sigma h_1 v \otimes h_2 w$ (in the sigma notation), where $\Delta h = \Sigma h_1 \otimes h_2$ ($\Delta : H \longrightarrow H \otimes_k H$ from the definition of a k-coalgebra). In terms of maps, if $\phi_V : H \otimes_k V \longrightarrow V$ and $\phi_W : H \otimes_k W \longrightarrow W$ are the two given module actions, then

$$\phi_{V \otimes_k W} = (\phi_V \otimes \phi_W) \circ (1_H \otimes \mu \otimes 1_W) \circ (\Delta \otimes 1_{V \otimes_k W})$$

(where $\mu : H \otimes_k V \longrightarrow V \otimes_k H$ is the twist map) is the left H-module structure on $V \otimes_k W$, as in the following diagram

$$\begin{array}{ccc} H \otimes_k V \otimes_k W & \xrightarrow{\Delta \otimes 1_{V \otimes_k W}} & H \otimes_k H \otimes_k V \otimes_k W \\ \Big\downarrow{\scriptstyle \phi_{V \otimes_k W}} & & \Big\downarrow{\scriptstyle 1_H \otimes \mu \otimes 1_W} \\ V \otimes_k W & \xleftarrow{\phi_V \otimes \phi_W} & (H \otimes_k V) \otimes_k (H \otimes_k W) \end{array} \tag{9}$$

We have: $h \cdot (v \otimes w) = \Sigma \varepsilon(h_1) v \otimes h_2 w = \Sigma v \otimes \varepsilon(h_1) h_2 w = v \otimes (\Sigma \varepsilon(h_1) h_2) w = v \otimes hw$ and $h \cdot (v \otimes w) = h \cdot v \otimes w$, similarly.

Verify that $\tau : V \times W \longrightarrow V \otimes_k W$ is a 2-morphism in ${}_H\mathcal{M}$. First, τ is a bilinear map over k. Also, $\forall (h \in H,\ v \in V,\ w \in W)$, $\tau(hv, w) = hv \otimes w = h \cdot (v \otimes w) = h\tau(v, w) = \tau(v, hw)$.

Let $\varphi : V \times W \longrightarrow M$ be a 2-morphism in ${}_H\mathcal{M}$. There exists exactly one morphism of k-linear spaces $\psi : V \otimes_k W \longrightarrow M$ such that $\psi\tau = \varphi$. ψ is a morphism in ${}_H\mathcal{M}$: $\psi(h \cdot (v \otimes w)) = \psi(hv \otimes w) = \varphi(hv, w) = h\varphi(v, w) = h \cdot \psi(v \otimes w) \Longrightarrow \forall (h \in H,\ \omega \in V \otimes_k W)$, $\psi(h\omega) = h\psi(\omega)$. Uniqueness is obvious.

REMARK 8. The concrete tensor product in ${}_H\mathcal{M}$ coincides with the tensor product of H-modules from [7] only in the situation of Proposition 7. In general, $V \otimes_k W \not\simeq W \otimes_k V$ as H-modules, so that the commutativity is not satisfied. Thus, the tensor product in [7] is not concrete.

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