DEMONSTRATIO MATHEMATICA
Vol. XXXIII No 4 2000

Ç. Asma, R. Çolak

ON THE KÖTHE–TOEPLITZ DUALS OF SOME GENERALIZED SETS OF DIFFERENCE SEQUENCES

Abstract. In this paper we define the sequence sets $\ell_\infty(u,\Delta,p)$, $c(u,\Delta,p)$ and $c_0(u,\Delta,p)$, and give $\alpha-$ and $\beta-$ duals of these sets and the sequence space $\ell_\infty(\Delta_r p)$ defined by Mursaleen et al. These sets generalizes some sets defined by Ahmad and Mursaleen [1], and Malkowsky [6].

1. Introduction

We shall write ω for the set of all sequences $x = \{x_k\}$ with complex terms and $p = \{p_k\}$ will denote a certain sequence of positive real numbers. The following sets introduced and investigated by various authors.

$$\ell_\infty(p) = \{x \in \omega : \sup_k |x_k|^{p_k} < \infty\},$$

$$c(p) = \{x \in \omega : |x_k - l|^{p_k} \longrightarrow 0, \text{ for some complex } l\},$$

$$c_0(p) = \{x \in \omega : |x_k|^{p_k} \longrightarrow 0\},$$

$$\ell(p) = \{x \in \omega : \sum_{k=1}^{\infty} |x_k|^{p_k} < \infty\} \quad ([3],[4],[5] \text{ and } [8]).$$

Let U be the set of all sequences $u = \{u_k\}_{k=1}^{\infty}$ such that $u_k \neq 0$ and complex for all k=1,2, $\ldots$. Throughout the paper we write $w_k = \frac{1}{|u_k|}$.

Given an arbitrary sequence $p = \{p_k\}$ of positive real numbers p_k and given $u \in U$ we define the sets

$$\ell_\infty(u,\Delta,p) = \{x \in \omega : \{u_k \Delta x_k\} \in \ell_\infty(p)\}$$

$$c(u,\Delta,p) = \{x \in \omega : \{u_k \Delta x_k\} \in c(p)\}$$

$$c_0(u,\Delta,p) = \{x \in \omega : \{u_k \Delta x_k\} \in c_0(p)\},$$

where $\Delta x_k = x_k - x_{k+1}$, $k = 1,2,\ldots$

Key words and phrases: difference sequences, $\alpha-$ and $\beta-$ duals.
1991 *Mathematics Subject Classification*: 40C05, 46A45.

If $p = \{p_k\}$ is a fixed bounded sequence of positive real numbers and $u \in U$, then $\ell_\infty(u, \Delta, p)$, $c(u, \Delta, p)$ and $c_0(u, \Delta, p)$ are linear spaces, under the usual operations

$$x + y = \{x_n + y_n\} \quad \text{and} \quad \alpha x = \{\alpha x_n\}$$

where α is any complex number.

THEOREM 1.1. *Let $p = \{p_k\}$ be a bounded sequence of strictly positive real numbers p_k and $u \in U$. Then $c_0(u, \Delta, p)$ is a paranormed space with paranorm $g(x) = \sup_k |u_k \Delta x_k|^{p_k/M}$, where $M = max\{1, H = \sup_k p_k\}$. If $\inf_k p_k > 0$ then $\ell_\infty(u, \Delta, p)$ and $c(u, \Delta, p)$ are paranormed spaces with the same norm as above.*

The proof of this theorem is easy therefore we omit it.

2. Köthe–Toeplitz duals

In [7] Mursaleen et al. defined and studied the sequence space

$$\ell_\infty(\Delta_r p) = \{x \in \omega : \Delta_r x \in \ell_\infty(p), \quad r > 0\}.$$

They give α-, β- duals of $\ell_\infty(\Delta_r p)$ in Theorem 2.1 in that paper. But they applied some argument which do not seem to hold : In the proof of Theorem 2.1 (in [7]) it is expressed that " obviously $\sum_{k=1}^\infty k^{-r}|a_k| \sum_{j=1}^{k-1} N^{1/p_j} < \infty$ implies $a \in \ell_1$". But this is not valid; for example, if we take $r = 2$, $p_k = 1$, $a_k = \frac{1}{k}$ $(k = 1, 2, \ldots)$ and $N > 1$, then as easily seen the series

$$\sum_{k=1}^\infty k^{-r}|a_k| \sum_{j=1}^{k-1} N^{1/p_j} = N \sum_{k=1}^\infty \frac{k-1}{k^3}$$

is convergent, but the series $\sum_{k=1}^\infty |a_k| = \sum_{k=1}^\infty \frac{1}{k}$ is divergent.

In this paper we also determine the α- and β-duals of the sequence space $\ell_\infty(\Delta_r p)$ in Corollary 2.3(i) and Corollary 2.5.

For any subset X of ω, the sets

$$X^\alpha = \left\{a \in \omega : \sum_{k=1}^\infty |a_k x_k| < \infty \text{ for all } x \in X\right\}$$

$$X^\beta = \left\{a \in \omega : \sum_{k=1}^\infty a_k x_k \quad \text{converges for all } x \in X\right\}$$

are called α- and β-dual of X. We shall write $X^{\alpha\alpha} = (X^\alpha)^\alpha$ and $X^{\beta\beta} = (X^\beta)^\beta$.

THEOREM 2.1. *For every sequence $p = \{p_k\}$ of strictly positive real numbers p_k, and for every $u \in U$, we have*

(i) $[\ell_\infty(u,\Delta,p)]^\alpha = M_\alpha(p)$,

(ii) $[c_0(u,\Delta,p)]^\alpha = D_\alpha(p)$,

(iii) $[c(u,\Delta,p)]^\alpha = D(p)$,

where

$$M_\alpha(p) = \bigcap_{N=2}^{\infty} \Big\{a \in \omega : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{1/p_j} w_j < \infty\Big\},$$

$$D_\alpha(p) = \bigcup_{N=2}^{\infty} \Big\{a \in \omega : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{-1/p_j} w_j < \infty\Big\}$$

and

$$D(p) = D_\alpha(p) \cap \Big\{a \in \omega : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} w_j < \infty\Big\}.$$

Proof. (i) Let $a \in M_\alpha(p)$ and $x \in \ell_\infty(u,\Delta,p)$. For any $N > \max\{1, \sup_k |u_k \Delta x_k|^{p_k}\}$, obviously $\sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{1/p_j} w_j < \infty$ implies $\sum_{k=1}^{\infty} |a_k| < \infty$ and hence we have

$$\sum_{k=1}^{\infty} |a_k x_k| \le \sum_{k=1}^{\infty} |a_k| |\sum_{j=1}^{k-1} \Delta x_j| + |x_1| \sum_{k=1}^{\infty} |a_k| \tag{1}$$

$$\le \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{1/p_j} w_j + |x_1| \sum_{k=1}^{\infty} |a_k| < \infty,$$

so we have $a \in [\ell_\infty(u,\Delta,p)]^\alpha$. Therefore $M_\alpha(p) \subset [\ell_\infty(u,\Delta,p)]^\alpha$.

Conversely let $a \notin M_\alpha(p)$. Then $\sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{1/p_j} w_j = \infty$ for some integer $N > 1$. If we take $x_k = \sum_{j=1}^{k-1} N^{1/p_j} w_j$ $(k = 1, 2, \ldots)$, then $x \in \ell_\infty(u,\Delta,p)$ and $\sum_{k=1}^{\infty} |a_k x_k| = \infty$. This implies that $a \notin [\ell_\infty(u,\Delta,p)]^\alpha$. Therefore $[\ell_\infty(u,\Delta,p)]^\alpha \subset M_\alpha(p)$ and hence $M_\alpha(p) = [\ell_\infty(u,\Delta,p)]^\alpha$.

(ii) Let $a \in D_\alpha(p)$ and $x \in c_0(u,\Delta,p)$. Then there is an integer k_0 such that $\sup_{k>k_0} |u_k \Delta x_k|^{p_k} \le N^{-1}$, where N is the number in $D_\alpha(p)$. We put $M := \max_{1\le k\le k_0} |u_k \Delta x_k|^{p_k}$, $m := \min_{1\le k\le k_0} p_k$, $L := (M+1)N$ and define the sequence y by $y_k := x_k L^{-1/m}$ $(k = 1, 2, \ldots)$. Then it is easy to see that $\sup_k |u_k \Delta y_k|^{p_k} \le N^{-1}$, and as in (1) with N replaced by N^{-1}, we have

$$\sum_{k=1}^{\infty} |a_k x_k| = L^{1/m} \sum_{k=1}^{\infty} |a_k y_k| < \infty.$$

Conversely let $a \notin D_\alpha(p)$. Then we may choose a sequence $\{k(s)\}$ of integers such that $k(1) = 1, k(1) < k(2) < \ldots < k(s) < k(s+1) < \ldots$ and

$$M(p,s) = \sum_{k=k(s)}^{k(s+1)-1} |a_k| \sum_{j=1}^{k-1} (s+1)^{-1/p_j} w_j > 1 \quad (s = 1, 2, \ldots).$$

If we take

$$x_k = \sum_{l=1}^{s-1} \sum_{j=k(l)}^{k(l+1)-1} (l+1)^{-1/p_j} w_j + \sum_{j=k(s)}^{k-1} (s+1)^{-1/p_j} w_j,$$
$$(k(s) \le k \le k(s+1) - 1; s = 1, 2, \ldots)$$

then $|u_k \Delta x_k|^{p_k} = \frac{1}{s+1}$ $(k(s) \le k \le k(s+1) - 1;\ s = 1, 2, \ldots)$ and so $x \in c_0(u, \Delta, p)$. Then we can easily see that $\sum_{k=1}^{\infty} |a_k x_k| \ge \sum_{s=1}^{\infty} 1 = \infty$. This implies that $a \notin [c_0(u, \Delta, p)]^\alpha$.

(iii) Let $a \in D(p)$ and $x \in c(u, \Delta, p)$. Then $|u_k \Delta x_k - l|^{p_k} \to 0$ $(k \to \infty)$ for some complex number l. We define the sequence $y = \{y_k\}$ by

$$y_k = x_k + l \sum_{j=1}^{k-1} u_j^{-1} \qquad (k = 1, 2, \ldots).$$

Then $y = \{y_k\} \in c_0(u, \Delta, p)$. Since $a \in D(p)$, then by (ii), we have

$$\sum_{k=1}^{\infty} |a_k x_k| \le \sum_{k=1}^{\infty} |a_k| |\sum_{j=1}^{k-1} \Delta y_j| + |y_1| \sum_{k=1}^{\infty} |a_k| + |l| \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} w_j < \infty$$

and hence $a \in [c(u, \Delta, p)]^\alpha$.

Now let $a \in [c(u, \Delta, p)]^\alpha$. Since

$$[c(u, \Delta, p)]^\alpha \subset [c_0(u, \Delta, p)]^\alpha \quad \text{and} \quad [c_0(u, \Delta, p)]^\alpha = D_\alpha(p)$$

by (ii), then $a \in D_\alpha(p)$. If we put $x_k = \sum_{j=1}^{k-1} w_j$ for $k = 1, 2, \ldots$, then $x \in c(u, \Delta, p)$ and therefore $\sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} \omega_j < \infty$. Thus $a \in D(p)$.

THEOREM 2.2. *For every sequence $p = \{p_k\}$ of strictly positive real numbers p_k, and for every $u \in U$, we have:*

(i) $[\ell_\infty(u, \Delta, p)]^{\alpha\alpha} = M_{\alpha\alpha}(p)$,
(ii) $[c_0(u, \Delta, p)]^{\alpha\alpha} = D_{\alpha\alpha}(p)$,

where

$$M_{\alpha\alpha}(p) = \bigcup_{N=2}^{\infty} \left\{ a \in \omega : \sup_{k \ge 2} |a_k| \Big(\sum_{j=1}^{k-1} N^{1/p_j} w_j \Big)^{-1} < \infty \right\}$$

and

$$D_{\alpha\alpha}(p) = \bigcap_{N=2}^{\infty} \Big\{ a \in \omega : \sup_{k \geq 2} |a_k| \Big(\sum_{j=1}^{k-1} N^{-1/p_j} w_j \Big)^{-1} < \infty \Big\}.$$

Proof. (i) Let $a \in M_{\alpha\alpha}(p)$ and $x \in [\ell_\infty(u, \Delta, p)]^\alpha = M_\alpha(p)$. Then we have

$$\sum_{k=2}^{\infty} |a_k x_k| = \sum_{k=2}^{\infty} |a_k| \Big(\sum_{j=1}^{k-1} N^{1/p_j} w_j \Big)^{-1} |x_k| \sum_{j=1}^{k-1} N^{1/p_j} w_j$$
$$\leq \Big[\sup_{k \geq 2} |a_k| \Big(\sum_{j=1}^{k-1} N^{1/p_j} w_j \Big)^{-1} \Big] \sum_{k=2}^{\infty} |x_k| \sum_{j=1}^{k-1} N^{1/p_j} w_j < \infty$$

for some $N > 1$, by Theorem 2.1(i). Hence $a \in [\ell_\infty(u, \Delta, p)]^{\alpha\alpha}$.

Conversely let $a \notin M_{\alpha\alpha}(p)$. Then for every $N > 1$ we have

$$\sup_{k \geq 2} |a_k| (\sum_{j=1}^{k-1} N^{1/p_j} w_j)^{-1} = \infty.$$

Therefore we may choose a strictly increasing sequence $\{k(i)\}$ of integers such that

$$|a_{k(i)}| \Big(\sum_{j=1}^{k(i)-1} N^{1/p_j} w_j \Big)^{-1} > i^2 \quad (i = 2, 3, \dots).$$

Put $x_k = |a_{k(i)}|^{-1}$ $(k = k(i))$, $x_k = 0$ $(k \neq k(i))$. Then

$$\sum_{k=1}^{\infty} |x_k| \sum_{j=1}^{k-1} N^{1/p_j} w_j = \sum_{i=1}^{\infty} |a_{k(i)}|^{-1} \sum_{j=1}^{k(i)-1} N^{1/p_j} w_j \leq \sum_{i=1}^{\infty} i^{-2} < \infty$$

for every integer $N > 1$. This yields that $x \in [\ell_\infty(u, \Delta, p)]^\alpha$ and $\sum_{k=1}^\infty |a_k x_k| = \sum_{i=1}^\infty 1 = \infty$ and so $a \notin [\ell_\infty(u, \Delta, p)]^{\alpha\alpha}$. Hence $[\ell_\infty(u, \Delta, p)]^{\alpha\alpha} = M_{\alpha\alpha}(p)$.

(ii) Let us define

$$E_N(p) = \{ a \in \omega : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{-1/p_j} w_j < \infty \}$$

and

$$F_N(p) = \Big\{ a \in \omega : \sup_{k \geq 2} |a_k| \Big(\sum_{j=1}^{k-1} N^{-1/p_j} w_j \Big)^{-1} < \infty \Big\}$$

where $N = 2, 3, \dots$

By Lemma 4(iv) in [3], $F_N(p) = [E_N(p)]^\alpha$ for each integer $N > 1$.

Let us define

$$X(\Delta_r p) = \{ x = (x_k) : \Delta_r x \in X(p), \quad r > 0 \},$$

where $X = \ell_\infty, c_0$ or c and $\Delta_r x = \{k^r \Delta x_k\}_{k=1}^\infty$.

If we put $u_k = k^r$ then we have $\ell_\infty(u,\Delta,p) = \ell_\infty(\Delta_r p)$, $c_0(u,\Delta,p) = c_0(\Delta_r p)$ and $c(u,\Delta,p) = c(\Delta_r p)$.

Now if we take $u_k = k^r$, $(r > 0)$ in Theorem 2.1 (i), (ii), (iii) then we have the following results.

COROLLARY 2.3. *For every strictly positive sequence of reals $p = \{p_k\}$ and $r > 0$, we have*

(i) $[\ell_\infty(\Delta_r p)]^\alpha = \bigcap_{N=2}^\infty \{a \in \omega : \sum_{k=1}^\infty |a_k| \sum_{j=1}^{k-1} j^{-r} N^{1/p_j} < \infty\}$,

(ii) $[c_0(\Delta_r p)]^\alpha = \bigcup_{N=2}^\infty \{a \in \omega : \sum_{k=1}^\infty |a_k| \sum_{j=1}^{k-1} j^{-r} N^{-1/p_j} < \infty\}$,

(iii) $[c(\Delta_r p)]^\alpha = [c_0(\Delta_r p)]^\alpha \bigcap \{a \in \omega : \sum_{k=1}^\infty |a_k| \sum_{j=1}^{k-1} j^{-r}$ *converges*$\}$.

THEOREM 2.4. *Let $p = \{p_k\}$ be a sequence of strictly positive real numbers. Then $[\ell_\infty(u,\Delta,p)]^\beta = D_\infty(p)$ where*

$$D_\infty(p) =$$

$$\bigcap_{N\geq 2} \left\{a \in \omega : \sum_{k=1}^\infty a_k \sum_{j=1}^{k-1} N^{1/p_j} w_j \text{ converges and } \sum_{k=1}^\infty N^{1/p_k} w_k |R_k| < \infty\right\}$$

and

$$R_k = \sum_{v=k+1}^\infty a_v \qquad (k = 1,2,\ldots).$$

Proof. Let $a \in D_\infty(p)$ and $x \in \ell_\infty(u,\Delta,p)$. We may write

$$(2) \quad \sum_{k=1}^n a_k x_k = -\sum_{k=1}^{n-1} \Delta x_k R_k + R_n \sum_{k=1}^{n-1} \Delta x_k + x_1 \sum_{k=1}^n a_k \quad (n = 1,2,\ldots).$$

Obviously $a \in D_\infty(p)$ implies the convergence of the series $\sum_{k=1}^\infty a_k$. Since $x \in \ell_\infty(u,\Delta,p)$ we may choose an integer $N > \max\{1, \sup_k |u_k \Delta x_k|^{p_k}\}$ so that

$$\sum_{k=1}^\infty |\Delta x_k||R_k| \leq \sum_{k=1}^\infty N^{1/p_k} w_k |R_k| < \infty.$$

Therefore $\sum_{k=1}^\infty \Delta x_k R_k$ is absolutely convergent. From Corollary 2 in [2], the convergence of $\sum_{k=1}^\infty a_k \sum_{j=1}^{k-1} N^{1/p_j} w_j$ implies that $R_k \sum_{j=1}^{k-1} N^{1/p_j} w_j \to 0$ $(k \to \infty)$, and hence (2) yields that $\sum_{k=1}^\infty a_k x_k$ is convergent. Therefore $a \in [\ell_\infty(u,\Delta,p)]^\beta$.

Conversely let $a \in [\ell_\infty(u,\Delta,p)]^\beta$. Since $e = \{1,1,\ldots\} \in \ell_\infty(u,\Delta,p)$ then we have that the series $\sum_{k=1}^\infty a_k$ is convergent. Define $x = \{x_k\}$ by $x_k = \sum_{j=1}^{k-1} N^{1/p_j} w_j$, then $x \in \ell_\infty(u,\Delta,p)$ and $\sum_{k=1}^\infty a_k \sum_{j=1}^{k-1} N^{1/p_j} w_j$ is convergent. Again by Corollary 2 in [2] we have $R_n \sum_{j=1}^{n-1} N^{1/p_j} w_j \to 0$ $(n \to \infty)$. By (2) we have the series $\sum_{k=1}^\infty \Delta x_k R_k$ converges for all $a \in$

$\ell_\infty(u, \Delta, p)$. Since $x \in \ell_\infty(u, \Delta, p)$ if and only if $\{y_k\} = \{u_k \Delta x_k\} \in \ell_\infty(p)$, then $\{u^{-1} R_k\} \in [\ell_\infty(p)]^\beta$. By Theorem 2 in [4] we have

$$\sum_{k=1}^{\infty} N^{1/p_k} \omega_k |R_k| < \infty$$

for all integer $N > 1$. Hence $a \in D_\infty(p)$.

This completes the proof of the theorem.

If we take $u_k = k^r$, $r > 0$ in Theorem 2.4, then we have the following result.

COROLLARY 2.5. *For every strictly positive sequence of reals* $p = \{p_k\}$ *and* $r > 0$, *we have*

$$[\ell_\infty(\Delta_r p)]^\beta = \bigcap_{N=2}^{\infty} \Big\{ a \in \omega : \sum_{k=1}^{\infty} a_k \sum_{j=1}^{k-1} j^{-r} N^{1/p_j} \text{ converges and } \sum_{k=1}^{\infty} k^{-r} N^{1/p_k} |R_k| < \infty \Big\}.$$

References

[1] Z. U. Ahmad and Mursaleen, *Köthe-Toeplitz duals of some new sequence spaces and their matrix maps*, Publ. Inst. Math. (Beograd) 42(56) (1987), 57–61.

[2] H. Kizmaz, *On certain sequence spaces*, Canadian Math. Bull. 24 (1981), 169–176.

[3] C. G. Lascarides, *A study of certain sequence spaces of Maddox and a generalization of a theorem of Iyer*, Pacific J. Math. 38 (1971), 487–500.

[4] C. G. Lascarides and I. J. Maddox, *Matrix transformations between some classes of sequences*, Proc. Cambridge Phil. Soc. 68 (1970), 99–104.

[5] I. J. Maddox, *Continuous and Köthe-Toeplitz duals of certain sequence spaces*, Proc. Cambridge Phil. Soc. 65 (1967), 471–475.

[6] E. Malkowsky, *A note on the Köthe-Toeplitz duals of generalized sets of bounded and convergent difference sequences*, J. Analysis 4 (1996), 81–91.

[7] Mursaleen, A. K. Gaur and A. H. Saifi, *Some new sequence spaces, their duals and matrix transformations*, Bull. Cal. Math. Soc. 88 (1996), 207–212.

[8] S. Simons, *The sequence spaces* $\ell(p_v)$ *and* $m(p_v)$, Proc. London Math.Soc. 15 (1965), 422–436.

DEPARTMENT OF MATHEMATICS
FIRAT UNIVERSITY
23119, ELAZIG, TURKEY
E-mail: cigdemas78@hotmail.com

Received February 7, 2000.