DEMONSTRATIO MATHEMATICA
Vol. XXXIII No 4 2000

Brian Fisher, Liu Zhenhai and Joel D. Nicholas

THE CONVOLUTION OF FUNCTIONS AND DISTRIBUTIONS

Abstract. The non-commutative convolution product $f*g$ of two distributions f and g in $\mathcal{D}'$ is defined to be the limit of the sequence $\{(f\tau_n)*g\}$, provided the limit exists, where $\{\tau_n\}$ is a certain sequence of functions τ_n in $\mathcal{D}$ converging to 1.

In the following, $\mathcal{D}$ denotes the space of infinitely differentiable functions with compact support and $\mathcal{D}'$ denotes the space of distributions defined on $\mathcal{D}$.

The convolution product of certain pairs of distributions in $\mathcal{D}'$ is usually defined as follows, see for example Gel'fand and Shilov [4].

DEFINITION 1. Let f and g be distributions in $\mathcal{D}'$ satisfying either of the following conditions:

(a) either f or g has bounded support,
(b) the supports of f and g are bounded on the same side.

Then the *convolution product* $f*g$ is defined by the equation

$$\langle (f*g)(x), \varphi(x)\rangle = \langle g(x), \langle f(t), \varphi(x+t)\rangle\rangle \tag{1}$$

for arbitrary test function φ in $\mathcal{D}$.

The classical definition of the convolution product is as follows :

DEFINITION 2. If f and g are locally summable functions then the *convolution product* $f*g$ is defined by

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt \tag{2}$$

for all x for which the integrals exist.

1991 *Mathematics Subject Classification*: 46F10.
Key words and phrases: distribution, Dirac delta function, convolution product.

Note that if f and g are locally summable functions satisfying either of the conditions (a) or (b) in Definition 1, then Definition 1 is in agreement with Definition 2.

It follows that if the convolution product $f * g$ exists by Definitions 1 or 2 then the following equations hold:

$$f * g = g * f, \tag{3}$$

$$(f * g)' = f * g' = f' * g. \tag{4}$$

Definition 1 is rather restrictive and so a neutrix convolution product was introduced in [2]. In order to define the neutrix convolution product, we first of all let τ be a function in $\mathcal{D}$ satisfying the following conditions:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \le \tau(x) \le 1$,
(iii) $\tau(x) = 1, |x| \le \frac{1}{2}$,
(iv) $\tau(x) = 0, |x| \ge 1$.

The function τ_n is now defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \le n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n. \end{cases}$$

DEFINITION 3. Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$. Then the *neutrix convolution product* $f \circledast g$ is defined to be the neutrix limit of the sequence $\{f_n * g\}$, provided the limit h exists in the sense that

$$\operatorname*{N-lim}_{n\to\infty} \langle f_n * g, \varphi \rangle = \langle h, \varphi \rangle$$

for all φ in $\mathcal{D}$, where N is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range the real numbers with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n \quad (\lambda > 0,\ r = 1, 2, \ldots)$$

and all functions which converge to zero as n tends to infinity.

Note that the convolution product $f_n * g$ in this definition is in the sense of Definition 1, the support of f_n being bounded. Note also that the neutrix convolution product in this definition, is in general non-commutative.

It was proved in [2] that if the convolution product $f * g$ exists by Definition 1, then the neutrix convolution product $f \circledast g$ exists and

$$f * g = f \circledast g,$$

showing that Definition 3 is a generalization of Definition 1.

We now give a definition of the convolution product which generalizes both Definitions 1 and 2 but is a particular case of Definition 3.

DEFINITION 4. Let f, f_1, g and g_1 be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ and $f_{1,n} = f_1\tau_n$ for $n = 1, 2, \ldots$. Then the *convolution product* $f * g$ is defined to be the limit of the sequence $\{f_n * g\}$, provided the limit h exists in the sense that

$$\lim_{n\to\infty} \langle f_n * g, \varphi\rangle = \langle h, \varphi\rangle$$

for all φ in $\mathcal{D}$.

Further, if $f * g$ and $f_1 * g_1$ do not exist, but

$$\lim_{n\to\infty} \langle f_n * g + f_{1,n} * g_1, \varphi\rangle = \langle h, \varphi\rangle$$

for all φ in $\mathcal{D}$, we will say that the *sum* $f * g + f_1 * g_1$ of the *convolution products* $f * g$ and $f_1 * g_1$ exists and is equal to h.

We now prove our main theorem.

THEOREM 1. *The convolution product* $x_-^\lambda * x_+^{-\lambda-r}$ *and the sum of the convolution products* $x_-^\lambda * x_+^{-\lambda-1}$ *and* $-x_-^\mu * x_+^{-\mu-1}$ *exist and*

$$x_-^\lambda * x_+^{-\lambda-r} = \frac{(-1)^{r-1}(r-2)!\,\Gamma(\lambda+1)}{\Gamma(\lambda+r)} x^{-r+1} + \frac{(-1)^{r-1}\pi\cot(\pi\lambda)\Gamma(\lambda+1)}{\Gamma(\lambda+r)}\delta^{(r-2)}(x), \tag{5}$$

$$x_-^\lambda * x_+^{-\lambda-1} - x_-^\mu * x_+^{-\mu-1} = \frac{\Gamma'(-\mu)}{\Gamma(-\mu)} - \frac{\Gamma'(-\lambda)}{\Gamma(-\lambda)} + \pi[\cot(\pi\lambda) - \cot(\pi\mu)]H(x) \tag{6}$$

for $\lambda, \mu \neq 0, \pm 1, \pm 2, \ldots$ *and* $r = 2, 3, \ldots$.

Proof. We put $(x_-^\lambda)_n = x_-^\lambda \tau_n(x)$. Then

$$(x_-^\lambda)_n * x_+^{-\lambda-r} = (x_-^\lambda)_n * [x_+^{-\lambda-r}H(1-x)] + [x_-^\lambda H(x+1)] * [x_+^{-\lambda-r}H(x-1)] + [(x_-^\lambda)_n H(-x-1)] * [x_+^{-\lambda-r}H(x-1)],$$

where H denotes Heaviside's function. The convolutions

$$x_-^\lambda * [x_+^{-\lambda-r}H(1-x)], \quad [x_-^\lambda H(x+1)] * [x_+^{-\lambda-r}H(x-1)] \tag{8}$$

exist by Definition 1 and

$$\lim_{n\to\infty}(x_-^\lambda)_n * [x_+^{-\lambda-r}H(1-x)] = x_-^\lambda * [x_+^{-\lambda-r}H(1-x)], \tag{9}$$

since $x_+^{-\lambda-r}H(1-x)$ and $x_-^\lambda H(x+1)$ have compact supports.

Further, $(x_-^\lambda)_n H(-x-1)$ is a locally summable function with compact support and

$$\begin{aligned}[(x_-^\lambda)_n H(-x-1)] * [x_+^{-\lambda-r}H(x-1)] &= \int_{-n}^{-1}(-t)^\lambda(x-t)^{-\lambda-r}H(x-t-1)\,dt \\ &+ \int_{-n-n^{-n}}^{-n}\tau_n(t)(-t)^\lambda(x-t)^{-\lambda-r}\,dt.\end{aligned} \tag{10}$$

If $-n < x < 0$, we have

$$\begin{aligned}\int_{-n}^{-1}(-t)^\lambda(x-t)^{-\lambda-r}H(x-t-1)\,dt &= \int_{-n}^{x-1}(-t)^\lambda(x-t)^{-\lambda-r}\,dt \\ &= (-1)^r\int_{-n}^{x-1}t^{-1}(1-x/t)^{-\lambda-r}\,dt \\ &= (-1)^r\int_{-n}^{x-1}[t^{-r}+(\lambda+r)xt^{-r-1}+\ldots]\,dt\end{aligned}$$

and it follows that

$$\text{d.p.}\int_{-n}^{-1}(-t)^\lambda(x-t)^{-\lambda-r}H(x-t-1)\,dt = 0, \tag{11}$$

$$\text{d.p.}\int_{-n}^{-1}(-t)^\lambda(x-t)^{-\lambda-1}H(x-t-1)\,dt = \ln n, \tag{12}$$

for $x < 0$, and $r = 2, 3, \ldots$, where d.p. denotes the *divergent part* of the integral.

If $x \geq 0$, we have

$$\int_{-n}^{-1}(-t)^\lambda(x-t)^{-\lambda-r}H(x-t-1)\,dt = \int_{-n}^{-1}(-t)^\lambda(x-t)^{-\lambda-r}\,dt$$

$$= \int_{-n}^{-x-1} (-t)^{\lambda}(x-t)^{-\lambda-r}\,dt + \int_{-x-1}^{-1} (-t)^{\lambda}(x-t)^{-\lambda-r}\,dt$$

$$= (-1)^r \int_{-n}^{-x-1} [t^{-r} + (\lambda+r)xt^{-r-1} + \ldots]\,dt + \int_{-x-1}^{-1} (-t)^{\lambda}(x-t)^{-\lambda-r}\,dt$$

and it follows that

$$\text{d.p.} \int_{-n}^{-1} (-t)^{\lambda}(x-t)^{-\lambda-r} H(x-t-1)\,dt = 0,$$

$$\text{d.p.} \int_{-n}^{-1} (-t)^{\lambda}(x-t)^{-\lambda-1} H(x-t-1)\,dt = \ln n,$$

for $x \geq 0$ and $r = 2, 3, \ldots$. Equations (11) and (12) therefore hold for all x and $r = 2, 3, \ldots$.

It is easily seen that

$$\lim_{n\to\infty} \int_{-n-n^{-n}}^{-n} \tau_n(t)(-t)^{\lambda}(x-t)^{-\lambda-r}\,dt = 0 \tag{13}$$

for all x and it now follows from equations (7) to (12) that

$$\text{d.p.}\,(x_-^{\lambda})_n * x_+^{-\lambda-r} = 0,$$
$$\text{d.p.}\,[(x_-^{\lambda})_n * x_+^{-\lambda-1} - (x_-^{\mu})_n * x_+^{-\mu-1}] = 0$$

and so

$$\lim_{n\to\infty} (x_-^{\lambda})_n * x_+^{-\lambda-r}, \tag{14}$$

$$\lim_{n\to\infty} [(x_-^{\lambda})_n * x_+^{-\lambda-1} - (x_-^{\mu})_n * x_+^{-\mu-1}] \tag{15}$$

exist for $\lambda, \mu \neq 0, \pm 1, \pm 2, \ldots$ and $r = 2, 3, \ldots$.

It was proved in [3] that

$$x_-^{\lambda} \circledast x_+^{-\lambda-r} = \frac{(-1)^{r-1}(r-2)!\Gamma(\lambda+1)}{\Gamma(\lambda+r)} x^{-r+1} + \frac{(-1)^{r-1}\pi\cot(\pi\lambda)\Gamma(\lambda+1)}{\Gamma(\lambda+r)} \delta^{(r-2)}(x), \tag{16}$$

$$x_-^{\lambda} \circledast x_+^{-\lambda-1} = -\gamma - \frac{\Gamma'(-\lambda)}{\Gamma(-\lambda)} + \pi\cot(\pi\lambda)H(x) - \ln|x|, \tag{17}$$

for $\lambda > -1$, $\lambda \neq 0, 1, 2, \ldots$ and $r = 2, 3, \ldots$, where γ denotes Euler's constant, and it was later proved in [4] that equations (16) and (17) held for $\lambda < -1$, $\lambda \neq -2, -3, \ldots$ and $r = 2, 3, \ldots$.

It follows from (14) and (15) that

$$\lim_{n\to\infty}(x_-^\lambda)_n * x_+^{-\lambda-r} = \mathrm{N}-\lim_{n\to\infty}(x_-^\lambda)_n * x_+^{-\lambda-r},$$

$$\lim_{n\to\infty}[(x_-^\lambda)_n * x_+^{-\lambda-1} - (x_-^\mu)_n * x_+^{-\mu-1}] = \mathrm{N}-\lim_{n\to\infty}[(x_-^\lambda)_n * x_+^{-\lambda-1} - (x_-^\mu)_n * x_+^{-\mu-1}]$$

and equations (5) and (6) follow from equations (16) and (17) for $\lambda, \mu \neq 0, \pm 1, \pm 2, \dots$ and $r = 2, 3, \dots$.

COROLLARY 1.1. *The convolution product* $x_+^\lambda * x_-^{-\lambda-r}$ *and the sum of the convolution products* $x_+^\lambda * x_-^{-\lambda-1}$ *and* $-x_+^\mu * x_-^{-\mu-1}$ *exist and*

$$(18)\quad x_+^\lambda * x_-^{-\lambda-r} = \frac{(r-2)!\Gamma(\lambda+1)}{\Gamma(\lambda+r)}x^{-r+1} - \frac{\pi\cot(\pi\lambda)\Gamma(\lambda+1)}{\Gamma(\lambda+r)}\delta^{(r-2)}(x),$$

$$(19)\quad x_+^\lambda * x_-^{-\lambda-1} - x_+^\mu * x_-^{-\mu-1} = \frac{\Gamma'(-\mu)}{\Gamma(-\mu)} - \frac{\Gamma'(-\lambda)}{\Gamma(-\lambda)} + \pi[\cot(\pi\lambda) - \cot(\pi\mu)]H(-x)$$

for $\lambda, \mu \neq 0, \pm 1, \pm 2, \dots$ *and* $r = 2, 3, \dots$.

Proof. Equations (18) and (19) follow immediately upon replacing x by $-x$ in equations (5) and (6).

COROLLARY 1.2. *The convolution products* $|x|^\lambda * |x|^{-\lambda-2r-1}$, $(\operatorname{sgn} x.|x|^\lambda) * (\operatorname{sgn} x.|x|^{-\lambda-2r-1})$, $(\operatorname{sgn} x.|x|^\lambda) * |x|^{-\lambda-2r}$ *and* $|x|^\lambda * (\operatorname{sgn} x.|x|^{-\lambda-2r})$ *exist and*

$$(20)\quad |x|^\lambda * |x|^{-\lambda-2r-1} = \frac{2(2r-1)!\Gamma(\lambda+1)}{\Gamma(\lambda+2r+1)}x^{-2r},$$

$$(21)\quad (\operatorname{sgn} x.|x|^\lambda) * (\operatorname{sgn} x.|x|^{-\lambda-2r-1}) = \frac{2(2r-1)!\Gamma(\lambda+1)}{\Gamma(\lambda+2r+1)}x^{-2r},$$

$$(22)\quad (\operatorname{sgn} x.|x|^\lambda) * |x|^{-\lambda-2r} = \frac{2(2r-2)!\Gamma(\lambda+1)}{\Gamma(\lambda+2r)}x^{-2r+1},$$

$$(23)\quad |x|^\lambda * (\operatorname{sgn} x.|x|^{-\lambda-2r}) = \frac{2(2r-2)!\Gamma(\lambda+1)}{\Gamma(\lambda+2r)}x^{-2r+1}$$

for $\lambda, \neq 0, \pm 1, \pm 2, \dots$ *and* $r = 1, 2, \dots$.

Proof. The convolution products $x_+^\lambda * x_+^{-\lambda-r}$ and $x_-^\lambda * x_-^{-\lambda-r}$ exist by Definition 1 and it is easily proved that

$$(24)\quad x_+^\lambda * x_+^{-\lambda-r} = \frac{(-1)^r\pi\operatorname{cosec}(\pi\lambda)\Gamma(\lambda+1)}{\Gamma(\lambda+r)}\delta^{(r-2)}(x),$$

$$x_-^\lambda * x_-^{-\lambda-r} = \frac{\pi \operatorname{cosec}(\pi\lambda)\Gamma(\lambda+1)}{\Gamma(\lambda+r)}\delta^{(r-2)}(x) \tag{25}$$

for $\lambda \neq 0, \pm 1, \pm 2, \dots$ and $r = 2, 3, \dots$.

Now

$$|x|^\lambda * |x|^{-\lambda-r} = x_+^\lambda * x_+^{-\lambda-r} + x_-^\lambda * x_+^{-\lambda-r} + x_+^\lambda * x_-^{-\lambda-r} + x_-^\lambda * x_-^{-\lambda-r} \tag{26}$$

and equation (20) follows from equations (5), (18), (24) and (25).

Equations (21), (22) and (23) follow similarly.

References

[1] J. G. van der Corput, *Introduction to the neutrix calculus*, J. Anal. Math. 7 (1959-60), 291–398.

[2] B. Fisher. *Neutrices and the convolution of distributions*, Zb. Rad. Prirod.-Mat. Fak., Ser. Mat., Novi Sad, 17 (1987), 119–135.

[3] B. Fisher, *On the neutrix convolution of distributions*, Wissenschaftlich Beiräge der TH Wismar, 5.I (1988), 25–35.

[4] B. Fisher, *The neutrix convolution product* $x_-^\lambda \circledast x_+^\mu$, Demonstratio Math., 25 (1992), 525–532.

[5] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Vol. I, Academic Press (1964).

B. Fisher
INSTITUTE OF SIMULATION SCIENCES, SERC, HAWTHORN BUILDING
DE MONTFORT UNIVERSITY, LEICESTER, LE1 9BH, ENGLAND

and

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
THE UNIVERSITY OF LEICESTER, LEICESTER, LE1 7RH, ENGLAND
E-mail: fbr@le.ac.uk

Z. Liu
DEPARTMENT OF MATHEMATICS AND COMPUTERS
CHANGSHA UNIVERSITY OF ELECTRIC POWER
CHANGSHA, HUNAN, 410077, P.R. CHINA
E-mail: zhhliu@hotmail.com

J. D. Nicholas
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
THE UNIVERSITY OF LEICESTER, LEICESTER, LE1 7RH, ENGLAND
E-mail: jdn3@le.ac.uk

Received April 3rd, 2000.