DEMONSTRATIO MATHEMATICA
Vol. XXXIII No 4 2000

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AN APPLICATION OF MODULAR SPACES TO APPROXIMATION PROBLEMS, VII

Abstract. By means of terms of a sequence (ρ_n), where ρ_n, $n = 1, 2, \ldots$, are a pseudomodulars, and by means of an infinite matrix $A = [a_{ni}]$ of non-negative numbers we shall construct the various modular spaces. Then we shall approximate elements of the spaces $X_{\rho^{A,s}}$ and $X_{\rho^A_s}$ by means of terms of a sequence $(\widetilde{\rho}_m)$, where $(\widetilde{\rho}_m)$, $m = 1, 2, \ldots$, are a pseudomodulars. In particular, we will investigate the special case when ρ_n and $\widetilde{\rho}_m$ are singular integrals.

Let (Ω, Σ, μ) denote a space with a finite measure μ, defined on Σ, a σ-algebra of subsets of the set Ω, $\rho_n(t, f) : \Omega \times \mathscr{X} \to \langle 0, \infty)$ for $n = 1, 2, \ldots$ and $f \in \mathscr{X}$ – the space of functions $f : \Omega \to (-\infty, \infty)$ which are Σ-measurable and almost everywhere finite, with equality μ-almost everywhere.

Let us assume:

(a) $\rho_n(t, f)$ is a pseudomodular in $\mathscr{X}$ for almost all t and for every $n = 1, 2, \ldots$,

(b) if for $n = 1, 2, \ldots$ $\rho_n(t, f) = 0$ for almost all t, then $f = 0$,

(c) $\rho_n(t, f)$ is measurable and almost everywhere finite with respect to t for every $f \in \mathscr{X}$ and every $n = 1, 2, \ldots$

Let us denote by $A = [a_{ni}]$ an infinite matrix of non-negative numbers such that none of the columns of the matrix A consists only of zeros.

Let

$$\rho^A_n(t, f) = \sum_{i=1}^{\infty} a_{ni}\rho_i(t, f), \qquad \rho^A_{n0}(t, f) = \sup_i a_{ni}\rho_i(t, f)$$

for $n = 1, 2, \ldots$ By means of terms of a sequence (ρ_n) and by means of a

Key words and phrases: modular space, Lebesgue measure, singular integral.
1991 *Mathematics Subject Classification*: 41A35.

matrix A we shall construct the following modulars in $\mathscr{X}$:

$$\rho^{A,s}(f)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\rho_n^A(f)}{1+\rho_n^A(f)},\quad \text{where}\quad \rho_n^A(f)=\int_\Omega \rho_n^A(t,f)d\mu,$$
$$\rho_s^A(f)=\int_\Omega\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\rho_n^A(t,f)}{1+\rho_n^A(t,f)}d\mu,$$
$$\rho^A(f)=\sup_n \rho_n^A(f),\qquad \rho_A(f)=\int_\Omega \sup_n \rho_n^A(t,f)d\mu,$$
$$\rho_0^{A,s}(f)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\rho_{n0}^A(f)}{1+\rho_{n0}^A(f)},\quad \text{where}\quad \rho_{n0}^A(f)=\int_\Omega \rho_{n0}^A(t,f)d\mu,$$
$$\rho_{0s}^A(f)=\int_\Omega\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\rho_{n0}^A(t,f)}{1+\rho_{n0}^A(t,f)}d\mu,$$
$$\rho_0^A(f)=\sup_n \rho_{n0}^A(f),\qquad \rho_1^A(f)=\int_\Omega \sup_n \rho_{n0}^A(t,f)d\mu.$$

Let us denote by: $X_{\rho^{A,s}}, X_{\rho_s^A}, X_{\rho^A}, X_{\rho_A}, X_{\rho_0^{A,s}}, X_{\rho_{0s}^A}, X_{\rho_0^A}$ and $X_{\rho_1^A}$ the respective modular spaces. There hold the following inclusions

$$X_{\rho^{A,s}}\subset X_{\rho_s^A},\ X_{\rho^A}\supset X_{\rho_A},\qquad X_{\rho_0^{A,s}}\subset X_{\rho_{0s}^A},\ X_{\rho_0^A}\supset X_{\rho_1^A}.$$

In the special case when $a_{nn}=1,\ a_{ni}=0$ for $n\neq i,\ n,i=1,2,\dots$, we have $\rho^{A,s}=\rho_0^{A,s}=\rho^s$ and $\rho_s^A=\rho_{0s}^A=\rho_s$. The modular spaces X_{ρ^s} and X_{ρ_s} was study in [1]–[3].

We shall approximate elements of the modular spaces $X_{\rho^{A,s}}$ and $X_{\rho_s^A}$ by means of terms of a sequence $(\widetilde{\rho}_m)$, where $\widetilde{\rho}_m:\Omega\times\mathscr{X}\to\langle 0,\infty\rangle$ for $m=1,2,\dots$ and the following conditions are satisfied:

($\tilde{\text{a}}$) $\widetilde{\rho}_m(t,f)$ is a pseudomodular in $\mathscr{X}$ for almost all t and for every $m=1,2,\dots$,

($\tilde{\text{b}}$) $\widetilde{\rho}_m(t,f)$ and $\widetilde{\rho}_m(t,f-f(t))$ are measurable and almost everywhere finite with respect to t for every $f\in\mathscr{X}$ and every $m=1,2,\dots$

In the following we shall suppose that besides conditions: (a)–(c) the following condition is satisfied:

(d) if $f,g\in\mathscr{X}$, $|f(t)|\le|g(t)|$ almost everywhere in Ω, then for $n=1,2,\dots$ $\rho_n(t,f)\le\rho_n(t,g)$ almost everywhere in Ω.

We say that a sequence $(\widetilde{\rho}_m)$ preserves constants if $\widetilde{\rho}_m(t,c)=c$ for all $t\in\Omega$ and for every $c\ge 0$, $m=1,2,\dots$

The sequence $(\widetilde{\rho}_m)$ is called singular at the point $f \in X_{\rho^{A,s}}$ iff for any two positive numbers a, b and for $n = 1, 2, \dots$

$$J_m^n(f) = \sum_{i=1}^{\infty} a_{ni} \int_{\Omega} \rho_i(t, a\widetilde{\rho}_m(\cdot, b(f - f(\cdot))))d\mu \to 0 \quad \text{for} \;\; m \to \infty.$$

THEOREM 1. *If the sequence $(\widetilde{\rho}_m)$ preserves constants and is singular at the point $f \in X_{\rho^{A,s}}$, $f \geq 0$, then for every $\lambda > 0$*

$$\rho^{A,s}\{\lambda[f(\cdot) - \widetilde{\rho}_m(\cdot, f)]\} \to 0 \quad \textit{with} \;\; m \to \infty.$$

Proof. Let $f \in X_{\rho^{A,s}}$, $f \geq 0$, and $\alpha, \beta > 0$, $\alpha + \beta = 1$. In manner, like in [1] and [2], we obtain

$$|\widetilde{\rho}_m(t, f) - f(t)| \leq \widetilde{\rho}_m\left(t, \frac{f - f(t)}{\beta}\right) + \frac{\beta}{\alpha} f(t)$$

for almost all $t \in \Omega$, $m = 1, 2, \dots$ Hence, for $n, m = 1, 2, \dots$ and $\lambda > 0$ we have

$$\int_{\Omega} \sum_{i=1}^{\infty} a_{ni}\rho_i(t, \lambda|\widetilde{\rho}_m(\cdot, f) - f(\cdot)|)d\mu \leq$$

$$\leq \sum_{i=1}^{\infty} a_{ni} \int_{\Omega} \rho_i\Big(t, 2\lambda\widetilde{\rho}_m\Big(\cdot, \frac{f - f(\cdot)}{\beta}\Big)\Big)d\mu + \sum_{i=1}^{\infty} a_{ni} \int_{\Omega} \rho_i\left(t, 2\lambda\frac{\beta}{\alpha} f(\cdot)\right) d\mu,$$

and so

$$\rho^{A,s}(\lambda(\widetilde{\rho}_m(\cdot, f) - f(\cdot))) \leq \rho^{A,s}\Big(2\lambda\widetilde{\rho}_m\Big(\cdot, \frac{f - f(\cdot)}{\beta}\Big)\Big) + \rho^{A,s}\left(2\lambda\frac{\beta}{\alpha} f(\cdot)\right).$$

Since $f \in X_{\rho^{A,s}}$, so, for every $\varepsilon > 0$, there exists $\beta = \beta(\varepsilon) > 0$ such that $\rho^{A,s}(2\lambda\frac{\beta}{\alpha}f(\cdot)) < \frac{\varepsilon}{2}$. The sequence $(\widetilde{\rho}_m)$ is singular at the point $f \in X_{\rho^{A,s}}$. Hence for this β we obtain $\rho^{A,s}\big(2\lambda\widetilde{\rho}_m\big(\cdot, \frac{f-f(\cdot)}{\beta}\big)\big) < \frac{\varepsilon}{2}$ for $m > M = M(\varepsilon) > 0$. Therefore for every $\lambda > 0$

$$\rho^{A,s}\{\lambda[\widetilde{\rho}_m(\cdot, f) - f(\cdot)]\} < \varepsilon \quad \text{for} \;\; m > M.$$

The sequence $(\widetilde{\rho}_m)$ is called singular at the point $f \in X_{\rho_s^A}$ iff for any two positive numbers a, b and for $n = 1, 2, \dots$

$$J_n^m(f)(t) = \sum_{i=1}^{\infty} a_{ni}\rho_i(t, a\widetilde{\rho}_m(\cdot, b(f - f(\cdot)))) \to 0 \quad \text{for} \;\; m \to \infty$$

in measure in Ω.

THEOREM 2. *If the sequence $(\widetilde{\rho}_m)$ preserves constants and is singular at the point $f \in X_{\rho_s^A}$, $f \geq 0$, then for every $\lambda > 0$*

$$\rho_s^A\{\lambda[f(\cdot) - \widetilde{\rho}_m(\cdot, f)]\} \to 0 \quad \textit{for} \;\; m \to \infty.$$

Proof. Let $f \in X_{\rho_s^A}$, $f \geq 0$, and $\alpha, \beta > 0$, $\alpha + \beta = 1$. In manner, like in [1] and [2], we obtain, for every $\lambda > 0$ and for almost all $t \in \Omega$, that

$$\sum_{i=1}^{\infty} a_{ni}\rho_i(t, \lambda|\widetilde{\rho}_m(\cdot, f) - f(\cdot)|) \leq$$
$$\leq \sum_{i=1}^{\infty} a_{ni}\rho_i\Big(t, 2\lambda\widetilde{\rho}_m\Big(\cdot, \frac{f - f(\cdot)}{\beta}\Big)\Big) + \sum_{i=1}^{\infty} a_{ni}\rho_i(t, 2\lambda\frac{\beta}{\alpha}f(\cdot))$$

for $n, m = 1, 2, \ldots$, and so

$$\rho_s^A(\lambda(\widetilde{\rho}_m(\cdot, f) - f(\cdot))) \leq \rho_s^A\Big(2\lambda\widetilde{\rho}_m\Big(\cdot, \frac{f - f(\cdot)}{\beta}\Big)\Big) + \rho_s^A(2\lambda\frac{\beta}{\alpha}f(\cdot)).$$

Since $f \in X_{\rho_s^A}$, then, for every $\varepsilon > 0$, there exists $\beta = \beta(\varepsilon) > 0$ such that $\rho_s^A(2\lambda\frac{\beta}{\alpha}f(\cdot)) < \frac{\varepsilon}{2}$. The sequence $(\widetilde{\rho}_m)$ is singular at the point $X_{\rho_s^A}$. Using the Beppo Levi Theorem for series and the Lebesgue bounded convergence Theorem, we have for this β, $\rho_s^A\big(2\lambda\widetilde{\rho}_m\big(\cdot, \frac{f-f(\cdot)}{\beta}\big)\big) < \frac{\varepsilon}{2}$ for $m > M = M(\varepsilon) > 0$. Hence for $m > M$ we obtain

$$\rho_s^A(\lambda(\widetilde{\rho}_m(\cdot, f) - f(\cdot))) < \varepsilon \quad \text{for} \quad m > M.$$

In the sequel we will consider the following special case. Let $\Omega = \langle 0, 1\rangle$, $\Sigma - \sigma$-algebra of Lebesgue measurable sets in $\langle 0, 1\rangle$, μ — the Lebesgue measure. Let $\mathscr{X}$ denote the set of Σ-measurable and almost everywhere finite functions in $\langle 0, 1\rangle$, extended periodically, with period 1, outside $\langle 0, 1\rangle$, with equality μ-almost everywhere. Let K_n, $\widetilde{K}_m$, $n, m = 1, 2, \ldots$, be functions which are Σ-measurable and positive almost everywhere in $\langle 0, 1\rangle$ and such that

$$\int_0^1 K_n(u)du < \infty \quad \text{for} \quad n = 1, 2, \ldots$$

and

$$\int_0^1 \widetilde{K}_m(u)du = 1 \quad \text{for} \quad m = 1, 2, \ldots$$

We define the following sequences of operators

$$\rho_n(t, f) = \varphi^{-1}\Big(\int_0^1 K_n(u)\varphi(|f(u + t)|)du\Big),$$

$$\text{(A)} \qquad \widetilde{\rho}_m(t, f) = \varphi^{-1}\Big(\int_0^1 \widetilde{K}_m(u)\varphi(|f(u + t)|)du\Big),$$

for $n, m = 1, 2, \ldots$ and for $t \in \langle 0, 1\rangle$, where φ is a convex φ-function and φ^{-1} is the function inverse to φ for $u \geq 0$.

THEOREM 3. *Assume that:* a) *a convex φ-function φ satisfies the condition* (Δ_2) *for large arguments,*

b) *for $f \in X_{\rho^{A,s}}$, $f \geq 0$, and for an arbitrary $b > 0$ the following condition*

$$\lim_{m\to\infty} \int_0^1 \widetilde{K}_m(v)\Big(\int_0^1 \varphi(b|f(v+s) - f(s)|)ds\Big)dv = 0$$

holds,

c) *for every $n = 1, 2, \ldots$*

$$\sum_{i=1}^{\infty} a_{ni}\delta_\varepsilon^i \to 0 \quad \textit{with} \ \ \varepsilon \to 0,$$

where

$$\delta_\varepsilon^i = v_\varepsilon^i \sup_{u \geq v_\varepsilon^i} \frac{\varphi^{-1}(u)}{u}, \quad v_\varepsilon^i = \varphi(a\varepsilon)\int_0^1 K_i(u)du \quad \textit{for} \ \ a > 0.$$

Then the sequence $(\widetilde{\rho}_m)$ of the form (A) *is singular at the point f.*

Proof. Since φ satisfies the condition (Δ_2) for large arguments, so for every $\varepsilon > 0$ and for $a > 0$ there exists $a' = a'(\varepsilon) > 0$ such that $\varphi(au) \leq a'\varphi(u)$ for $u \geq \varepsilon$. Hence it follows

$$\begin{aligned} J_m^n(f) \leq \sum_{i=1}^{\infty} a_{ni} \int_0^1 \varphi^{-1}\Big\{\varphi(a\varepsilon)\int_0^1 K_i(u)\,du + \\ + a'\int_0^1 K_i(u)\Big[\int_0^1 \widetilde{K}_m(v)\varphi(b|f(u+v+t) - f(u+t)|)dv\Big]du\Big\}dt. \end{aligned}$$

It can be easily seen that a convex φ-function φ satisfies the following condition

$$\text{(W)} \qquad v \sup_{u \geq v} \frac{\varphi^{-1}(u)}{u} \to 0 \quad \text{as} \ \ v \to 0^+.$$

Let us put

$$v_\varepsilon^i = \varphi(a\varepsilon)\int_0^1 K_i(u)du, \quad \delta_\varepsilon^i = v_\varepsilon^i \sup_{u \geq v_\varepsilon^i} \frac{\varphi^{-1}(u)}{u}, \quad c_\varepsilon^i = \frac{\delta_\varepsilon^i}{v_\varepsilon^i}.$$

Then, by the condition (W), it follows the following estimation $\varphi^{-1}(u) \leq c_\varepsilon^i u$

for $u \geq v_\varepsilon^i$. Hence we obtain that for $m, n = 1, 2, \ldots$ and for $b > 0$

$$J_m^n(f) \leq \Big(\sum_{i=1}^{\infty} a_{ni}\delta_\varepsilon^i\Big)\Big(1 + \frac{a'}{\varphi(a\varepsilon)}\int_0^1 \widetilde{K}_m(v)\Big(\int_0^1 \varphi(b|f(v+s) - f(s)|)ds\Big)dv\Big).$$

The assumptions b) and c) imply that for every $n = 1, 2, \ldots$ $J_m^n(f) \to 0$ for $m \to \infty$. This proves the theorem.

We say that $(\widetilde{K}_m)$ is a singular kernel if

$$\lim_{m\to\infty} \int_\delta^1 \widetilde{K}_m(v)\, dv = 0$$

for every $\delta \in (0, 1)$.

From Theorems 1 and 3 it follows the following

THEOREM 4. *Assume that:* a) *a convex φ-function φ satisfies the condition* (Δ_2) *for large arguments,*

b) *for an arbitrary $a > 0$ and for every $n = 1, 2, \ldots$*

$$\sum_{i=1}^{\infty} a_{ni}\varphi^{-1}(\varphi(a\varepsilon)\int_0^1 K_i(u)du) \to 0 \quad \textit{with} \ \ \varepsilon \to 0,$$

c) *the sequence $(\widetilde{K}_m)$ is a singular kernel.*

Then for $f \in X_{\rho^{A,s}} \cap L^\varphi(0,1)$ and for every $\lambda > 0$ we have

$$\rho^{A,s}\{\lambda[f(\cdot) - \widetilde{p}_m(\cdot, f)]\} \to 0 \quad \textit{for} \ \ m \to \infty.$$

Let us denote for a bounded function $f \in \mathscr{X}$

$$g(t) = \sup_{|v|\leq\delta} \int_0^1 \varphi(|f(v+u+t) - f(u+t)|)\, du,$$

where $t \in \mathbb{R}, \delta \geq 0$, φ-function φ satisfies the condition (Δ_2) for large arguments. It is known (see [4]) that g is a measurable function. We define the φ-integral modulus of continuity in measure for a bounded function $f \in \mathscr{X}$

$$\omega_\mu^\varphi(\eta, \delta; f) = \mu(\{t \in \langle 0, 1\rangle : \sup_{|v|\leq\delta} \int_0^1 \varphi(|f(u+v+t) - f(u+t)|)\, du \geq \eta\}),$$

where $\eta \geq 0, \delta \geq 0$. In [4] the properties of $\omega_\mu^\varphi(\eta, \delta :, f)$ were shown.

For $f \in X_{\rho_s^A}$, $f \geq 0$, let us denote

$$f_k(t) = \begin{cases} f(t) & \text{for } t \in \{t \in \langle 0, 1\rangle : f(t) \leq k\}, \\ k & \text{for the remaining } t \in \langle 0, 1\rangle, \end{cases}$$

where k is a positive integer.

We say that $f \in X_{\rho_s^A}$, $f \geq 0$, is a μ-regular function if for every $k = 1, 2, \ldots$ and for every $\eta > 0$

$$\omega_\mu^\varphi(\eta, \delta; f_k) \to 0 \quad \text{as} \quad \delta \to 0.$$

A sequence $(\widetilde{\rho}_m)$ of the form (A) is called regular at the point $f \in X_{\rho_s^A}$, $f \geq 0$, if for every $n = 1, 2, \ldots$ and for an arbitrary $a > 0$,

$$\sum_{i=1}^{\infty} a_{ni} \rho_i(t, a|\widetilde{\rho}_m(\cdot, f_k) - \widetilde{\rho}_m(\cdot, f)|) \to 0 \quad \text{as} \quad k \to \infty,$$

for almost every $t \in \langle 0, 1 \rangle$ and uniformly with respect to $m = 1, 2, \ldots$

We say that $\rho_i(\cdot, f)$, $i = 1, 2, \ldots$, are equiabsolutely continuous at the point $f \in X_{\rho_s^A}$, $f \geq 0$, if for almost every $t \in \langle 0, 1 \rangle$ and for an arbitrary $\varepsilon > 0$ there exists a $\Delta > 0$ such that for every $i = 1, 2, \ldots$ and for every $A \subset \langle 0, 1 \rangle$, $A \in \Sigma$, such that $\mu(A) < \Delta$, we have

$$\int_A K_i(u) \varphi(f(u + t)) du < \varepsilon.$$

Let us denote by $l_i = l_i(\varepsilon)$ the least positive integer such that

$$(*) \qquad \int_{\{u \in \langle 0,1 \rangle : K_i(u) > l_i\}} K_i(u) du < \varepsilon, \quad \text{where} \quad \varepsilon > 0.$$

THEOREM 5. *Let $f \in X_{\rho_s^A}$, $f \geq 0$, is a μ-regular function. Assume that:* a) *a convex φ-function φ satisfies the condition (Δ_2) for large arguments,*

b) *the sequence*

$$\left(\int_0^1 K_i(u) du \right)$$

is bounded and for every $n = 1, 2, \ldots$ and for an arbitrary $\varepsilon > 0$ the series $\sum_{i=1}^{\infty} a_{ni} l_i(\varepsilon)$, where l_i is defined by the condition $(*)$, is convergent, where $\varphi^{-1}(\varepsilon) \sum_{i=1}^{\infty} a_{ni} l_i(\varepsilon) \to 0$ with $\varepsilon \to 0$,

c) *$\rho_i(\cdot, f)$, $i = 1, 2, \ldots$, are equiabsolutely continuous at the point f,*

d) *the sequence $(\widetilde{\rho}_m)$ of the form* (A) *is regular at the point f and $(\widetilde{K}_m)$ is a singular kernel.*

Then for every $\lambda > 0$

$$\rho_s^A\{\lambda[f(\cdot) - \widetilde{\rho}_m(\cdot, f)]\} \to 0 \quad \textit{for} \quad m \to \infty.$$

Proof. Let $f \in X_{\rho_s^A}$, $f \geq 0$, is a μ-regular function and (f_k) is the sequence of truncated functions of f. For $\lambda > 0$, $m, k = 1, 2, \ldots$ we have

$$\rho_s^A\{\lambda[f(\cdot)-\widetilde{\rho}_m(\cdot,f)]\} \le \rho_s^A\{3\lambda[f(\cdot)-f_k(\cdot)]\}+ \\ +\rho_s^A\{3\lambda[f_k(\cdot)-\widetilde{\rho}_m(\cdot,f_k)]\}+ \\ +\rho_s^A\{3\lambda[\widetilde{\rho}_m(\cdot,f_k)-\widetilde{\rho}_m(\cdot,f)]\}. \tag{1}$$

Since for almost every $t \in \langle 0,1\rangle$ we have $\varphi(3\lambda[f(u+t)-f_k(u+t)]) \to 0$ if $k\to\infty$, for almost every $u\in\langle 0,1\rangle$, so, from the Egorov Theorem, it follows that for an arbitrary $\varepsilon>0$ there exists a $\Delta=\Delta(\varepsilon)>0$, where Δ is defined by equiabsolutely continuity $\rho_i(\cdot,f)$, $i=1,2,\ldots$, at the point f, such that there exists a set A, $A\in\Sigma$, with $\mu(A)<\Delta$, and $\varphi(3\lambda[f(u+t) -f_k(u+t)])<\varepsilon$ for every $k>K=K(t,\varepsilon,\lambda)$, uniformly with respect to $u\in\langle 0,1\rangle\setminus A$. Using the Beppo Levi Theorem for series and the Lebesgue bounded convergence Theorem, we obtain that there exists $K_1=K_1(\varepsilon,\lambda)>0$ such that for $k>K_1$ we have

$$\rho_s^A\{3\lambda[f(\cdot)-f_k(\cdot)]\}<\frac{\varepsilon}{3}. \tag{2}$$

From the Theorem 2 we can conclude that if for any two positive numbers a,b and for every $n=1,2,\ldots$

$$J_n^m(f_k)(t)=\sum_{i=1}^{\infty}a_{ni}\rho_i(t,a\widetilde{\rho}_m(\cdot,b(f_k-f_k(\cdot))))\to 0 \quad \text{with} \quad m\to\infty$$

in measure in $\langle 0,1\rangle$, then $\rho_s^A\{3\lambda[f_k(\cdot)-\widetilde{\rho}_m(\cdot,f_k)]\}\to 0$ with $m\to\infty$. Since φ satisfies the condition (Δ_2) for large arguments, i.e. for an arbitrary $\varepsilon>0$ and for every $a>0$ there exists $a'=a'(\varepsilon,a)>0$ such that $\varphi(au)\le a'\varphi(u)$ for every $u\ge\varepsilon$, so the following estimation

$$J_n^m(f_k)(t)\le\sum_{i=1}^{\infty}a_{ni}\varphi^{-1}\Big(\varphi(a\varepsilon)\int_0^1 K_i(u)\,du+ \\ +a'\int_0^1 K_i(u)\Big(\int_0^1 \widetilde{K}_m(v)\varphi(b|f_k(v+u+t)-f_k(u+t)|)\,dv\Big)du\Big)$$

for $n=1,2,\ldots,$ $t\in\langle 0,1\rangle$, holds. In the following we obtain that for $\delta\in(0,1)$

$$\int_0^1 K_i(u)\Big(\int_0^1\widetilde{K}_m(v)\varphi(b|f_k(v+u+t)-f_k(u+t)|)dv\Big)du\le \\ \le\int_0^\delta\widetilde{K}_m(v)\Big(\int_0^1 K_i(u)\varphi(b|f_k(v+u+t)-f_k(u+t)|)du\Big)dv+ \\ +\varphi(2bk)\Big(\int_\delta^1\widetilde{K}_m(v)dv\Big)\Big(\int_0^1 K_i(u)du\Big)=I_1+I_2$$

and

$$I_1 \leq \left(l_i^k \sup_{0 \leq v \leq \delta} \int_0^1 \varphi(b|f_k(v+u+t) - f_k(u+t)|)du + \frac{\varepsilon}{a'}\right) \int_0^\delta \widetilde{K}_m(v)dv,$$

where l_i^k is the least positive integer such that

$$\int_{\{u \in \langle 0,1\rangle : K_i(u) > l_i^k\}} K_i(u)du < \frac{\varepsilon}{a'\varphi(2bk)}.$$

By the assumption a) it follows that for a' there exists $a'' = a''(\varepsilon, a', b) > 0$ such that for every $u \geq (1/b)\varphi^{-1}(\varepsilon/a')$ we have $\varphi(bu) \leq a''\varphi(u)$. Hence we obtain

$$\int_0^1 \varphi(b|f_k(v+u+t) - f_k(u+t)|)du \leq a'' \int_0^1 \varphi(|f_k(v+u+t) - f_k(u+t)|)du + \frac{\varepsilon}{a'}.$$

Therefore for an arbitrary $\eta > 0$ and for every $n = 1, 2, \ldots$ we have

$$
\begin{aligned}
(3) \qquad & \mu(\{t \in \langle 0,1\rangle : J_n^m(f_k)(t) \geq \eta\}) \leq \\
& \leq \mu\left(\left\{t \in \langle 0,1\rangle : \varphi^{-1}(C\varphi(a\varepsilon))D_n \geq \frac{\eta}{5}\right\}\right) + \\
& + \mu\left(\left\{t \in \langle 0,1\rangle : \varphi^{-1}(\varepsilon)D_n \geq \frac{\eta}{5}\right\}\right) + \\
& + \mu\left(\left\{t \in \langle 0,1\rangle : \varphi^{-1}\left(a'\varphi(2bk)C\int_\delta^1 \widetilde{K}_m(v)dv\right)D_n \geq \frac{\eta}{5}\right\}\right) + \\
& + \omega_\mu^\varphi\left((1/(a'a''))\varphi\left(\eta/\left(5\sum_{i=1}^\infty a_{ni}l_i^k\right)\right), \delta; f_k\right) + \\
& + \mu\left(\left\{t \in \langle 0,1\rangle : \varphi^{-1}(\varepsilon)\sum_{i=1}^\infty a_{ni}l_i^k \geq \frac{\eta}{5}\right\}\right),
\end{aligned}
$$

where

$$\int_0^1 K_i(u)du \leq C, \quad \sum_{i=1}^\infty a_{ni} \leq D_n, \quad i, n = 1, 2, \ldots$$

For $k = 1, 2, \ldots$ the following estimation

$$\rho_s^A\{3\lambda[f_k(\cdot) - \widetilde{p}_m(\cdot, f_k)]\} \leq \int_0^1 \sum_{n=1}^\infty \frac{1}{2^n} \frac{J_n^m(f_k)(t)}{1 + J_n^m(f_k)(t)}dt + \rho_s^A\left(6\lambda\frac{\beta}{\alpha}f_k(\cdot)\right)$$

holds. Since $f \in X_{\rho_s^A}$, so there exists β such that $\rho_s^A(6\lambda(\beta/\alpha)f_k(\cdot)) < \frac{\varepsilon}{6}$. Because f is a μ-regular function and $(\widetilde{K}_m)$ is a singular kernel, by the estimation (3) we obtain that for every $n = 1, 2, \dots$ $J_n^m(f_k) \to 0$ with $m \to \infty$ in measure in $\langle 0, 1\rangle$. Therefore, using the Lebesgue bounded convergence Theorem, we have that for $k = 1, 2, \dots$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \int_0^1 \frac{J_n^m(f_k)(t)}{1 + J_n^m(f_k)(t)} dt < \frac{\varepsilon}{6} \quad \text{for } m > M = M(\varepsilon, k),$$

and so

$$\rho_s^A\{3\lambda[f_k(\cdot) - \widetilde{\rho}_m(\cdot, f_k)]\} < \frac{\varepsilon}{3} \quad \text{for } m > M. \tag{4}$$

Since the sequence $(\widetilde{\rho}_m)$ is regular at the point f, so, using the Lebesgue in measure convergence Theorem, we obtain

$$\rho_s^A\{3\lambda[\widetilde{\rho}_m(\cdot, f_k) - \widetilde{\rho}_m(\cdot, f)]\} < \frac{\varepsilon}{3} \tag{5}$$

for $k > K_2 = K_2(\varepsilon, \lambda)$, uniformly with respect to $m = 1, 2, \dots$

Let $k_0 > \max(K_1, K_2)$. Then, putting in (1) $k = k_0$, we have from (1) and (2), (4), (5) that

$$\rho_s^A\{\lambda[f(\cdot) - \widetilde{\rho}_m(\cdot, f)]\} < \varepsilon$$

for $m > M = M(\varepsilon, k_0)$. This completes the proof.

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Received February 2nd, 2000.