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ON CERTAIN DIFFERENTIAL SUBORDINATION FOR SECTORS

Abstract. Let $\mathbb{D} = \{z : |z| < 1\}$ denote the unit disk. For $\alpha \in (0,1]$, $\gamma \neq 0$, $\operatorname{Re}\gamma \geq 0$, we study some properties of the differential subordination of the form

$$p(z) + \gamma z p'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (z \in \mathbb{D}).$$

1. Introduction

For each $n \in \mathbb{N}$ let $\mathcal{A}(n)$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

For functions g and h analytic in $\mathbb{D}$ a function g is called subordinate to h, written $g \prec h$ (or $g(z) \prec h(z),\ z \in \mathbb{D}$), if h is univalent in $\mathbb{D}$, $g(0) = h(0)$ and $g(\mathbb{D}) \subset h(\mathbb{D})$.

For each $\alpha \in (0,1]$ let us define

$$H_\alpha(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (z \in \mathbb{D}).$$

For $\gamma \neq 0$ being such that $\operatorname{Re}\gamma \geq 0$ and $\alpha \in (0,1]$ we consider some properties of the differential subordination of the form

$$p(z) + \gamma z p'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (z \in \mathbb{D}),$$

where p is an analytic function in $\mathbb{D}$ with $p(0) = 1$, $p'(0) = \ldots = p^{(n-1)}(0) = 0$. There is calculated a constant $\delta(\alpha,\gamma,n)$ such that the above subordination implies $\operatorname{Re} p(z) > \delta(\alpha,\gamma,n),\ z \in \mathbb{D}$.

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2. Main results

The following lemma proved by Hallenbeck and Ruscheweyh [1] will be basic for our investigations.

LEMMA 2.1 ([1]). *Let H be a convex univalent function in $\mathbb{D}$ with $H(0) = 1$. Let h be a function analytic in $\mathbb{D}$ with $h(0) = 1$, $h'(0) = \ldots = h^{(n-1)}(0) = 0$. If $h \prec H$ in $\mathbb{D}$, then*

$$\lambda z^{-\lambda} \int_0^z t^{\lambda-1} h(t)\, dt \prec \lambda z^{-\lambda/n} \int_0^{z^{1/n}} t^{\lambda-1} H(t^n)\, dt,$$

$z \in \mathbb{D}$, for all $\lambda \neq 0$, $\operatorname{Re}\lambda \geq 0$.

Now we will prove the following theorem.

THEOREM 2.1. *Let $\gamma \neq 0$ be such that $\operatorname{Re}\gamma \geq 0$ and $\alpha \in (0, 1]$. Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$, $p'(0) = \ldots = p^{(n-1)}(0) = 0$. If*

$$p(z) + \gamma z p'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (z \in \mathbb{D}), \tag{2.1}$$

then

$$\operatorname{Re} p(z) > \delta(\alpha; \gamma, n) = \int_0^1 \left(\frac{1 - t^{n \operatorname{Re}\gamma}}{1 + t^{n \operatorname{Re}\gamma}}\right)^{\alpha} dt \quad (z \in \mathbb{D}). \tag{2.2}$$

Proof. Let us fix $\alpha \in (0, 1]$. We have

$$\begin{aligned} q_\alpha(z) &= \frac{1}{\gamma} z^{-1/(n\gamma)} \int_0^{z^{1/n}} u^{1/\gamma - 1} H_\alpha(u^n)\, du \\ &= \frac{1}{\gamma} z^{-1/(n\gamma)} \int_0^{z^{1/n}} u^{1/\gamma - 1} \left(\frac{1+u^n}{1-u^n}\right)^{\alpha} du \\ &= \frac{1}{\gamma} \int_0^1 v^{1/\gamma - 1} \left(\frac{1+v^n z}{1-v^n z}\right)^{\alpha} dv \\ &= \int_0^1 \left(\frac{1+t^{\gamma n} z}{1-t^{\gamma n} z}\right)^{\alpha} dt = \int_0^1 H_\alpha(t^{\gamma n} z)\, dt \quad (z \in \mathbb{D}). \end{aligned} \tag{2.3}$$

Using the general result from [1] we state that the function q_α is convex and univalent in $\mathbb{D}$. Setting

$$h(z) = p(z) + \gamma z p'(z) \quad (z \in \mathbb{D}) \tag{2.4}$$

we see from (2.1) that $h \prec H_\alpha$ in $\mathbb{D}$. It is clear also from (2.4) that

$$p(z) = \frac{1}{\gamma} z^{-1/\gamma} \int_0^z u^{1/\gamma - 1} h(u) du \quad (z \in \mathbb{D}).$$

So applying now Lemma 2.1 and (2.3) we obtain

$$\begin{aligned}
(2.5) \qquad p(z) &= \frac{1}{\gamma} z^{-1/\gamma} \int_0^z u^{1/\gamma - 1} h(u) du \\
&\prec \frac{1}{\gamma} z^{-1/(n\gamma)} \int_0^{z^{1/n}} u^{1/\gamma - 1} H_\alpha(u^n) du \\
&= \int_0^1 \left(\frac{1 + t^{\gamma n} z}{1 - t^{\gamma n} z} \right)^\alpha dt = q_\alpha(z) \quad (z \in \mathbb{D}).
\end{aligned}$$

Observe that for $t \in [0,1]$ holds

$$\begin{aligned}
(2.6) \qquad \min_{z \in \partial\mathbb{D} \setminus \{1\}} \operatorname{Re} \left\{ \left(\frac{1 + t^{\gamma n} z}{1 - t^{\gamma n} z} \right)^\alpha \right\} &= \min_{z \in \partial\mathbb{D} \setminus \{1\}} \operatorname{Re} H_\alpha(t^{\gamma n} z) \\
&= H_\alpha(-|t^{\gamma n}|) = H_\alpha(-t^{n \operatorname{Re} \gamma}) = \left(\frac{1 - t^{n \operatorname{Re} \gamma}}{1 + t^{n \operatorname{Re} \gamma}} \right)^\alpha.
\end{aligned}$$

By (2.5) and (2.6) we have

$$\inf_{z \in \mathbb{D}} \operatorname{Re} p(z) \geq \inf_{z \in \mathbb{D}} \operatorname{Re} q_\alpha(z) = \min_{z \in \partial\mathbb{D} \setminus \{1\}} \operatorname{Re} q_\alpha(z) = \delta(\alpha, \gamma, n),$$

where

$$(2.7) \qquad \delta(\alpha, \gamma, n) = \int_0^1 \left(\frac{1 - t^{n \operatorname{Re} \gamma}}{1 + t^{n \operatorname{Re} \gamma}} \right)^\alpha dt \quad (z \in \mathbb{D}),$$

so the thesis of the theorem follows.

Remark 2.1. The constant $\delta(\alpha, \gamma, n)$ given by the integral (2.7) is the best possible. In fact the integral (2.7) can be aproximated for every fixed systems of parameters α, γ and n with an arbitrary precision, for example by using the numerical methods. Naturally, in the case when we are able to find a primary function of this integral we can calculate a constant $\delta(\alpha, \gamma, n)$ exactly. In the next considerations we propose lower estimates of $\delta(\alpha, \gamma, n)$, where we eliminate the integral by applying Hermite-Hadamard's inequality. Presented estimates are not sharp. To compare an error let $\alpha = 1$ and $\gamma = 1/n$. Then an easy computation of the integral (2.7) yields us to the sharp result

$$\delta(1, 1/n, n) = 2 \log 2 - 1 \approx 0.38629436.$$

On the other hand from (2.10) with $\beta = 1$ we have only $\delta(1, 1/n, n) \geq 1/3$. The constant $\delta(1, 1/n, n)$ was calculated by Ponnusamy [3] (see also Owa and Nunokawa [2]).

Now we will estimate the constant $\delta(\alpha, \gamma, n)$ by using the Hermite-Hadamard's inequality

$$\varphi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \varphi(t)\,dt \leq \frac{\varphi(a)+\varphi(b)}{2} \tag{2.8}$$

for every convex function $\varphi : I \to \mathbb{R}$ on the interval I, where $a,\ b \in I$ with $a < b$. In the case when the function φ is concave in a certain segment we estimate it by the most simple method, i.e. by the area of the triangle or of the trapezium under the graph of φ.

To this end let us define for each $\alpha \in (0, 1]$ and $\beta > 0$ the function

$$\varphi(t) = \left(\frac{1-t^\beta}{1+t^\beta}\right)^\alpha \qquad (t \in [0,1]).$$

By an elementary calculations we have

$$\varphi''(t) = -2\alpha\beta t^{\beta-2}\frac{(1-t^\beta)^{\alpha-2}}{(1+t^\beta)^{\alpha+2}}((\beta+1)t^{2\beta} - 2\alpha\beta t^\beta + \beta - 1) \qquad (t \in (0,1)).$$

Hence $\varphi''(t) = 0$ iff

$$(\beta+1)u^2 - 2\alpha\beta u + \beta - 1 = 0 \qquad (u \in (0,1)), \tag{2.9}$$

where $u = t^\beta$. Since $\Delta = 4(\alpha^2\beta^2 - \beta^2 + 1)$ so we distinguish the following cases:

Case 1. $\alpha = 1$ and $\beta \geq 1$. Then the equation (2.9) has two solutions

$$t = t_1 = \left(\frac{\beta-1}{\beta+1}\right)^{1/\beta}, \qquad t = t_2 = 1$$

lying in the interval $[0,1]$. Moreover $\varphi''(t) < 0$ for $t \in (0, t_1)$ and $\varphi''(t) > 0$ for $t \in (t_1, 1)$. Consequently, by (2.8) we have

$$\begin{aligned} \int_0^1 \varphi(t)\,dt &= \int_0^{t_1} \varphi(t)\,dt + \int_{t_1}^1 \varphi(t)\,dt \\ &\geq \frac{1+\varphi(t_1)}{2}t_1 + \varphi\left(\frac{1+t_1}{2}\right)(1-t_1) \end{aligned} \tag{2.10}$$

$$= \frac{1}{2}\left(1+\frac{1}{\beta}\right)\left(\frac{\beta-1}{\beta+1}\right)^{1/\beta} + \left(1 - \left(\frac{\beta-1}{\beta+1}\right)^{1/\beta}\right)\frac{2^\beta - \left(\left(\frac{\beta-1}{\beta+1}\right)^{1/\beta}+1\right)^\beta}{2^\beta + \left(\left(\frac{\beta-1}{\beta+1}\right)^{1/\beta}+1\right)^\beta}.$$

CASE 2. $\alpha = 1$ and $\beta \in (0,1)$. Then $\varphi''(t) > 0$ for $t \in (0,1)$ so the function φ is convex in $(0,1)$. Hence and by (2.8) we state that

$$\int_0^1 \varphi(t)\,dt \geq \varphi\left(\frac{1}{2}\right) = \frac{2^\beta - 1}{2^\beta + 1}.$$

Let now $\alpha \in (0,1)$.

CASE 3. $\beta > 1/\sqrt{1-\alpha^2}$. Then the function φ is concave so

$$\int_0^1 \varphi(t)\,dt \geq \frac{\varphi(0) + \varphi(1)}{2} = \frac{1}{2}.$$

CASE 4. $\beta = 1/\sqrt{1-\alpha^2}$. Then the equation (2.9) has a double solution

$$t = t_0 = \left(\frac{\alpha\beta}{\beta+1}\right)^{1/\beta} = \left(\frac{\alpha}{1+\sqrt{1-\alpha^2}}\right)^{\sqrt{1-\alpha^2}} \in (0,1)$$

and therefore we have

$$\begin{aligned}\int_0^1 \varphi(t)\,dt &= \int_0^{t_0} \varphi(t)\,dt + \int_{t_0}^1 \varphi(t)\,dt \\ &\geq \frac{1+\varphi(t_0)}{2} t_0 + \frac{\varphi(t_0)}{2}(1-t_0) = \frac{t_0 + \varphi(t_0)}{2} \\ &= \frac{1}{2}\left(\left(\frac{\alpha}{1+\sqrt{1-\alpha^2}}\right)^{\sqrt{1-\alpha^2}} + \left(\frac{1-\alpha+\sqrt{1-\alpha^2}}{1+\alpha+\sqrt{1-\alpha^2}}\right)^{\alpha}\right).\end{aligned}$$

CASE 5. $\beta < 1/\sqrt{1-\alpha^2}$ and $\beta > 1$. Then there are two solutions of (2.9) of the form

$$t = t_1 = \left(\frac{\alpha\beta - \sqrt{1-\beta^2(1-\alpha^2)}}{\beta+1}\right)^{1/\beta}$$

and

$$t = t_2 = \left(\frac{\alpha\beta + \sqrt{1-\beta^2(1-\alpha^2)}}{\beta+1}\right)^{1/\beta} \tag{2.11}$$

both of them lying in the interval $(0,1)$. Hence $\varphi''(t) < 0$ for $t \in (0,t_1) \cup (t_1, 1)$ and $\varphi''(t) > 0$ for $t \in (t_1, t_2)$. Therefore using (2.8) we have

$$\begin{aligned}\int_0^1 \varphi(t)\,dt &= \int_0^{t_1} \varphi(t)\,dt + \int_{t_1}^{t_2} \varphi(t)\,dt + \int_{t_2}^1 \varphi(t)\,dt \\ &\geq \frac{1+\varphi(t_1)}{2} t_1 + \varphi\left(\frac{t_1+t_2}{2}\right)(t_2 - t_1) + \frac{(1-t_2)\varphi(t_2)}{2}.\end{aligned}$$

CASE 6. $\beta = 1$. Then the function φ is convex in the interval $(0, \alpha)$ and concave in the interval $(\alpha, 1)$. Using (2.8) we have

$$\int_0^1 \varphi(t)\,dt = \int_0^\alpha \varphi(t)\,dt + \int_\alpha^1 \varphi(t)\,dt$$
$$\geq \alpha\left(\frac{2-\alpha}{2+\alpha}\right)^\alpha + \frac{1}{2}\frac{(1-\alpha)^{\alpha+1}}{(1+\alpha)^\alpha}.$$

CASE 7. $\beta < 1$. Then the equation (2.9) has a unique solution $t = t_2$ given by (2.11) in the interval $(0,1)$. Therefore $\varphi''(t) < 0$ for $t \in (0, t_2)$ and $\varphi''(t) > 0$ in $(t_2, 1)$. Consequently applying again (2.8) we have

$$\int_0^1 \varphi(t)\,dt \geq \frac{1+\varphi(t_2)}{2} t_2 + (1-t_2)\varphi\left(\frac{t}{2}\right).$$

For $\alpha = 1$ and $\operatorname{Re}\gamma = 1/n$ we obtain the result mentioned in Remark 2.1 due to [3] and [2].

COROLLARY 2.1. *Let γ be such that* $\operatorname{Re}\gamma = 1/n$. *If p is an analytic function in $\mathbb{D}$ with* $p(0) = 1$, $p'(0) = \ldots = p^{(n-1)}(0) = 0$ *and*

$$\operatorname{Re}\{p(z) + \gamma z p'(z)\} > 0 \quad (z \in \mathbb{D}),$$

then

$$\operatorname{Re} p(z) > 2\log 2 - 1 \quad (z \in \mathbb{D}).$$

Remark 2.2. By setting in Theorem 2.1 the expressions $p(z) = f(z)/z$, $p(z) = f'(z)$, $p(z) = zf'(z)/f(z)$, $(z \in \mathbb{D})$, or the another ones, where $f \in \mathcal{A}(n)$, we obtain the results concernig the inclusion relations between various classes of analytic or univalent functions.

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