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Noori F. A. AL-Mayahi

ON INFINITE DIMENSIONAL GENERALIZATION OF YAN'S THEOREM

1. Introduction

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be any probability space and denote by B_+ the set of nonnegative bounded random variables. Classical theorem of Yan (1980) state that, for any convex subset K in $L^1(\Omega, \mathcal{F}, P)$ such that $0 \in K$, the following three conditions are equivalent:

(a) for each $f \in L^1_+(\Omega, \mathcal{F}, P)$, $f \neq 0$, there exists $c > 0$ such that cf $\notin \overline{K - B_+}$,

(b) for each $A \in \mathcal{F}$ such that $P(A) > 0$, there exists $c > 0$ such that $cI_A \notin \overline{K - B_+}$,

(c) there exists a bounded random variable $Z \in L^\infty(\Omega, \mathcal{F}, P)$ such that $Z > 0$ a. e. and $\mathrm{Sup}_{Y \in K} \mathbb{E}[ZY] < +\infty$.

In the proof of this theorem there is only one serious difficulty to overcome: to show that (b) implies (c), This is done by the use of 'second separation theorem'(Schaefer 1971 page 65): Let A, B be non-empty, disjoint convex subsets of a locally convx space X such that A is closed and B is compact.There exists a closed real hyperplane in X strictly separating A and B. This theorem is used with $A = \overline{K - B_+}$ and $B = \{cI_A\}$.

A rather simple theorem of Yan is interesting because of applications. We only mention the followimg reasoning which gives the existence of a martingale measure. We use an $L^2(\Omega, \mathcal{F}, P)$ version of Yan's theorem.Let $\xi_1, \xi_2, \ldots \in L^2(\Omega, \mathcal{F}, P)$ be a random process adapted to the filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots$ With $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Take the following linear subspace

$$K = \mathrm{lin}\{I_A \xi_i;\ A \in \mathcal{F}_{i-1},\ i = 1, 2, \ldots\}.$$

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Then the assumption that, for any $A \in \mathcal{F}$, $P(A) > 0$, there exists $c > 0$ such that $cI_A \notin \overline{K - B_+}$ (the bar denotes the closure in L^2), implies the existence of density function $Z \in L^2$ which is positive with probability one and $\xi_1, \xi_2, \ldots$ are martingale increment for probability Q, $\frac{dQ}{dP} = Z$.

An equivalent martingale measure is proved to be an important tool in applied probability (Musiela, Rutkowski 1997). In particular, d-dimensional generalization of Yan's theorem was used in (Pham, Touzi Theorem 3.1). On the other hand, it is natural to discribe some complicated data by random process taking values in infinite dimensional spaces. In particular, in a Hilbert space. Theory of distribution of infinite dimensional random vector is one of the most classical toppics in probability (Vakhania 1981, Parthasathy 1967, Bryc 1995, Janson 1997, Fellah, Pardoux 1992).

In this paper some natural genralization of Yan's theorem, for random variables taking values in a Hilbert space equiped with an orthonormal basis, is given. For simplicity, we use ℓ^2 with the natural orthonormal basis. (e_m), $e_m = (0, 0, \ldots, 1, 0, \ldots)$.

2. Notations and results

By ℓ^2_+ we denote the "positive cone" of ℓ^2; i.e. $\ell^2_+ = \{x \in \ell^2 : \langle x, e_i \rangle \geq 0$ for $i = 1, 2, \ldots\}$ where (e_m) is the natural orthonormal basis in ℓ^2.

An ℓ^2-valued random variable (defined on $(\Omega, \mathcal{F}, P)$ is a measurable function $f : \Omega \to \ell^2$ where ℓ^2 is equipped with its Borel σ-field $\beta(\ell^2)$, in ℓ^2 the norm topology is always taken. Denote by $L^0(\ell^2) = L^0(\Omega, \mathcal{F}, P; \ell^2)$ the space of all ℓ^2-valued random variables with the topology of convergence in probability. By $L^0_+(\ell^2) = L^0_+(\Omega, \mathcal{F}, P; \ell^2)$, we denote the positive cone of $L^0(\ell^2)$. Namely $f \in L^0_+(\ell^2) \Leftrightarrow f(\omega) \in \ell^2_+$ for almost all $\omega \in \Omega$. We also use the standard notation $L^p(\ell^2) = L^p(\Omega, \mathcal{F}, P; \ell^2)$, for $1 \leq p \leq \infty$, and $L^p_+(\ell^2) = L^p(\ell^2) \cap L^0_+(\ell^2)$, $1 \leq p < \infty$, $B_+ = L^\infty(\ell^2) \cap L^0_+(\ell^2)$.

As the main theorem of this paper, we prove.

THEOREM 2.1. *Let K be a convex subset of $L^1(\ell^2)$ such that $0 \in K$. Then the following three conditions are equivalent:*

(a) *For each $f \in L^1_+(\ell^2)$, $f \neq 0$, there exist $\lambda > 0$ such that $\lambda f \notin \overline{K - B_+}$,*

(b) *For each $A \in \mathcal{F}$, $P(A) > 0$, and for each $i = 1, 2, \ldots$, there exists $\lambda > 0$ such that $\lambda I_A e_i \notin \overline{K - B_+}$,*

(c) *there exists a $Z \in L^\infty(\ell^2)$ such that $P\{\omega : \langle Z(\omega), e_i \rangle > 0\} = 1$ for each $i = 1, 2, \ldots$ and $\mathrm{Sup}_{Y \in K}\, \mathbb{E}[\langle Z, Y \rangle] < +\infty$.*

P r o o f. The implication (a) implies (b) is obvious. We prove that (b) implies (c).

Let $A \in \mathcal{F}$ be such that $P(A) > 0$, by condition (b) for each $i = 1, 2, \ldots,$ there exists $\lambda > 0$ such that $\lambda I_A e_i \notin \overline{K - B_+}$. Since $K - B_+$ is convex and $L^\infty(\ell^2)$ is the dual of $L^1(\ell^2)$, by separation theorem, there exists $g \in L^\infty(\ell^2)$ such that

$$\operatorname*{Sup}_{Y \in K, \eta \in B_+} \mathbb{E}(\langle g, Y - \eta \rangle) < \lambda \mathbb{E}\langle g, I_A e_i \rangle.$$

Replacing η by $a\eta$ where $a \in \mathbb{R}_+$ and taking $Y = 0$ we have $a\mathbb{E}(\langle g, -\eta \rangle) < \lambda \mathbb{E}\langle g, I_A e_i \rangle$ for any $a \in \mathbb{R}_+$. If $\eta = I_{\{\omega : \langle g(\omega), e_i \rangle < 0\}}$, then

$$\mathbb{E}(\langle g, -I_{\{\omega : \langle g(\omega), e_i \rangle < 0\}} \rangle) < \frac{\lambda}{a} \mathbb{E}\langle g, I_A e_i \rangle.$$

Taking $a \to \infty$, we obtain

$$\mathbb{E}(\langle g, I_{\{\omega : \langle g(\omega), e_i \rangle < 0\}} \rangle) \geq 0$$

which implies that $g \geq 0$ a.e. By putting $\eta = 0$, we get that

$$\operatorname*{Sup}_{Y \in K} (\mathbb{E}\langle g, Y \rangle) < \mathbb{E}(\langle g, I_A e_i \rangle) < +\infty.$$

Now, let $G = \{g \in L^\infty_+(\ell^2) : \operatorname{Sup}_{Y \in K} \mathbb{E}(\langle g, Y \rangle) < +\infty\}$. From the above arguments, $G \neq \emptyset$.

Let $\mathcal{G}$ be the family of all subsets of Ω being the supports of the elements g of G. Note that $\mathcal{G}$ is closed under countable union, as for a sequence $(g_n)_{n \geq 1} \in G$ we may find strictly positive scalars $(\lambda_n)_{n \geq 1}$ such that $\sum_{n=1}^\infty \lambda_n g_n \in G$.

Hence there is $Z \in G$ such that for $A_0 = \{\omega : \langle Z(\omega), e_i \rangle > 0$ for $i = 1, 2, \ldots\}$ we have $P(A_0) = \operatorname{Sup}\{P(A) : A \in \mathcal{G}\}$ for example,

$$Z = \sum_{n=1}^\infty \frac{1}{2^n \|g_n\|_{L^\infty(\ell^2)}} g_n$$

for some $g_n \in G$ with $P\{\omega : \langle g_n(\omega), e_i \rangle > 0\} > P(A_0) - \frac{1}{n}$.

We shall show that $P(A_0) = 1$, thus that $Z > 0$ a.s. supposing that $P(A_0) < 1$, we can find $Z_0 \in G$ such that $\mathbb{E}(\langle Z_0, I_{\Omega \setminus A_0} e_i \rangle) > 0$ and hence $Z_0 + Z \in G$ with $P\{\omega : \langle (Z_0 + Z)(\omega), e_i \rangle > 0\} > P(A_0)$. This gives a contradiction.

To prove that (c) implies (a) by contradiction, Suppose that there exists $f \in L^1_+(\ell^2)$, $f \neq 0$ and for each n we have $nf \in \overline{K - B_+}$. So $nf = Y_n - h_n - \delta_n$ where $Y_n \in K$, $h_n \in B_+$, $\|\delta_n\|_{L^1(\ell^2)} < \frac{1}{n}$. If Z is bounded ℓ^2-valued random variable, say $\|Z\| < 1$, such that $Z > 0$ a. e. we have that

$$\mathbb{E}(\langle Z, Y_n \rangle) \geq n\mathbb{E}\langle Z, f \rangle - \frac{1}{n}$$

and hence

$$\operatorname*{Sup}_{Y \in K} \mathbb{E}\langle Z, Y\rangle = +\infty.$$

This gives a contradiction. ∎

COROLLARY 2.2. *Let K be a convex subset of $L^1(\ell^2)$ such that $0 \in K$, and let for every $\varepsilon > 0$ and for each $i = 1, 2, \ldots$ there exists $\lambda > 0$ such that $P\{\omega : \langle Y(\omega), e_i\rangle > \lambda\} \leq \varepsilon$ for each $Y \in K$. Then each one of conditions in theorem* 2.1 *holds.*

Proof. We have to show that condition (b) of theorem 2.1.1 holds. Let $A \in \mathcal{F}$ be such that $P(A) > 0$, and take $\varepsilon = \frac{1}{2}P(A)$, then for each $i = 1, 2, \ldots$ there exists $\lambda > 0$ such that

$$P\{\omega : \langle Y(\omega), e_i\rangle > \lambda\} \leq \frac{1}{2}P(A)$$

which implies that for each $i = 1, 2, \ldots$ We have $\lambda I_A e_i \notin \overline{K - B_+}$. By assuming that $0 \in K$ the condition (b) of Theorem 2.1.1 holds. ∎

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FACUTLY OF MATHEMATICS
UNIVERSITY OF ŁÓDŹ
Banacha 22
90-238 ŁÓDŹ, POLAND

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