DEMONSTRATIO MATHEMATICA
Vol. XXXIII No 4 2000

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ANALYTIC SOLUTIONS OF A NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATION WITH PROPORTIONAL DELAYS

Abstract. This paper is concerned with a nonlinear functional differential equation of the form

$$G(z)\frac{dF(z)}{dz} = \sum_{j=1}^{m} p_j G[F(q_j z)].$$

By means of the method of majorant series conditions are given for existence of analytic solutions of the above equation.

It is well-known that the third Jabotinsky equation

$$G(z)\frac{dF(z)}{dz} = G[F(z)] \tag{1}$$

plays an important role in iteration theory. This equation has been studied by many authors [1-3]. In this paper, we will be concerned with a more general class of equation of the form

$$G(z)\frac{dF(z)}{dz} = p_1 G(F(q_1 z)) + p_2 G(F(q_2 z)) + \ldots + p_m G(F(q_m z)), \tag{2}$$

where p_j and q_j are complex numbers. It is clear that (2) includes equation (1) as a special case. In the linear case, such equations with proportional delays also have been studied to some extent by many authors [4–6]. By means of the method of majorant series, we will discuss the existence of analytic solutions of equation (2) in a neighborhood of the origin. Some ideas in the proof of our main results are similar to that used in [3, 7, 8].

We now state and prove our main results in this note.

1991 *Mathematics Subject Classification*: 34K25.

Key words and phrases: Functional differential equation, analytic solution.

THEOREM 1. *Suppose that $G(z)$ is analytic on a neighborhood of zero, $G(0)=0, G'(0)\neq 0, \sum_{j=1}^{m}|p_j|\leq 1$ and $|q_j|\leq 1$, then for any complex number η, equation (2) has an analytic solution $F(z)$ in a neighborhood of the origin such that $F(0)=0, F'(0)=\eta+\sum_{j=1}^{m}p_j$.*

Proof. Let

$$G(z)=\sum_{n=0}^{\infty}a_n z^n, \tag{3}$$

Since $G(z)$ is analytic on a neighborhood of the origin, there exists a positive β such that

$$|a_n|\leq \beta^{n-1}, n=2,3,\ldots \tag{4}$$

Introducing new functions $f(z) == \beta F(\beta^{-1}z)$ and $g(z)=\beta G(\beta^{-1}z)$, we obtain from (2)

$$g(z)\frac{df(z)}{dz}=\sum_{j=1}^{m}p_j g[f(q_j z)],$$

which is an equation of the form (2). From (3) and $g(z)=\beta G(\beta^{-1}z)$, we have

$$g(z)=sz+\sum_{n=2}^{\infty}a_n\beta^{1-n}z^n.$$

By (4) it follows that

$$\left|\frac{a_n}{\beta^{n-1}}\right|\leq 1, n=2,3,\ldots$$

Therefore, without loss of generality, we assume that

$$|a_n|\leq 1, n=2,3,\ldots \tag{5}$$

Let

$$F(z)=\sum_{n=1}^{\infty}b_n z^n \tag{6}$$

be the expansion of a formal solution $F(z)$ of equation (2). Inserting (3) and (6) into (2) and equating the coefficients we obtain

$$a_0\Big(b_1-\sum_{j=1}^{m}p_j\Big)=0,$$

and

$$(7) \qquad \Big(n - \sum_{j=1}^{m} p_j q_j^n\Big) a_1 b_n$$

$$= \sum_{j=1}^{m} p_j q_j^n \sum_{\substack{l_1+\ldots+l_t=n;\\ t=2,3,\ldots,n}} a_t b_{l_1} b_{l_2} \ldots b_{l_t} - \sum_{k=0}^{n-2} (k+1) a_{n-k} b_{k+1}, , n = 2, 3, \ldots$$

Noting that $a_0 = G(0) = 0$, we choose $b_1 - \sum_{j=1}^{m} p_j = \eta$. Since $a_1 = G'(0) \neq 0$ and $|\sum_{j=1}^{m} p_j q_j^n| \leq \sum_{j=1}^{m} |p_j||q_j|^n \leq \sum_{j=1}^{m} |p_j| \leq 1$, we see that the sequence $\{b_n\}_{n=2}^{\infty}$ is successively determined by the relation (7) in a unique manner. Now, we show that the power series (6) converges in a neighborhood of the origin. First of all, since

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{m} p_j q_j^n}{n - \sum_{j=1}^{m} p_j q_j^n} = 0,$$

there exists a positive number M, such that for $n \geq 2$,

$$\left| \frac{\sum_{j=1}^{m} p_j q_j^n}{n - \sum_{j=1}^{m} p_j q_j^n} \right| \leq M.$$

Thus if we define a sequence $\{B_n\}_{n=1}^{\infty}$ by $B_1 = |\eta| + \sum_{j=1}^{m} |p_j|$ and

$$B_n = \frac{M}{|a_1|} \sum_{\substack{l_1+\ldots+l_t=n;\\ t=2,3,\ldots,n}} B_{l_1} B_{l_2} \ldots B_{l_t} + \frac{1}{|a_1|} \sum_{k=0}^{n-2} B_{k+1}, n = 2, 3, \ldots,$$

then in view of (7) and the inequality (5),

$$(8) \qquad |b_n| \leq B_n, n = 1, 2, \ldots$$

Now if we define

$$W(z) = \sum_{n=1}^{\infty} B_n z^n,$$

then

$$W(z) = \sum_{n=1}^{\infty} B_n z^n = \Big(|\eta| + \sum_{j=1}^{m} |p_j|\Big) z + \sum_{n=2}^{\infty} B_n z^n$$

$$= \Big(|\eta| + \sum_{j=1}^{m} |p_j|\Big) z$$

$$+ \sum_{n=2}^{\infty} \left(\frac{M}{|a_1|} \sum_{\substack{l_1+\ldots+l_t=n;\\ t=2,3,\ldots,n}} B_{l_1} B_{l_2} \ldots B_{l_t} + \frac{1}{|a_1|} \sum_{k=0}^{n-2} B_{k+1} \right) z^n$$

$$= \Big(|\eta| + \sum_{j=1}^{m} |p_j|\Big) z$$

$$+\frac{M}{|a_1|}\sum_{n=2}^{\infty}\Big(\sum_{\substack{l_1+\ldots+l_t=n;\\ t=2,3,\ldots,n}} B_{l_1}B_{l_2}\ldots B_{l_t}\Big)z+\frac{1}{|a_1|}\sum_{n=2}^{\infty}\Big(\sum_{k=0}^{n-2}B_{k+1}\Big)z^n$$

$$= \Big(|\eta| + \sum_{j=1}^{m} |p_j|\Big) z + \frac{M}{|a_1|}\frac{[W(z)]^2}{1-W(z)} + \frac{1}{|a_1|}\frac{z}{1-z}W(z).$$

That is, $W(z)$ satisfies the implicit equation

$$W(z) - \Big(|\eta| + \sum_{j=1}^{m} |p_j|\Big) z - \frac{M}{|a_1|}\frac{[W(z)]^2}{1-W(z)} - \frac{1}{|a_1|}\frac{z}{1-z}W(z) = 0.$$

Note that the left hand side of the above equation, when regarded as a function $R(z, W) = 0$, satisfies $R(0,0) = 0$ and

$$R'_W(0,0) = 1 \neq 0.$$

According to the implicit function theorem, we see that the power series $W(z)$ converges in a neighborhood of the origin. This implies that (6) is also convergent in a neighborhood of the origin. The proof is complete.

THEOREM 2. *Suppose that $G(z)$ is analytic on a neighbourhood of zero, $G(0) \neq 0, \sum_{j=1}^{m} |p_j| \leq 1$ and $|q_j| \leq 1$, then equation (2) has an analytic solution $F(z)$ in a neighborhood of the origin such that $F(0) = 0, F'(0) = \sum_{j=1}^{m} p_j$.*

Proof. As in the proof of Theorem 1, we seek a power series solution of the form (6), and (5) again holds. Since $a_0 = G(0) \neq 0$, by defining $b_1 = \sum_{j=1}^{m} p_j$ and then substituting (3) and (6) into (2), we see that the sequence $\{b_n\}_{n=2}^{\infty}$ is successively determined by the condition

$$(9) \qquad (n+1)a_0 b_{n+1} + \Big(n - \sum_{j=1}^{m} p_j q_j^n\Big) a_1 b_n$$

$$= \sum_{j=1}^{m} p_j q_j^n \sum_{\substack{l_1+\ldots+l_t=n;\\ t=2,3,\ldots,n}} a_t b_{l_1} b_{l_2}\ldots b_{l_t} - \sum_{k=0}^{n-2}(k+1)a_{n-k}b_{k+1},, n = 2, 3, \ldots$$

in a unique manner. Furthermore, it is easy to see by (5) and (9) that

$$|b_{n+1}| \leq \left|\frac{n - \sum_{j=1}^m p_j q_j^n}{n+1}\right|\left|\frac{a_1}{a_0}\right||b_n|$$

$$+\frac{\left|\sum_{j=1}^m p_j q_j^n\right|}{(n+1)|a_0|}\sum_{\substack{l_1+\ldots+l_t=n;\\ t=2,3,\ldots,n}} |a_t||b_{l_1}||b_{l_2}|\ldots|b_{l_t}| + \frac{1}{|a_0|}\sum_{k=0}^{n-2}\frac{k+1}{n+1}|a_{n-k}||b_{k+1}|$$

$$\leq \left|\frac{a_1}{a_0}\right||b_n| + \frac{1}{|a_0|}\sum_{\substack{l_1+\ldots+l_t=n;\\ t=2,3,\ldots,n}} |b_{l_1}||b_{l_2}|\ldots|b_{l_t}| + \frac{1}{|a_0|}\sum_{k=0}^{n-2}|a_{n-k}||b_{k+1}|.$$

Thus if we define a sequence $\{B_n\}_{n=1}^{\infty}$ by $B_1 = \sum_{j=1}^m |p_j|$ and

$$B_{n+1} = \left|\frac{a_1}{a_0}\right|B_n + \frac{1}{|a_0|}\sum_{\substack{l_1+\ldots+l_t=n;\\ t=2,3,\ldots,n}} B_{l_1}B_{l_2}\ldots B_{l_t} + \frac{1}{|a_0|}\sum_{k=0}^{n-2}B_{k+1}, n = 2,3,\ldots,$$

then $|b_n| \leq B_n$ for $n = 1, 2, \ldots$. Now if we define

$$\Phi(z) = \sum_{n=0}^{\infty} B_{n+1}z^{n+1},$$

then

$$\Phi(z) = \Big(\sum_{j=1}^m |p_j|\Big)z + B_2z^2 + \sum_{n=2}^{\infty} B_{n+1}z^{n+1} = \Big(\sum_{j=1}^m |p_j|\Big)z$$

$$+B_2z^2 + \sum_{n=2}^{\infty}\left(\left|\frac{a_1}{a_0}\right|B_n + \frac{1}{|a_0|}\sum_{\substack{l_1+\ldots+l_t=n;\\ t=2,3,\ldots,n}} B_{l_1}B_{l_2}\ldots B_{l_t} + \frac{1}{|a_0|}\sum_{k=0}^{n-2}B_{k+1}\right)z^{n+1}$$

$$= \Big(\sum_{j=1}^m |p_j|\Big)z + B_2z^2 + \left|\frac{a_1}{a_0}\right|\sum_{n=2}^{\infty}B_nz^{n+1}$$

$$+\frac{1}{|a_0|}\sum_{n=2}^{\infty}\Big(\sum_{\substack{l_1+\ldots+l_t=n;\\ t=2,3,\ldots,n}} B_{l_1}B_{l_2}\ldots B_{l_t}\Big)z^{n+1} + \frac{1}{|a_0|}\sum_{n=2}^{\infty}\Big(\sum_{k=0}^{n-2}B_{k+1}\Big)z^{n+1}$$

$$= \Big(\sum_{j=1}^m |p_j|\Big)z + B_2z^2 + \left|\frac{a_1}{a_0}\right|z\Big(\Phi(z) - (\sum_{j=1}^m |p_j|)z\Big)$$

$$+\frac{1}{|a_0|}\frac{z[\Phi(z)]^2}{1-\Phi(z)} + \frac{1}{|a_0|}\frac{z^2}{1-z}\Phi(z).$$

That is, $\Phi(z)$ satisfies the implicit equation

$$\Phi - \Big(\sum_{j=1}^{m} |p_j|\Big)z - \Big(B_2 - \Big|\frac{a_1}{a_0}\Big| \sum_{j=1}^{m} |p_j|\Big)z^2$$
$$-\Big|\frac{a_1}{a_0}\Big| z\Phi - \frac{1}{|a_0|}\frac{z[\Phi]^2}{1-\Phi} - \frac{1}{|a_0|}\frac{z^2}{1-z}\Phi = 0.$$

Note that the left hand side of the above equation, when regarded as a function $R(z, \Phi) = 0$, satisfies $R(0,0) = 0$ and

$$\frac{\partial}{\partial \Phi} R(0,0) = 1 \neq 0.$$

According to the implicit function theorem, we see that the power series Φ converges in a neighborhood of the origin. This implies that (6) is also convergent in a neighborhood of the origin. The proof is complete.

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Received July 12, 1999; revised version June 14, 2000.