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Soon–Mo Jung, Prasanna K. Sahoo

ON THE STABILITY OF A MEAN VALUE TYPE FUNCTIONAL EQUATION

In this paper, we prove the stability results of a mean value type functional equation, namely $f(x) - g(y) = (x - y)h(x + y)$ which arises from the mean value theorem.

1. Introduction

The starting point of studying the stability problems of functional equations seems to be the famous talk of S. M. Ulam in 1940, in which he discussed a number of important unsolved problems (see [11]). Among those was the following question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In 1941, D. H. Hyers [4] affirmatively answered the question of Ulam for the case of approximately additive functions under the assumption that G_1 and G_2 are Banach spaces.

Taking this historical backgrounds into account, we say that the additive Cauchy equation $f(x + y) = f(x) + f(y)$ is stable on (G_1, G_2), in the sense of Hyers and Ulam. This terminology is also applied to the case of other functional equations. For more detailed definitions of such terminologies, we refer the reader to [2], [5], [6] and [7].

The functional equation

$$f(x) - g(y) = (x - y)h(x + y) \tag{1}$$

where $x, y \in \mathbb{R}$ (the set of reals), arises from the mean value theorem and

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characterizes polynomials of degree one and two. This functional equation was originally treated by J. Aczél in 1963 (see J. Aczél (1985)) and also independently by Sh. Haruki (1979). A generalization of the functional equation (1) was treated by Pl. Kannappan, P. K. Sahoo and M. S. Jacobson [8] (see also [10, Theorem 2.5], [1] or [3]). We summarize the result of Aczél (1985) regarding the equation (1) in the following theorem.

THEOREM 1. *Let F be a field of characteristic different from two. The functions $f, g, h : F \to F$ satisfy the functional equation* (1) *for all $x, y \in F$ if and only if there exist $a, b, c \in F$ such that $f(x) = g(x) = ax^2 + bx + c$ and $h(x) = ax + b$ for all $x \in F$.*

2. Stability of the functional equation (1)

Th. M. Rassias extended in [9] the stability result of Hyers for the additive functions by considering the case that the Cauchy difference is not bounded. By generalizing the idea of Rassias, we will prove a general theorem for the stability of the functional equation (1).

First, assume that $\varphi : F \times F \to [0, \infty)$ is a symmetric function with the property $\varphi(-x, y) = \varphi(x, y)$ for all x and y in F where F be a normed algebra with a unit element 1 (or a normed field of characteristic different from 2). We will use the following notation

$$\Phi(x,y) = 2\varphi\Big(\frac{x+y}{2}, 0\Big) + 4\varphi\Big(0, \frac{x-y}{2}\Big) + 2\varphi\Big(\frac{x+y}{2}, \frac{x-y}{2}\Big) + \varphi\Big(\frac{x-y}{2}, \frac{x-y}{2}\Big) + 3\varphi(0,0)$$

for all $x, y \in F$.

THEOREM 2. *Let F be a normed algebra with a unit element* 1 *(or a normed field of characteristic different from* 2*). If functions $f, g, h : F \to F$ satisfy the functional inequality*

$$\|f(x) - g(y) - (x-y)h(x+y)\| \le \varphi(x,y) \tag{2}$$

for all $x, y \in F$, then there exist constants $a, b, c, d \in F$ with $\|c-d\| \le \varphi(0,0)$ such that

$$\|f(x) - ax^2 - bx - c\| \le \varphi(x,0) + \|x\|\Phi(x,1),$$
$$\|g(x) - ax^2 - bx - d\| \le \varphi(x,0) + \|x\|\Phi(x,1),$$
$$\|h(x) - ax - b\| \le \Phi(x,1)$$

for all $x \in F$.

Proof. If we put $y = 0$ in (2), then we have

$$\|f(x) - g(0) - xh(x)\| \le \varphi(x,0). \tag{3}$$

Similarly, if we set $x = 0$ in (2), we get

$$\|f(0) - g(y) + yh(y)\| \le \varphi(0, y). \tag{4}$$

By (2), (3) and (4), we obtain

$$\begin{aligned} &\|xh(x) - yh(y) - (x - y)h(x + y)\| \\ &\le \| - f(x) + g(0) + xh(x)\| + \|g(y) - f(0) - yh(y)\| \\ &\quad + \|f(x) - g(y) - (x - y)h(x + y)\| + \|f(0) - g(0)\| \\ &\le \varphi(x, 0) + \varphi(0, y) + \varphi(x, y) + \varphi(0, 0) \end{aligned} \tag{5}$$

for all $x, y \in F$. By replacing x by $-y$ in (5), we obtain

$$\|-yh(-y)-yh(y)+2yh(0)\| \le \varphi(-y, 0)+\varphi(0, y)+\varphi(-y, y)+\varphi(0, 0) \tag{6}$$

and by substituting $-y$ for y in (5), we have

$$\begin{aligned} &\|xh(x) + yh(-y) - (x + y)h(x - y)\| \\ &\qquad \le \varphi(x, 0) + \varphi(0, -y) + \varphi(x, -y) + \varphi(0, 0) \end{aligned} \tag{7}$$

for all $x, y \in F$. By (5), (6) and (7), we obtain

$$\begin{aligned} &\|(x - y)h(x + y) - (x + y)h(x - y) + 2yh(0)\| \\ &\le \|(x - y)h(x + y) - xh(x) + yh(y)\| \\ &\quad + \|xh(x) + yh(-y) - (x + y)h(x - y)\| \\ &\quad + \| - yh(-y) - yh(y) + 2yh(0)\| \\ &\le 2\varphi(x, 0) + 4\varphi(0, y) + 2\varphi(x, y) + \varphi(y, y) + 3\varphi(0, 0) \end{aligned} \tag{8}$$

for any x and y in F. Substituting $u = x + y$ and $v = x - y$ in (8), we get $\|vH(u) - uH(v)\| \le \Phi(u, v)$ where we define $H(u) = h(u) - h(0)$. Let $v = 1$ and replace u by x to obtain $\|h(x) - ax - b\| \le \Phi(x, 1)$ where $a = H(1) = h(1) - h(0)$ and $b = h(0)$. It follows from (3) that

$$\begin{aligned} \|f(x) - ax^2 - bx - c\| &\le \|f(x) - c - xh(x)\| + \|xh(x) - ax^2 - bx\| \\ &\le \varphi(x, 0) + \|x\|\Phi(x, 1) \end{aligned}$$

for any x in F, where we let $c = g(0)$. Analogously, by (4), we get

$$\begin{aligned} \|g(x) - ax^2 - bx - d\| &\le \|g(x) - d - xh(x)\| + \|xh(x) - ax^2 - bx\| \\ &\le \varphi(x, 0) + \|x\|\Phi(x, 1) \end{aligned}$$

where we set $d = f(0)$.

If we assume $\varphi(x, y) = \varepsilon$ in Theorem 2, we see the stability (like the type of Hyers and Ulam) of the functional equation (1) directly from Theorem 2.

COROLLARY 3. *Let F be a normed algebra with a unit element* 1 *(or a normed field of characteristic different from* 2*). If functions $f, g, h : F \to F$ satisfy the functional inequality $\|f(x) - g(y) - (x - y)h(x + y)\| \le \varepsilon$ for all*

$x, y \in F$, then there exist constants $a, b, c, d \in F$ with $\|c - d\| \le \varepsilon$ such that

$$\|f(x) - ax^2 - bx - c\| \le \varepsilon + 12\varepsilon\,\|x\|,$$
$$\|g(x) - ax^2 - bx - d\| \le \varepsilon + 12\varepsilon\,\|x\|,$$
$$\|h(x) - ax - b\| \le 12\varepsilon$$

for all $x \in F$.

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MATHEMATICS SECTION
COLLEGE OF SCIENCE & TECHNOLOGY
HONG-IK UNIVERSITY
339-800 CHOCHIWON, KOREA
E-mail address: smjung@wow.hongik.ac.kr

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LOUISVILLE
LOUISVILLE, KY 40292, U.S.A.
E-mail address: sahoo@louisville.edu

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