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THE CAUCHY PROBLEM FOR CERTAIN GENERALIZED DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH SINGULARITY

1. Introduction

Let X, Y be Banach spaces over the field $\mathbb{R}$ and let U and V be open subsets of X and Y, respectively.

Let h_1, h_2 be the mappings from U into X and F a mapping from $U \times V$ into Y.

We shall start with defining a directional derivative of second order of a function f in a direction of the pair of mappings h_1, h_2 on U, denoted by $(\nabla^2_{(h_2,h_1)} f)(x)$ for $x \in U$, and generalizing the well known notion of the directional derivative of second order [3]. Then we consider the Cauchy problem

$$(\nabla^2_{(h_2,h_1)} f)(x) + a(\nabla_{h_1} f)(x) = F(x, f(x))$$
$$f(0) = 0, \qquad (\nabla_{h_1} f)(0) = 0,$$

where $a \geq 0$, for mappings from a subset of a Banach space into a Banach space, which are defined in class of continuous mappings C, with the assumption that 0 is a singular point (i.e. $h_1(0) = 0$ and $h_2(0) = 0$).

The subject matter refers to studies of generalized differential equations of first order introduced in [4] and continued in [5].

In lemmas and theorems presented in this paper the real Banach space X will be considered with a semi-inner product, defined as follows [2], [4], [5].

Let X^* be the dual space of Y and

$$T(x) = \{x^* \in X^*;\ \|x^*\| = 1,\ x^*(x) = \|x\|\}$$

for $x \in X$.

Let X_0 be a set of nonzero elements with norm equal to 1, chosen one by one from each line in X through zero.

Let $\Im_0$ be any (fixed in further considerations) mapping from X_0 into X^* such that $\Im_0(y) \in T(y)$ for $y \in X_0$. Let $\Im$ be the homogenous extension of $\Im_0$ to the whole space X defined as follows

$$\Im(\lambda y) = \lambda \Im_0(y) \quad \text{for} \quad y \in X_0, \lambda \in \mathbb{R}.$$

Now we define a semi-inner product by

$$\langle x, y \rangle = \Im(y)(x) \quad \text{for} \quad x, y \in X.$$

This product has the following properties:

(a) it maps $X \times X$ into $\mathbb{R}$,

(b) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$, $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$, $\langle x, \lambda y \rangle = \lambda \langle x, y \rangle$ for $x, y, z \in X$, $\lambda \in \mathbb{R}$

(c) $\langle x, x \rangle = \|x\|^2$ for $x \in X$

(d) $|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$ for $x, y \in X$.

Denote by B_ρ the open ball in X with radius ρ and centre zero, i. e., $B_\rho = \{x \in X : \|x\| < \rho\}$.

The following definitions, lemmas and theorem will be needed throughout the paper.

DEFINITION 1. *A mapping $h : B_\rho \to X$ of the class C^1 will be called a regular mapping (in zero) if:*

(i) *is bounded with its first derivative Dh in B_ρ,*

(ii) $h(0) = 0$,

(iii) *there exists such a constant $C > 0$ that*

$$y^*(Dh(0)y) \geq C \tag{1}$$

for $y^* \in T(y)$ and for every $y \in X$ such that $\|y\| = 1$.

LEMMA 1. *If $h : B_\rho \to X$ is a regular mapping (in zero), then for every $\alpha \in (0, C)$ there exists a constant $r \in (0, \rho)$ such that a Cauchy problem*

$$\frac{\partial}{\partial t} v(t, x) = -h(v(t, x)), \qquad v(0, x) = x \tag{2}$$

has in the domain $[0, \infty) \times B_r$ exactly one continuously differentiable solution $v = v(t, x)$, having the properties

$$\|v(t, x)\| \leq e^{-\alpha t} \|x\| \quad \textit{for} \quad t \in [0, \infty), x \in B_r, \tag{3}$$

and

$$v(t, v(\tau, x)) = v(t + \tau, x) \quad \textit{for} \quad t, \tau \in [0, \infty), x \in B_r. \tag{4}$$

LEMMA 2. *If $h : B_\rho \to X$ is a regular mapping (in zero) of class C^2, then for every $\beta \in (0, C)$ there exists such $\tilde{r} \in (0, r]$ (where r denotes the constant from Lemma 1), that*

$$\|D_2 v(t, x)\| \leq e^{-\beta t} \quad \textit{for} \quad t \in [0, \infty), x \in B_{\tilde{r}}. \tag{5}$$

The proofs of Lemma 1 and Lemma 2 are presented in [5].

Let $C_*^{0,1}(B_{\tau_0}, Y)$ be the space of the continuous mappings $f : B_{\tau_0} \to Y$, $B_{\tau_0} \subset X$, such that for every mapping f there exists such a constant $C_* > 0$ that $\|f(x)\| \le C_*\|x\|$ for $x \in B_{\tau_0}$.
Let

$$\|f\|_* = \inf\{C_* > 0; \|f(x)\| \le C_*\|x\| \quad \text{for} \quad x \in B_{\tau_0}\}.$$

This functional is a norm.

THEOREM 1. *The space $C_*^{0,1}(B_{\tau_0}, Y)$ with the norm $\|\cdot\|_*$ is the Banach space (compare [5]).*

Let X, Y be a real Banach spaces, U an open subset of X and $h : U \longrightarrow X$ be a function of class C^1, bounded together with its first derivative on U.

Lemma 1 specifies the additional properties of the solution of the problem (2) for x in a neighbourhood of zero in the case of regular mapping h.

DEFINITION 2. *We say that a mapping $f : U \to Y$ has in a point $x \in U$ a derivative in a direction of the mapping h if there exists a limit*

$$(\nabla_h f)(x) := \lim_{t \to 0} \frac{f(v(0,x)) - f(v(t,x))}{t} \quad for \quad t \in \mathbb{R},$$

where $v = v(t,x)$ is the solution of the problem (2) in a neighbourhood of a point $(0,x)$.

We can use in Definition 2 the natural transformation generated by the regular mapping h (for $t \ge 0$), since

$$(\nabla_h) f(x) = -\left[\frac{\partial}{\partial t} f(v(t,x))\right]_{t=0}$$

in a neighbourhood of a point $x = 0$.

2. The Cauchy problem for the generalized differential equations of second order with singularity in a point zero

Let X, Y be real Banach spaces, U an open subset of X and $h_1, h_2 : U \to X$ be the functions of class C^1, bounded together with their first derivatives on U.

DEFINITION 3. *Let $f : U \to Y$. A directional derivative of a function*

$$X \supset U \ni x \to (\nabla_{h_1} f)(x) \in Y$$

in a direction of a mapping h_2 (if it exists) we shall denote by the symbol $(\nabla^2_{(h_2,h_1)} f)(x)$ and call the directional derivative of the second order of f in the direction of the pair of mappings (h_2, h_1),

$$(\nabla^2_{(h_2,h_1)} f)(x) = (\nabla_{h_2}(\nabla_{h_1} f))(x) \quad for \quad x \in U.$$

From Theorem 10.4.5 and Theorem 10.8.2 in [1] it follows that for any $x_0 \in U$ there exist the constants $s_0, t_0 > 0$ and a neighbourhood $U_0 \subset U$ of a point x_0 such that each of the following Cauchy problems

$$(2') \quad \begin{array}{l} \frac{\partial}{\partial s} v_1(s,x) = -h_1(v_1(s,x)) \\ v_1(0,x) = x \end{array} \quad \text{and} \quad \begin{array}{l} \frac{\partial}{\partial t} v_2(t,x) = -h_2(v_2(t,x)) \\ v_2(0,x) = x \end{array}$$

has exactly one continuously differentiable solution, respectively: $v_1 = v_1(s,x)$ in the domain $(-s_0, s_0) \times U_0$ and $v_2 = v_2(t,x)$ in the domain $(-t_0, t_0) \times U_0$.

From the Definition 3 it follows that

$$(\nabla^2_{(h_2,h_1)} f)(x) = [\frac{\partial}{\partial t}[\frac{\partial}{\partial s} f(v_1(s, v_2(t,x)))]_{s=0}]_{t=0},$$

where $v_1 = v_1(s,x)$, $v_2 = v_2(t,x)$ are the solutions of the Cauchy problems (2′) for $s \in (-s_0, s_0), t \in (-t_0, t_0)$ and $x \in U_0$. The above equality is true for the natural transformations generated by the regular mappings h_1, h_2 in a neighbourhood of a point $x = 0$ (for $s, t \geq 0$).

COROLLARY 1. *If a mapping $f : U \to Y$ is twice differentiable at a point $x \in U$, then the following equality is true*

$$(\nabla^2_{(h_2,h_1)} f)(x) = D^2 f(x)(h_2(x), h_1(x)) + Df(x)(Dh_1(x)(h_2(x))).$$

Proof. Since f is twice differentiable, by Definition 3, we obtain

$$\begin{aligned} (\nabla^2_{(h_2,h_1)} f)(x) &= \left[\frac{\partial}{\partial t}\left[\frac{\partial}{\partial s} f(v_1(s, v_2(t,x)))\right]_{s=0}\right]_{t=0} \\ &= \left[\frac{\partial}{\partial t}[Df(v_1(s, v_2(t,x)))(-h_1(v_1(s, v_2(t,x))))]_{s=0}\right]_{t=0} \\ &= -\left[\frac{\partial}{\partial t}[Df(v_2(t,x))(h_1(v_2(t,x)))]_{t=0}\right] \\ &= [D^2 f(v_2(t,x))(h_2(v_2(t,x)), h_1(v_2(t,x))) \\ &\quad - Df(v_2(t,x))(Dh_1(v_2(t,x))(-h_2(v_2(t,x))))]_{t=0} \\ &= D^2 f(x)(h_2(x), h_1(x)) + Df(x)(Dh_1(x)(h_2(x))). \ \blacksquare \end{aligned}$$

DEFINITION 4. *Let U and V be open subsets of Banach spaces X and Y, respectively. Let $h_1, h_2 : U \to X$ be mappings of class C^1 and F be any function from $U \times V$ into Y.*

Every function $f : U \to V$ which has a derivative of a second order in a direction of a pair of mappings (h_1, h_2) in U and fulfills the equation

$$(*) \qquad (\nabla^2_{(h_2,h_1)} f)(x) + a(\nabla_{h_1} f)(x) = F(x, f(x)) \quad \text{for} \quad x \in U,$$

where $a \geq 0$, will be called a solution of $(*)$.

We shall introduce the following assumption

ASSUMPTION 1. *Let $h_i : B_{\rho_1} \to X$ $(i = 1,2)$, with $B_{\rho_1} \subset X$, be regular mappings and let $C_i > 0$ $(i = 1,2)$ be the constants such that the following conditions are fulfilled*

$$(1') \qquad y^*(Dh_i(0)(y)) \geq C_i \quad (i = 1,2)$$

for $y^* \in T(y)$ and for every $y \in X$ such that $\|y\| = 1$. We shall assume that α_i are some constants from $(0, C_i)$, $v_i : [0,\infty) \times B_{r_1} \to X$, where $r_1 \in (0, \rho_1)$, are the natural transformations generated by the mappings h_i $(i = 1,2)$ and the following inequalities take places

$$(3') \qquad \|v_i(s,x)\| \leq e^{-\alpha_i s}\|x\| \quad \text{for} \quad s \in [0,\infty), x \in B_{r_1} \ (i = 1,2).$$

For the mapping $F : B_{\rho_1} \times B_{\rho_2} \to Y$, with $B_{\rho_1} \times B_{\rho_2} \subset X \times Y$, and for $a \geq 0$ we consider the Cauchy problem

$$(6) \qquad \begin{gathered} (\nabla^2_{(h_2,h_1)} f)(x) + a(\nabla_{h_1} f)(x) = F(x, f(x)) \quad \text{for} \quad x \in B_{\rho_1} \\ f(0) = 0, \qquad (\nabla_{h_1} f)(0) = 0, \end{gathered}$$

under Assumption 1.

LEMMA 3. *Let $F : B_{\rho_1} \times B_{\rho_2} \to Y$, with $B_{\rho_1} \times B_{\rho_2} \subset X \times Y$, be a continuous mapping such that for certain constans $\tilde{\rho}_1 \in (0, \rho_1], \tilde{\rho}_2 \in (0, \rho_2]$ and $K > 0$ we have*

(i) $\|F(x,y)\| \leq K\|x\|$ *for* $(x,y) \in B_{\tilde{\rho}_1} \times B_{\tilde{\rho}_2}$.

Then for $\tau_1 = \min(r_1, \tilde{\rho}_1)$ (where r_1 is a constant from Assumption 1) and for any continuous function $f : B_{\tau_1} \to B_{\tilde{\rho}_2}$ two following conditions are equivalent:

I. *f is the solution of Cauchy problem (6) (in particular f is continuously differentiable in a direction of the mapping h_1 and twice differentiable in a direction of the pair of mappings (h_2, h_1)).*

II. *f fulfills the equation*

$$(7) \qquad f(x) = \int_0^\infty \int_0^\infty e^{-at} F(v_2(t, v_1(s,x)), f(v_2(t, v_1(s,x))))\, dt\, ds$$

for $x \in B_{\tau_1}$, where the above integral is absolutely convergent.

Proof. Let $\tau_1 = \min(r_1, \tilde{\rho}_1)$. By (i) and (3′), any function $f \in \mathcal{B}_{\tilde{\rho}_2} \subset C(B_{\tau_1}, Y)$ fulfills the following inequalities

$$(8) \qquad \|e^{-at} F(v_2(t,x), f(v_2(t,x)))\| \leq e^{-(\alpha_2 + a)t} K\|x\|$$

and

$$(9) \qquad \|e^{-at} F(v_2(t, v_1(s,x)), f(v_2(t, v_1(s,x))))\| \leq e^{-\alpha_1 s} e^{-(\alpha_2 + a)t} K\|x\|$$

for $t, s \in [0,\infty)$, $x \in B_{\tau_1}$.

Let $f \in C(B_{\tau_1}, Y)$ be the solution of the problem (6). From Definition 3 it follows that

$$(\nabla_{h_2}(\nabla_{h_1} f))(x) + a(\nabla_{h_1} f)(x) = F(x, f(x)) \quad \text{for} \quad x \in B_{\tau_1}.$$

Hence

$$-\frac{\partial}{\partial t}[e^{-at}(\nabla_{h_1} f)(v_2(t,x))] = e^{-at} F(v_2(t,x), f(v_2(t,x))) \qquad \text{for} \quad t \in [0,\infty), x \in B_{\tau_1}.$$

Integrating above equation with respect to t on the interval $[0,t]$, we obtain

$$\begin{aligned} (10) \quad & -e^{-at}(\nabla_{h_1} f)(v_2(t,x)) + (\nabla_{h_1} f)(v_2(0,x)) \\ & = \int_0^t e^{-a\tau} F(v_2(\tau,x), f(v_2(\tau,x)))\, d\tau \quad \text{for} \quad t \in [0,\infty), x \in B_{\tau_1}. \end{aligned}$$

Since $\lim_{t\to\infty} v_2(t,x) = 0$ for every $x \in B_{\tau_1}$ by Lemma 1, from the continuouity of the mapping $\nabla_{h_1} f$, due to the condition $(\nabla_{h_1} f)(0) = 0$, we conclude that

$$\lim_{t\to\infty} e^{-at}(\nabla_{h_1} f)(v_2(t,x)) = 0 \quad (a \geq 0).$$

Consequently, by (10), we obtain

$$(11) \qquad (\nabla_{h_1} f)(x) = \int_0^\infty e^{-a\tau} F(v_2(\tau,x), f(v_2(\tau,x)))d\tau \quad \text{for} \quad x \in B_{\tau_1},$$

where the above integral is absolutely convergent by (8). The following equation is true

$$-\frac{\partial}{\partial s} f(v_1(s,x)) = \int_0^\infty e^{-a\tau} F(v_2(\tau, v_1(s,x)), f(v_2(\tau, v_1(s,x))))d\tau$$

for $s \in [0,\infty), x \in B_{\tau_1}$. Integrating the above equation with respect to s on the interval $[0,s]$, we have

$$\begin{aligned} (12) \qquad & -f(v_1(s,x)) + f(v_1(0,x)) \\ & = \int_0^s \int_0^\infty e^{-a\tau} F(v_2(\tau, v_1(\sigma,x)), f(v_2(\tau, v_1(\sigma,x))))\, d\tau\, d\sigma \end{aligned}$$

for $s \in [0,\infty), x \in B_{\tau_1}$.

Since $\lim_{t\to\infty} v_1(t,x) = 0$ for every $x \in B_{\tau_1}$ by Lemma 1, from the continuouity of the mapping f and (12), we obtain

$$f(x) = \int_0^\infty \int_0^\infty e^{-a\tau} F(v_2(\tau, v_1(\sigma,x)), f(v_2(\tau, v_1(\sigma,x))))d\tau d\sigma \quad \text{for} \quad x \in B_{\tau_1},$$

where the above integral is absolutely convergent by (9).

Now we shall show that every solution $f \in \mathcal{B}_{\tilde{\rho}_2} \subset C(B_{r_1}, Y)$ of the above integral equation is the solution of the problem (6). Notice that

$$f(v_1(s, v_2(t,x)))$$
$$= \int_0^\infty \int_0^\infty e^{-a\tau} F(v_2(\tau, v_1(\sigma, v_1(s, v_2(t,x)))), f(v_2(\tau, v_1(\sigma, v_1(s, v_2(t,x)))))) \, d\tau \, d\sigma$$

for $s, t \in [0, \infty), x \in B_{r_1}$. By Lemma 1 we have $v_1(\sigma, v_1(s,x)) = v_1(\sigma + s, x)$ for $\sigma, s \in [0, \infty)$, $x \in B_{r_1}$. Hence

$$f(v_1(s, v_2(t,x)))$$
$$= \int_s^\infty \int_0^\infty e^{-a\tau} F(v_2(\tau, v_1(\sigma, v_2(t,x))), f(v_2(\tau, v_1(\sigma, v_2(t,x))))) \, d\tau \, d\sigma$$

for $s, t \in [0, \infty), x \in B_{r_1}$.

Differentiating above equation with respect to s at a point $s = 0$, we obtain

$$\left[\frac{\partial}{\partial s} f(v_1(s, v_2(t,x)))\right]_{s=0} = -\int_0^\infty e^{-a\tau} F(v_2(\tau, v_2(t,x)), f(v_2(\tau, v_2(t,x)))) \, d\tau.$$

By Lemma 1 we have $v_2(\tau, v_2(t,x)) = v_2(\tau + t, x)$ for $\tau, t \in [0, \infty), x \in B_{r_1}$. Hence

$$(\nabla_{h_1} f)(v_2(t,x)) = -e^{at} \int_t^\infty e^{-a\tau} F(v_2(\tau,x), f(v_2(\tau,x))) \, d\tau \tag{13}$$

for $t \in [0, \infty), x \in B_{r_1}$. In particular

$$(\nabla_{h_1} f)(x) = \int_0^\infty e^{-a\tau} F(v_2(\tau,x), f(v_2(\tau,x))) d\tau \quad \text{for} \quad x \in B_{r_1}$$

is a continuous function. Differentiating the equation (13) with respect to t, we obtain

$$\frac{\partial}{\partial t}(\nabla_{h_1} f)(v_2(t,x)) = -ae^{at} \int_t^\infty e^{-a\tau} F(v_2(\tau,x), f(v_2(\tau,x))) \, d\tau$$
$$+ F(v_2(t,x), f(v_2(t,x))) \quad \text{for} \quad x \in B_{r_1}.$$

Therefore, by (13), we have for $t = 0$

$$(\nabla_{h_2}(\nabla_{h_1} f))(x) = -a(\nabla_{h_1} f)(x) + F(x, f(x)) \quad \text{for} \quad x \in B_{r_1}.$$

Now, it is sufficient to show that $f(0) = 0$ and $(\nabla_{h_1} f)(0) = 0$. Since $v_2(t, 0) = 0$ for $t \in [0, \infty)$, from the inequality (8) it follows that the condition $(\nabla_{h_1} f)(0) = 0$ is fulfilled. Moreover, since $v_1(s, 0) = 0$ for $s \in [0, \infty)$, from the inequality (9) it follows that the condition $f(0) = 0$ is fulfilled. ∎

COROLLARY 2. *Let $F : B_{\rho_1} \to Y$, with $B_{\rho_1} \subset X$, be a continuous mapping such that for certain constants $\tilde{\rho}_1 \in (0, \rho_1]$ and $K > 0$ the following condition takes place*

(i) $\|F(x)\| \leq K\|x\| \quad for \quad x \in B_{\tilde{\rho}_1}$.

If $a \geq 0$ than for $\tau_1 = min(r_1, \tilde{\rho}_1)$ the problem (6) under the Assumption 1 has exactly one solution $f \in C(B_{\tau_1}, Y)$ of the form

$$f(x) = \int_0^\infty \int_0^\infty e^{-at} F(v_2(t, v_1(s, x)))\, dt\, ds \quad for \quad x \in B_{\tau_1}.$$

This solution is in $C_*^{0,1}(B_{\tau_1}, Y)$.

Proof. This remark follows immediately from Lemma 3 and $f \in C_*^{0,1}(B_{\tau_1}, Y)$ by (9). ∎

The following theorem presents the sufficient conditions for the existence of the solution of the problem (6) in the class C.

THEOREM 2. *Let $F : B_{\rho_1} \times B_{\rho_2} \to Y$, with $B_{\rho_1} \times B_{\rho_2}$, be the continuous mapping such that for certain constants $\tilde{\rho}_1 \in (0, \rho_1], \tilde{\rho}_2 \in (0, \rho_2]$ and K_1, K_2, $(K_1^2 + K_2^2 > 0), L > 0$ the following conditions take places:*

(i) $\|F(x, y)\| \leq K_1\|x\| + K_2\|y\| \quad for \quad (x, y) \in B_{\tilde{\rho}_1} \times B_{\tilde{\rho}_2}$.

(ii) $\|F(x, y_1) - F(x, y_2)\| \leq L\|y_1 - y_2\|\|x\|$ *for* $x \in B_{\tilde{\rho}_1}$ *and* $y_1, y_2 \in B_{\tilde{\rho}_2}$.

For $K_1 \neq 0$ let

$$\tau_2 = \begin{cases} \min(r_1, \tilde{\rho}_1, \gamma \frac{1}{L}) & if \quad 0 \leq \frac{K_2}{a} \leq 1 - \frac{K_1/\tilde{\rho}_2}{L} \text{ and } a \neq 0 \\ \min(r_1, \tilde{\rho}_1, \gamma \frac{1 - K_2/a}{K_1/\tilde{\rho}_2}) & if \ 1 - \frac{K_1/\tilde{\rho}_2}{L} \leq \frac{K_2}{a} \leq \frac{L}{K_1/\tilde{\rho}_2 + L} \text{ and } a \neq 0 \\ \min(r_1, \tilde{\rho}_1, \gamma \frac{1}{K_1/\tilde{\rho}_2 + L}) & if \quad \frac{K_2}{a} \geq \frac{L}{K_1/\tilde{\rho}_2 + L} \text{ and } a \neq 0 \\ \min(r_1, \tilde{\rho}_1, \frac{\alpha_1 \alpha_2}{K_1/\tilde{\rho}_2 + L}) & if \quad a = 0, \end{cases}$$

and for $K_1 = 0$ let $\tau_2 = \min(r_1, \tilde{\rho}_1, \gamma \frac{1}{L})$ (where $\gamma = \alpha_1(\alpha_2 + a)$ and α_1, α_2, r_1 denote constants from Assumption 1).

Then the problem (6) has exactly one solution f in the ball $\mathcal{B}_{\tilde{\rho}_2} \subset C(B_{\tau_2}, Y)$. This solution is in $C_*^{0,1}(B_{\tau_2}, Y)$.

Proof. Let $\tau_1 = \min(r_1, \tilde{\rho}_1)$. From (i) and (ii) it follows that for every function $f \in \mathcal{B}_{\tilde{\rho}_2} \subset C(B_{\tau_1}, Y)$ the inequalities are true:

(14) $\|e^{-at} F(v_2(t, x), f(v_2(t, x)))\| \leq (K_1 + L\|f(v_2(t, x))\|)e^{-(\alpha_2 + a)t}\|x\|$

for $t \in [0, \infty)$, $x \in B_\tau$ and

$$\begin{aligned}(15)\qquad &\|e^{-at}F(v_2(t,v_1(s,x)),f(v_2(t,v_1(s,x))))\| \\ &\le (K_1 + L\|f(v_2(t,v_1(s,x)))\|)e^{-\alpha_1 s}e^{-(\alpha_2+a)t}\|x\|\end{aligned}$$

for $s,t \in [0,\infty)$, $x \in B_\tau$.

The analysis similar to that in the proof of Lemma 3 (when (14) replaces (8) and (15) replaces (9)) shows that the problem (6) in the ball $\mathcal{B}_{\tilde{\rho}_2} \subset C(B_\tau, Y), \tau \le \tau_1$, is equivalent to the equation

$$f(x) = \int_0^\infty e^{-at}F(v_2(t,v_1(s,x)),f(v_2(t,v_1(s,x))))dt \quad \text{for} \quad x \in B_\tau.$$

Let τ_2 be as assumed. For every $\tau < \tau_2$ we prove existence and uniqueness of the solution $f \in \mathcal{B}_{\tilde{\rho}_2} \subset C(B_\tau, Y)$ of the equation (7):

$$f(x) = \int_0^\infty\int_0^\infty e^{-at}F(v_2(t,v_1(s,x)),f(v_2(t,v_1(s,x))))dtds, \quad x \in B_\tau,$$

equivalent to the problem (6).

The condition (i) implies that for every function $f \in \mathcal{B}_{\tilde{\rho}_2} \subset C(B_\tau, Y)$ we have

$$(16)\qquad \|e^{-at}F(v_2(t,x),f(v_2(t,x)))\| \le K_1 e^{-(\alpha_2+a)t}\|x\| + K_2\tilde{\rho}_2 e^{-at}$$

for $t \in [0,\infty), x \in B_\tau$ and

$$\begin{aligned}(17)\qquad &\|e^{-at}F(v_2(t,v_1(s,x)),f(v_2(t,v_1(s,x))))\| \\ &\le K_1 e^{-\alpha_1 s}e^{-(\alpha_2+a)t}\|x\| + K_2\tilde{\rho}_2 e^{-at}\end{aligned}$$

for $s,t \in [0,\infty)$, $x \in B_\tau$.

Using the Banach fixed point theorem ([3] Theorem VIII.1), consider a mapping S on the closed ball $\bar{\mathcal{B}}_{\tilde{\rho}_2} \subset C(B_\tau, Y)$ defined by

$$S(f)(x) = \int_0^\infty\int_0^\infty e^{-at}F(v_2(t,v_1(s,x)),f(v_2(t,v_1(s,x))))\,dt\,ds$$

for $x \in B_\tau, f \in \bar{\mathcal{B}}_{\tilde{\rho}_2}$. From the inequality (15) we obtain

$$\begin{aligned}(18)\ \|S(f)(x)\| &\le \int_0^\infty\int_0^\infty (K_1 + L\|f(v_2(t,v_1(s,x)))\|)e^{-\alpha_1 s}e^{-(\alpha_2+a)t}\|x\|\,dt\,ds \\ &\le \frac{(K_1 + L\tilde{\rho}_2)\|x\|}{\alpha_1(\alpha_2+a)} \quad \text{for} \quad x \in B_\tau, f \in \bar{\mathcal{B}}_{\tilde{\rho}_2}.\end{aligned}$$

We shall prove that S is a contraction. Let $f_1, f_2 \in \bar{\mathcal{B}}_{\tilde{\rho}_2} \subset C(B_\tau, Y)$. Then from the assumption (ii) we have

$$\begin{aligned}
(19) \qquad & \|S(f_1)(x) - S(f_2)(x)\| \\
&\le \int_0^\infty \int_0^\infty Le^{-\alpha_1 s} e^{-(\alpha_2+a)t} \|f_1(v_2(t,v_1(s,x))) - f_2(v_2(t,v_1(s,x)))\| \|x\| \,dt\,ds \\
&\le \frac{L\|x\|}{\alpha_1(\alpha_2+a)} \|f_1 - f_2\| \quad \text{for} \quad x \in B_\tau.
\end{aligned}$$

In the case of $K_1 \neq 0$ and $0 \le \frac{K_2}{a} \le \frac{L}{K_1/\tilde{\rho}_2 + L}$, with $a \neq 0$ we have

$$\tau_2 = \min\left(r_1, \tilde{\rho}_1, \alpha_1(\alpha_2+a)\frac{1}{L}, \alpha_1(\alpha_2+a)\frac{1-K_2/a}{K_1/\tilde{\rho}_2}\right),$$

because the condition $\frac{K_2}{a} \ge 1 - \frac{K_1/\tilde{\rho}_2}{L}$ is equivalent to $\frac{1}{L} \ge \frac{1-K_2/a}{K_1/\tilde{\rho}_2}$ and from the condition $\frac{K_2}{a} \le \frac{L}{K_1/\tilde{\rho}_2+L}$ it follows that $\frac{1}{K_1/\tilde{\rho}_2+L} \le \frac{1-K_2/a}{K_1/\tilde{\rho}_2}$. By (17) and the condition $\tau_2 \le \alpha_1(\alpha_2+a)\frac{1-K_2/a}{K_1/\tilde{\rho}_2}$ we obtain

$$\begin{aligned}
\|S(f)(x)\| &\le \int_0^\infty \int_0^\infty (K_1 e^{-\alpha_1 s} e^{-(\alpha_2+a)t}\|x\| + K_2\tilde{\rho}_2 e^{-at})\,dt\,ds \\
&< \int_0^\infty \int_0^\infty (K_1 e^{-\alpha_1 s} e^{-(\alpha_2+a)t}\tau + K_2\tilde{\rho}_2 e^{-at})\,dt\,ds \\
&\le \int_0^\infty \int_0^\infty (K_1 e^{-\alpha_1 s} e^{-(\alpha_2+a)t}\alpha_1(\alpha_2+a)\frac{1-K_2/a}{K_1/\tilde{\rho}_2} + K_2\tilde{\rho}_2 e^{-at})\,dt\,ds \\
&= (1-K_2/a)\tilde{\rho}_2 + \frac{K_2\tilde{\rho}_2}{a} \\
&= \tilde{\rho}_2 \qquad \text{for} \quad x \in B_\tau, f \in \bar{\mathcal{B}}_{\tilde{\rho}_2}.
\end{aligned}$$

The inequality $\tau_2 \le \alpha_1(\alpha_2+a)\frac{1}{L}$ is true, hence we conclude from (19) that the mapping S is a contraction.

Now let $\frac{K_2}{a} \ge \frac{L}{K_1/\tilde{\rho}_2+L}$ (and $a \neq 0$) or $a = 0$ or $K_1 = 0$. Then

$$\tau_2 = \min\left(r_1, \tilde{\rho}_1, \alpha_1(\alpha_2+a)\frac{1}{K_1/\tilde{\rho}_2 + L}\right).$$

From (18) and the condition $\tau_2 \le \alpha_1(\alpha_2+a)\frac{1}{K_1/\tilde{\rho}_2+L}$ it follows that

$$\|S(f)\| < \tilde{\rho}_2 \quad \text{for} \quad f \in \bar{\mathcal{B}}_{\tilde{\rho}_2}\ .$$

Thus S is a well-defined mapping and $S : \bar{\mathcal{B}}_{\tilde{\rho}_2} \longrightarrow \mathcal{B}_{\tilde{\rho}_2} \subset \bar{\mathcal{B}}_{\tilde{\rho}_2}$.

Since $\tau_2 \le \frac{\alpha_1(\alpha_2+a)}{K_1/\tilde{\rho}_2+L} \le \frac{\alpha_1(\alpha_2+a)}{L}$, then, from the inequality (19) it follows that S is a contraction. So, the assumptions of the Banach fixed point theorem hold. Therefore, there exists only one mapping $f_\tau \in \bar{\mathcal{B}}_{\tilde{\rho}_2} \subset C(B_\tau, Y)$ which is a fixed point of the mapping S and so it is the solution of the problem (6); in addition $f_\tau = S(f_\tau)$ belongs to the open ball $\mathcal{B}_{\tilde{\rho}_2}$. From the uniqueness of the solution we obtain the equality $f_\tau = f_{\tau'} \mid B_\tau$ for $\tau < \tau' < \tau_2$. The searched solution $f \in \mathcal{B}_{\tilde{\rho}_2} \subset C(B_{\tau_2}, Y)$ coincide with the function f_τ on every ball B_τ, i.e.

$$f(x) = f_\tau(x) \quad \text{for} \quad x \in B_\tau, \tau < \tau_2$$

By (18), $f \in C_*^{0,1}(B_{\tau_2}, Y)$ what completes our considerations.

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